Characterization of magnetorotatory thermohaline instability in porous medium: Darcy Brinkman model

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Abstract: The present paper prescribes upper bounds for oscillatory motions of neutral or growing amplitude in magnetorotatory thermohaline configurations of Veronis (Veronis, G., J. Mar. Res., **23**(1965)1) and Stern types (Stern, M.E., Tellus **12**(1960)172) in porous medium(Darcy-Brinkman model) in such a way that also result in sufficient conditions of stability for an initially bottom-heavy as well as initially top-heavy configuration.

Keywords: Thermohaline instability, oscillatory motions, initially bottom-heavy configuration, initially top-heavy configuration, Porous medium.

I.INTRODUCTION

The convective phenomenon which is driven by the differential diffusion of two properties such as heat and salt is known as thermohaline convection or, more generally, double diffusive convection. The onset of motion in thermohaline convection is fundamentally different from Rayleigh – Benard convection and is of direct relevance to the fields of astrophysics, oceanography, limnology and chemical engineering etc. For a broad view of the subject one may be referred to Turner [1] and Brandt and Fernando [2]. Two fundamental configurations have been studied in the context of thermohaline convection problem, one by Veronis [3], wherein the temperature gradient is destabilizing and the concentration gradient is stabilizing and another by Stern [4] wherein the temperature gradient is stabilizing and the concentration gradient is destabilizing. The main results derived by Veronis and Stern for their respective problems are that instability might occur in the configuration through a stationary pattern of motions or oscillatory motions provided the destabilizing temperature gradient or the concentration gradient is sufficiently large but compatible with the conditions that the total density field is gravitationally stable. Thus thermohaline configurations of the Veronis and the Stern type can be classified into the following two classes:
1. The first class, in which thermohaline instability manifests itself when the total density field is initially bottom heavy, and
2. The second class, in which thermohaline instability manifests itself when the total density field is initially top heavy.
Banerjee et al. [5] derived a characterization theorem for thermohaline convection of the Veronis type that disallow the existence of oscillatory motions of neutral or growing amplitude in an initially bottom heavy configuration for the certain parameter regime.
The problem of thermohaline instability in porous media has attracted considerable interest during the past few decades because of its wide range of applications including the ground water contamination, disposal of waste material, food processing, prediction of ground water movement in aquifers, the energy extraction process from the geothermal reservoirs, assessing the effectiveness of fibrous insulations etc.([6],[7]).The thermohaline instability problem in porous media has been extensively investigated and the growing volume of work devoted to this area is well documented by Nield and Bezan[6] and Vafai [8].Prakash and Vinod[9] have proved the nonexistence of nonoscillatory motions in thermohaline convection of Stern type in porous medium. In the early researches most of the researchers have studied double diffusive convection in porous medium by considering the Darcy flow model which is relevant to densely packed, low permeability porous medium. However, experiments conducted with several combinations of solids and

fluids covering wide ranges of governing parameters indicate that most of the experimental data do not agree with the theoretical predictions based on the Darcy flow model. Hence, non-Darcy effects on double diffusive convection in porous media have received a great deal of attention in recent years. Poulikakos [10] has used the Brinkman extended Darcy flow model for the problem to investigate the linear stability analysis. Recently, Givler and Altobelli [11] have demonstrated that for high permeability porous media the effective viscosity is about ten times the fluid viscosity. Therefore, the effect of viscosities on the stability analysis is of practical interest. Thus in the present paper the Brinkman extended Darcy model has been used to investigate the thermohaline convection in porous medium.

The work of Banerjee et al.[5] have been extended to magnetorotatory thermohaline instability problem in a porous medium by Prakash et al. [12] in the form of characterization theorems for magnetorotatory thermohaline convection of Veronis type and Stern type that disapprove the existence of oscillatory motions of growing amplitude in initially bottom-heavy configurations of the two types respectively and left open the possibility for the derivation of the analogous theorems for an initially top-heavy configuration of the Veronis type and an initially top-heavy configuration of the Stern type.

Moreover, when compliment of the sufficient conditions contained in the characterization theorems of Prakash et al. [12] holds good, oscillatory motions of growing amplitude can exist, and thus it is important to derive bounds for the complex growth rate of such motions when both the boundaries are not dynamically free, so that exact solutions in the closed form are not obtained. Thus present communication, which prescribes upper bounds for the oscillatory motions of neutral or growing amplitude in magnetorotatory thermohaline configurations of Veronis and Stern types in porous medium in such a manner that also results in sufficient conditions for stability for an initially top heavy or initially bottom configuration, may be regarded as a further step in this scheme of extended investigations.
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II. MATHEMATICAL FORMULATION AND ANALYSIS

An infinite horizontal porous layer filled with a viscous fluid is statically confined between two horizontal boundaries z = 0 and z = d maintained at constant temperatures T_0 and T_1 and solute concentrations S_0 and S_1 at the lower and upper boundaries respectively, where $T_1 < T_0$ and $S_1 < S_0$(as shown in Fig.1) in the presence of uniform vertical magnetic field. The layer is rotating about its vertical axis with constant angular velocity $\vec{\omega}$. It is further assumed that the saturating fluid and the porous layer are incompressible and that the porous medium is a constant porosity medium. Darcy-Brinkman model has been used to investigate the present problem. Let the origin be taken on the lower boundary z=0 with z-axis perpendicular to it.
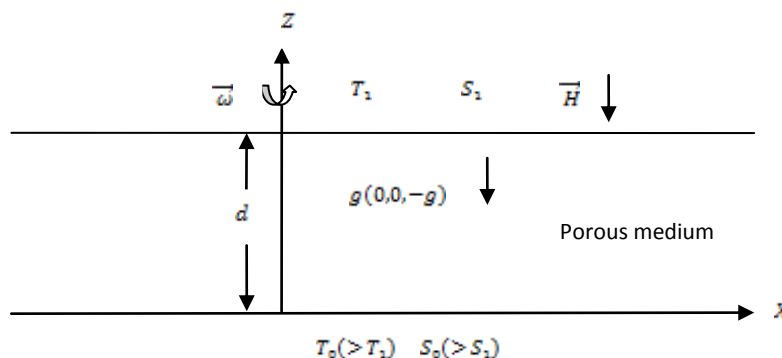

Fig.1 Physical Configuration

The governing hydrodynamic equations in the non dimensional form are given by [12]:

$$\Lambda(D^2 - a^2)^2 w - \left(\frac{\sigma}{\epsilon} + D_a^{-1}\right)(D^2 - a^2)w = Ra^2\theta - R_s a^2\phi - QD(D^2 - a^2)h_z + \frac{T_a}{\epsilon}D\zeta, \qquad (1)$$

$$(D^2 - a^2 - E\sigma P_r)\theta = -w, \qquad (2)$$

$$\left(D^2 - a^2 - \frac{E'\sigma P_r}{\tau}\right)\phi = -\frac{1}{\tau}w, \qquad (3)$$

$$(D^2 - a^2 - \sigma p_2)h_z = -\frac{1}{\epsilon}Dw, \qquad (4)$$

$$\Lambda(D^2 - a^2)\zeta - \left(\frac{\sigma}{\epsilon} + D_a^{-1}\right)\zeta = -QD\xi - \frac{1}{\epsilon}Dw, \tag{5}$$

$$(D^2 - a^2 - \sigma p_2)\xi = -\frac{1}{\epsilon}D\zeta. \tag{6}$$

with boundary conditions

$$w = \theta = \phi = Dw = h_z = \zeta = D\xi = 0 \text{ at } z = 0 \text{ and at } z = 1 \tag{7}$$

(both boundary are rigid and perfectly conducting)

Theorem 1:- If $(\sigma, w, \theta, \phi, h_z)$, $(\sigma = \sigma_r + i\sigma_i)$, $\sigma_r \geq 0, \sigma_i \neq 0$ is a non-trivial solution of equations (1)-(6), with boundary condition (7) and $R > 0, R_s > 0, T_a \geq 0$ and $Q \geq 0$, then

$$|\sigma| < \frac{\lambda R_s}{EP_r(4\pi^2(\Lambda + \delta) + D_a^{-1})}\sqrt{\Omega^2 - 1},$$

where $\Omega = \frac{\lambda R_s}{\left(\frac{27}{4}\pi^4(\Lambda + \delta) + 4\pi^2 D_a^{-1}\right)}$, if $\delta = \min\left(\frac{\tau}{\epsilon E'P_r}, \Lambda, \frac{1}{\epsilon p_2}\right)$ and $\lambda = \frac{R}{R_s}$

Proof: Multiply equation (1) by w^* throughout, integrating the resulting equation over vertical range of z and utilizing equations (2)-(6), we get

$$\Lambda\int_0^1 w^*(D^2 - a^2)^2 w\, dz - \left(\frac{\sigma}{\epsilon} + D_a^{-1}\right)\int_0^1 w^*(D^2 - a^2)w\, dz +$$
$$Ra^2\int_0^1 \theta(D^2 - a^2 - EP_r\sigma^*)\theta^*\, dz - R_s a^2\tau\int_0^1 \phi\left(D^2 - a^2 - \frac{E'\sigma^* P_r}{\tau}\right)\phi^*dz + \epsilon Q\int_0^1 (D^2 - a^2 - \sigma^* p_2)h_z^*(D^2 -$$
$$a^2)h_z\, dz + T_a\left(\frac{\sigma^*}{\epsilon} + D_a^{-1}\right)\int_0^1 \zeta^*\zeta\, dz - T_a Q\epsilon\int_0^1 \xi^*(D^2 - a^2 - \sigma p_2)\xi\, dz - T_a\Lambda\int_0^1 \zeta(D^2 - a^2)\zeta^*dz = 0. \tag{8}$$

Integrate various term of equation (8) an appropriate number of times and using the boundary condition (7), we get

$$\Lambda\int_0^1 (|D^2 w|^2 + 2a^2|Dw|^2 + a^4|w|^2)\, dz + \left(\frac{\sigma}{\epsilon} + D_a^{-1}\right)\int_0^1 (|Dw|^2 + a^2|w|^2)\, dz - Ra^2\int_0^1 (|D\theta|^2 + a^2|\theta|^2)dz -$$
$$Ra^2 E\sigma^* P_r\int_0^1 |\theta|^2 + R_s a^2\tau\int_0^1 (|D\phi|^2 + a^2|\phi|^2)dz + R_s a^2 E'\sigma^* P_r\int_0^1 |\phi|^2\, dz +$$
$$\epsilon Q\int_0^1 |(D^2 - a^2)h_z|^2 dz + \epsilon Q\sigma^* p_2\int_0^1 (|Dh_z|^2 + a^2|h_z|^2)dz + T_a Q\epsilon\int_0^1 (|D\xi|^2 + a^2|\xi|^2 + \sigma p_2|\xi|^2)\, dz + T_a\Lambda\int_0^1 (|D\zeta|^2 +$$
$$a^2|\zeta|^2)\, dz + T_a\left(\frac{\sigma^*}{\epsilon} + D_a^{-1}\right)\int_0^1 |\zeta|^2\, dz = 0. \tag{9}$$

Equating real and imaginary parts of (9) equal to zero and cancelling $\sigma_i(\neq 0)$ throughout from imaginary part, we have

$$\Lambda\int_0^1 (|D^2 w|^2 + 2a^2|Dw|^2 + a^4|w|^2)\, dz + \left(\frac{\sigma_r}{\epsilon} + D_a^{-1}\right)\int_0^1 (|Dw|^2 + a^2|w|^2)\, dz - Ra^2\int_0^1 (|D\theta|^2 + a^2|\theta|^2)dz -$$
$$Ra^2 EP_r\sigma_r\int_0^1 |\theta|^2 dz + R_s a^2\tau\int_0^1 (|D\phi|^2 + a^2|\phi|^2)dz + R_s a^2 E'P_r\sigma_r\int_0^1 |\phi|^2\, dz + \epsilon Q\int_0^1 |(D^2 - a^2)h_z|^2 dz +$$
$$\epsilon Q p_2\sigma_r\int_0^1 (|Dh_z|^2 + a^2|h_z|^2)dz + T_a Q\epsilon\int_0^1 (|D\xi|^2 + a^2|\xi|^2)\, dz + T_a Q\epsilon\sigma_r p_2\int_0^1 |\xi|^2 dz + T_a\Lambda\int_0^1 (|D\zeta|^2 + a^2|\zeta|^2)\, dz +$$
$$T_a\left(\frac{\sigma_r}{\epsilon} + D_a^{-1}\right)\int_0^1 |\zeta|^2\, dz = 0, \tag{10}$$

and

$$\frac{1}{\epsilon}\int_0^1 (|Dw|^2 + a^2|w|^2)\, dz + Ra^2 EP_r\int_0^1 |\theta|^2\, dz - R_s a^2 E'P_r\int_0^1 |\phi|^2\, dz -$$
$$\epsilon Q p_2\int_0^1 (|Dh_z|^2 + a^2|h_z|^2)dz + T_a Q\epsilon p_2\int_0^1 |\xi|^2 dz - \frac{T_a}{\epsilon}\int_0^1 |\zeta|^2\, dz = 0\, . \tag{11}$$

Multiplying equation (11) by σ_r and adding resulting equation to equation (10), we get

$$\Lambda\int_0^1 (|D^2 w|^2 + 2a^2|Dw|^2 + a^4|w|^2)\, dz + \left(\frac{2\sigma_r}{\epsilon} + D_a^{-1}\right)\int_0^1 (|Dw|^2 + a^2|w|^2)\, dz -$$
$$Ra^2\int_0^1 (|D\theta|^2 + a^2|\theta|^2)dz + R_s a^2\tau\int_0^1 (|D\phi|^2 + a^2|\phi|^2)dz +$$
$$\epsilon Q\int_0^1 |(D^2 - a^2)h_z|^2 dz + T_a Q\epsilon\int_0^1 (|D\xi|^2 + a^2|\xi|^2)\, dz + 2T_a Q\epsilon\sigma_r p_2\int_0^1 |\xi|^2 dz + T_a\Lambda\int_0^1 (|D\zeta|^2 + a^2|\zeta|^2)\, dz +$$
$$T_a D_a^{-1}\int_0^1 |\zeta|^2\, dz = 0. \tag{12}$$

Also, since w, θ, ϕ, ζ and h_z vanish at $z = 0$ and $z = 1$, the Rayleigh-Ritz inequality [13] gives

$$\int_0^1 |Dw|^2 dz \geq \pi^2\int_0^1 |w|^2\, dz, \tag{13}$$

$$\int_0^1 |D\theta|^2 dz \geq \pi^2\int_0^1 |\theta|^2\, dz, \tag{14}$$

$$\int_0^1 |D\phi|^2 dz \geq \pi^2 \int_0^1 |\phi|^2 \, dz, \tag{15}$$

$$\int_0^1 |Dh_z|^2 dz \geq \pi^2 \int_0^1 |h_z|^2 \, dz, \tag{16}$$

$$\int_0^1 |D\zeta|^2 dz \geq \pi^2 \int_0^1 |\zeta|^2 \, dz. \tag{17}$$

Furthermore, utilizing the Schwartz inequality, we have

$$\left(\int_0^1 |w|^2 dz\right)^{\frac{1}{2}} \left(\int_0^1 |D^2 w|^2 dz\right)^{\frac{1}{2}} \geq \left| -\int_0^1 w^* D^2 w \, dz \right| = \int_0^1 |Dw|^2 dz \geq \pi^2 \int_0^1 |w|^2 dz.$$

Consequently,

$$\int_0^1 |D^2 w|^2 dz \geq \pi^4 \int_0^1 |w|^2 dz, \tag{18}$$

and thus

$$\int_0^1 |(D^2 - a^2)w|^2 dz = \int_0^1 (|D^2 w|^2 + 2a^2 |Dw|^2 + a^4 |w|^2) dz$$

$$\geq (\pi^2 + a^2)^2 \int_0^1 |w|^2 dz \tag{19}$$

Further,

$$\int_0^1 |w|^2 dz = \int_0^1 w \, w^* \, dz$$

$$= \int_0^1 (D^2 - a^2 - E\sigma P_r)\theta (D^2 - a^2 - E\sigma^* P_r)\theta^* \, dz$$

$$= \int_0^1 |(D^2 - a^2)\theta|^2 dz + 2E P_r \sigma_r \int_0^1 (|D\theta|^2 + a^2 |\theta|^2) dz + E^2 |\sigma|^2 P_r^2 \int_0^1 |\theta|^2 dz. \tag{20}$$

Since $\sigma_r \geq 0$, it follows from inequality (20) that

$$\int_0^1 |w|^2 dz \geq \int_0^1 |(D^2 - a^2)\theta|^2 dz + E^2 |\sigma|^2 P_r^2 \int_0^1 |\theta|^2 \, dz \tag{21}$$

and

$$\int_0^1 |w|^2 dz \geq \int_0^1 |(D^2 - a^2)\theta|^2 dz. \tag{22}$$

Also, emulating the derivation of inequality (18), we have

$$\int_0^1 |(D^2 - a^2)\theta|^2 dz = \int_0^1 (|D^2 \theta|^2 + 2a^2 |D\theta|^2 + a^4 |\theta|^2) dz$$

$$\geq (\pi^2 + a^2)^2 \int_0^1 |\theta|^2 dz. \tag{23}$$

Combining inequality (21) and (23), we obtain

$$\int_0^1 |w|^2 dz \geq \left((\pi^2 + a^2)^2 + E^2 |\sigma|^2 P_r^2\right) \int_0^1 |\theta|^2 \, dz. \tag{24}$$

Further,

$$\int_0^1 |w|^2 dz = \left(\int_0^1 |w|^2 dz\right)^{\frac{1}{2}} \left(\int_0^1 |w|^2 dz\right)^{\frac{1}{2}}$$

$$\geq \left\{(\pi^2 + a^2)^2 + E^2 |\sigma|^2 P_r^2\right\}^{\frac{1}{2}} \left(\int_0^1 |\theta|^2 \, dz\right)^{\frac{1}{2}} \left(\int_0^1 |(D^2 - a^2)\theta|^2 dz\right)^{\frac{1}{2}} \quad \text{\{Using inequalities (22) and (24)\}}$$

$$\geq (\pi^2 + a^2) \left\{1 + \frac{E^2 |\sigma|^2 P_r^2}{(\pi^2 + a^2)^2}\right\}^{\frac{1}{2}} \left| -\int_0^1 \theta^* (D^2 - a^2)\theta \, dz \right| \quad \text{(Using Schwartz inequality)}$$

$$= (\pi^2 + a^2) \left\{1 + \frac{E^2 |\sigma|^2 P_r^2}{(\pi^2 + a^2)^2}\right\}^{\frac{1}{2}} \int_0^1 (|D\theta|^2 + a^2 |(\theta)|^2) dz. \tag{25}$$

Further, since $h_z(0) = 0 = h_z(1)$, we have

$$\int_0^1 |Dh_z|^2 dz = -\int_0^1 h_z^* D^2 h_z \, dz$$

$$\leq \left| -\int_0^1 h_z^* D^2 h_z \, dz \right|$$

$$\leq \int_0^1 |h_z^* D^2 h_z| \, dz$$

$$\leq \int_0^1 |h_z^*| |D^2 h_z| \, dz$$

$$\leq \int_0^1 |h_z| |D^2 h_z| \, dz$$

$$\leq \left[\int_0^1 |h_z|^2 \ dz\right]^{1/2} \left[\int_0^1 |D^2 h_z|^2 \ dz\right]^{1/2} \qquad \text{(Using Schwartz inequality)}$$

$$\leq \frac{1}{\pi}\left[\int_0^1 |Dh_z|^2 \ dz\right]^{1/2} \left[\int_0^1 |D^2 h_z|^2 \ dz\right]^{1/2}, \qquad \{\text{using inequality (16)}\}$$

so that we have

$$\int_0^1 |D^2 h_z|^2 \ dz \geq \pi^2 \int_0^1 |Dh_z|^2 dz \ , \tag{26}$$

and thus we can write

$$Q\epsilon \int_0^1 |(D^2 - a^2)h_z|^2 \ dz \geq Q\epsilon(\pi^2 + a^2) \int_0^1 (|Dh_z|^2 + a^2 |h_z|^2) \ dz \tag{27}$$

Using inequalities (13),(15),(17),(18),(25) and (27) in equation (12)and utilizing the fact $\sigma_r \geq 0, T_a Q \geq 0$ we have

$$\{\Lambda(\pi^2 + a^2)^2 + D_a^{-1}(\pi^2 + a^2)\} \int_0^1 |w|^2 dz + +R_s a^2 \tau(\pi^2 + a^2)\int_0^1 |\phi|^2 dz + \epsilon Q(\pi^2 + a^2) \int_0^1 (|Dh_z|^2 + a^2 |h_z|^2) \ dz +$$

$$T_a\{\Lambda(\pi^2 + a^2) + D_a^{-1}\} \int_0^1 |\zeta|^2 \ dz < \frac{Ra^2}{(\pi^2 + a^2)}\left\{1 + \frac{E^2 |\sigma|^2 P_r^2}{(\pi^2 + a^2)^2}\right\}^{-\frac{1}{2}} \int_0^1 |w|^2 dz. \tag{28}$$

Equation (11) upon using inequality (13) and the fact $T_a Q \geq 0$ implies

$$R_s a^2 \int_0^1 |\phi|^2 \ dz > \frac{(\pi^2 + a^2)}{\epsilon E' P_r}\int_0^1 |w|^2 dz - \frac{\epsilon Q p_2}{E' P_r}\int_0^1 (|Dh_z|^2 + a^2 |h_z|^2) dz - \frac{T_a}{\epsilon E' P_r}\int_0^1 |\zeta|^2 \ dz, \tag{29}$$

$$T_a \int_0^1 |\zeta|^2 dz > (\pi^2 + a^2)\int_0^1 |w|^2 \ dz - R_s a^2 E' P_r \epsilon \int_0^1 |\phi|^2 \ dz - \epsilon^2 Q p_2 \int_0^1 (|Dh_z|^2 + a^2 |h_z|^2) dz, \tag{30}$$

and

$$\epsilon Q \int_0^1 (|Dh_z|^2 + a^2 |h_z|^2) dz > \frac{(\pi^2 + a^2)}{\epsilon p_2}\int_0^1 |w|^2 \ dz - \frac{R_s a^2 E' P_r}{P_2}\int_0^1 |\phi|^2 \ dz - \frac{T_a}{\epsilon P_2}\int_0^1 |\zeta|^2 \ dz. \tag{31}$$

Inequality (28) coupled with each of inequality (29),(30) and (31) respectively yield the following inequalities

$$\left\{(\pi^2 + a^2)^2 \left(\Lambda + \frac{\tau}{\epsilon E' P_r}\right) + (\pi^2 + a^2)D_a^{-1}\right\}\int_0^1 |w|^2 dz + \epsilon Q(\pi^2 + a^2)\left\{1 - \frac{\tau p_2}{E' P_r}\right\}\int_0^1 (|Dh_z|^2 + a^2 |h_z|^2) \ dz +$$

$$T_a\left[\{(\pi^2 + a^2)\Lambda + D_a^{-1}\} - \frac{(\pi^2 + a^2)\tau}{\epsilon E' P_r}\right]\int_0^1 |\zeta|^2 dz < \frac{Ra^2}{(\pi^2 + a^2)}\left\{1 + \frac{E^2 |\sigma|^2 P_r^2}{(\pi^2 + a^2)^2}\right\}^{-\frac{1}{2}}\int_0^1 |w|^2 dz, \tag{32}$$

$$2\{\Lambda(\pi^2 + a^2)^2 + (\pi^2 + a^2)D_a^{-1}\}\int_0^1 |w|^2 dz + R_s a^2 E' P_r \epsilon\left[\frac{(\pi^2 + a^2)\tau}{E' P_r \epsilon} - \{\Lambda(\pi^2 + a^2) + D_a^{-1}\}\right]\int_0^1 |\phi|^2 \ dz + \epsilon Q[(\pi^2 +$$

$$a^2) - \epsilon p_2\{\Lambda(\pi^2 + a^2) + D_a^{-1}\}]\int_0^1 (|Dh_z|^2 + a^2 |h_z|^2) \ dz < \frac{Ra^2}{(\pi^2 + a^2)}\left\{1 + \frac{E^2 |\sigma|^2 P_r^2}{(\pi^2 + a^2)^2}\right\}^{-\frac{1}{2}}\int_0^1 |w|^2 dz, \tag{33}$$

and

$$\left\{(\pi^2 + a^2)^2 \left(\Lambda + \frac{1}{\epsilon p_2}\right) + (\pi^2 + a^2)D_a^{-1}\right\}\int_0^1 |w|^2 dz + R_s a^2(\pi^2 + a^2)\left\{\tau - \frac{E' P_r}{p_2}\right\}\int_0^1 |\phi|^2 \ dz + \frac{T_a}{\epsilon p_2}[\epsilon p_2\{(\pi^2 + a^2)\Lambda +$$

$$D_a^{-1}\} - (\pi^2 + a^2)]\int_0^1 |\zeta|^2 dz < \frac{Ra^2}{(\pi^2 + a^2)}\left\{1 + \frac{E^2 |\sigma|^2 P_r^2}{(\pi^2 + a^2)^2}\right\}^{-\frac{1}{2}}\int_0^1 |w|^2 dz. \tag{34}$$

Now if $\delta = \min\left(\frac{\tau}{\epsilon E' P_r}, \Lambda, \frac{1}{\epsilon p_2}\right)$,then depending on the value of δ exactly one of the inequalities (32)-(34) will imply that

$$\{(\pi^2 + a^2)^2(\Lambda + \delta) + (\pi^2 + a^2)D_a^{-1}\}\int_0^1 |w|^2 dz < \frac{Ra^2}{(\pi^2 + a^2)}\left\{1 + \frac{E^2 |\sigma|^2 P_r^2}{(\pi^2 + a^2)^2}\right\}^{-\frac{1}{2}}\int_0^1 |w|^2 dz. \tag{35}$$

Since the minimum value of $\frac{(\pi^2 + a^2)^2}{a^2}$ is $4\pi^2$ (for $a^2 = \pi^2$) and the minimum value of $\frac{(\pi^2 + a^2)^3}{a^2}$ is $\frac{27\pi^4}{4}$ (for $a^2 = \frac{\pi^2}{2}$),therefore it follows from inequality (35) that

$$\left\{\frac{27}{4}\pi^4(\Lambda + \delta) + 4\pi^2 D_a^{-1}\right\}\left\{1 + \frac{E^2 |\sigma|^2 P_r^2}{(\pi^2 + a^2)^2}\right\}^{\frac{1}{2}} < R = \lambda R_s. \tag{36}$$

Inequality (36) implies that

$$|\sigma| < \frac{(\pi^2 + a^2)}{E P_r}\sqrt{\Omega^2 - 1}, \tag{37}$$

where $\Omega = \frac{\lambda R_s}{\left(\frac{27}{4}\pi^4(\Lambda + \delta) + 4\pi^2 D_a^{-1}\right)}$.

Further it follows from inequality (35) that

$$(\pi^2 + a^2)\left\{\frac{(\pi^2+a^2)^2}{a^2}(\Lambda + \delta) + D_a^{-1}\right\} < R = \lambda R_s \tag{38}$$

Since the minimum value of $\frac{(\pi^2+a^2)^2}{a^2}$ is $4\pi^2$ (for $a^2 = \pi^2$),therefore it follows from inequality (38) that

$$(\pi^2 + a^2) < \frac{\lambda R_s}{\{4\pi^2(\Lambda+\delta)+D_a^{-1}\}} \tag{39}$$

Combining inequalities (37) and (39), we get

$$|\sigma| < \frac{\lambda R_s}{EP_r(4\pi^2(\Lambda+\delta)+D_a^{-1})}\sqrt{\Omega^2 - 1},$$

which completes the proof of the theorem.

Theorem 1, from the physical point of view of hydrodynamic stability theory, may be stated as: the complex growth rate $\sigma = \sigma_r + i\sigma_i$ of an arbitrary oscillatory perturbation $(\sigma_i \neq 0)$ of neutral or growing amplitude $(\sigma_r \geq 0)$ in magnetorotatory thermohaline convection in porous medium of Veronis' type lies inside a semicircle in the right half of the $\sigma_r\sigma_i$-plane whose center is at the origin and whose radius is $\frac{\lambda R_s}{EP_r(4\pi^2(\Lambda+\delta)+D_a^{-1})}\sqrt{\Omega^2 - 1}$.

This result is uniformly valid for an initially top heavy $(\lambda > 1)$ as well as an initially bottom heavy $(\lambda < 1)$ configuration.

Corollary 1: If $(\sigma, w, \theta, \phi, h_z), \sigma = \sigma_r + i\sigma_i, \sigma_i \neq 0$ is a non trivial solution of equations (1) - (6), together with the boundary condition (7) and R>0, $R_s > 0, Q \geq 0, T_a \geq 0$, and $\lambda < \frac{\frac{27}{4}\pi^4(\Lambda + \delta) + 4\pi^2 D_a^{-1}}{R_s}$ then $\sigma_r < 0$.

Proof: Follows from Theorem 1

Corollary 1 implies that oscillatory motions of growing amplitude are not allowed in magnetorotatory thermohaline convection in porous medium of Veronis' type if the initial stability parameter λ does not exceed the value $\frac{\frac{27}{4}\pi^4(\Lambda+\delta)+4\pi^2 D_a^{-1}}{R_s}$. Further this result is uniformly valid for an initially top heavy $(\lambda > 1)$ as well as an initially bottom heavy $(\lambda < 1)$ configuration.

Remarks: The following remarks, now deserve attention

1.   If $0 < R \leq R_s \leq \frac{27}{4}\pi^4(\Lambda + \delta) + 4\pi^2 D_a^{-1}$ and $\sigma_i \neq 0$ even then Corollary 1 implies that $\sigma_r < 0$. This result is the characterization theorem of Prakash et al. [12], which we see is built into our characterization Corollary 1.

2.   If $0 < R \leq \frac{27}{4}\pi^4(\Lambda + \delta) + 4\pi^2 D_a^{-1} < R_s$ and $\sigma_i \neq 0$, even then Corollary 1 implies that $\sigma_r < 0$, a new result that obviously cannot be averred from characterization theorem of Prakash et al. [12].

Theorem 2: If $(\sigma, w, \theta, \phi, h_z), \sigma = \sigma_r + i\sigma_i, \sigma_r \geq 0, \sigma_i \neq 0$ is a non-trivial solution of equations (1)-(6), together with the boundary condition (7) and $R < 0, R_s < 0$, then

$$|\sigma| < \frac{\bar{\lambda}|R|}{E'P_r(4\pi^2(\Lambda+\bar{\delta})+D_a^{-1})}\sqrt{\bar{\Omega}^2 - 1}$$

where $\bar{\Omega} = \frac{\bar{\lambda}|R|}{\tau\{\frac{27}{4}\pi^4(\Lambda+\bar{\delta})+4\pi^2 D_a^{-1}\}}, \bar{\delta} = \min(\frac{1}{\varepsilon EP_r}, \Lambda, \frac{1}{\varepsilon p_2})$ and $\bar{\lambda} = \frac{|R_s|}{|R|}$

Proof: Replace R with $-|R|$ and R_s with $-|R_s|$ in equation (1) and adopting a similar procedure used in proving Theorem 1, we obtain the desired result.

Corollary 2: If $(\sigma, w, \theta, \phi, h_z), \sigma = \sigma_r + i\sigma_i, \sigma_i \neq 0$ is a non trivial solution of equations (1) -(6), together with the boundary condition (7) and R<0, $R_s < 0$ and $\bar{\lambda} < \frac{\tau\{\frac{\varepsilon'}{4}\pi^4(\Lambda+\bar{\delta})+4\pi^2 D_a^{-1}\}}{|R|}$ then

$$\sigma_r < 0.$$

Proof: Follows from theorem 2

Corollary 2 implies that oscillatory motions of growing amplitude are not allowed in magnetorotatory thermohaline convection in porous medium of Stern type if the initial stability parameter λ does not exceed the value $\frac{\tau\{\frac{\varepsilon'}{4}\pi^4(\Lambda+\bar{\delta})+4\pi^2 D_a^{-1}\}}{|R|}$. Further this result is uniformly valid for an initially top heavy $(\lambda > 1)$ as well as an initially bottom heavy $(\lambda < 1)$ configuration. We also have

3. If $0 < |R_s| \leq |R| \leq \tau\left\{\frac{27}{4}\pi^4(\Lambda + \overline{\delta}) + 4\pi^2 D_a^{-1}\right\}$ and $\sigma_i \neq 0$ then $\sigma_r < 0$, a more general result than that following from theorem 2 of Prakash et al. [12] for the present problem.

4. If $0 < |R_s| \leq \tau\left\{\frac{27}{4}\pi^4(\Lambda + \overline{\delta}) + 4\pi^2 D_a^{-1}\right\} < |R|$ and $\sigma_i \neq 0$, then $\sigma_r < 0$, a new result that obviously cannot be averred from characterization theorem 2 of Prakash et al. [12].

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