

# A Subset of the Space of the Orlicz Space of $\chi^\pi$ Sequences

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**ABSTRACT:** Let  $\chi_M^\pi$  denote the space of all Orlicz space gai rate sequences. Let  $\wedge_M$  denote the space of all Orlicz space of analytic rate sequences. Mainly in this paper in studied the general properties of Sectional space  $(\chi_M^\pi)_S$  of  $(\chi_M^\pi)$ .

**KEYWORDS:** Orlicz space, rate sequence, FK space, AK space

## I.INTRODUCTION

A complex sequence, whose  $k^{\text{th}}$  term is  $x_k$  is denoted by  $\{x_k\}$  or simply  $x$ . Let  $\phi$  be the set of all finite sequences. A sequences  $x = (x_k)$  is said to be analytic if  $\sup_k |x_k|^{1/k} < \infty$ . The vector space of all analytic sequences will be denoted by  $\wedge$ .  $\chi$  was discussed in Kamthan [11]. Matrix transformations involving  $\chi$  were characterized by sridhar [12]and Sirajudeen

[13]. A sequence  $x$  is called entire sequence if  $\lim_{k \rightarrow \infty} (k!|x_k|)^{1/k} = 0$ . The vector space of all gai of phi sequences will be denoted by  $\chi$ . Kizmaz [17] defined the following difference sequence spaces  $Z(\Delta) = \{x = (x_k) : \Delta$

$x \in Z\}$  for  $Z = l_\infty, c, c_0$ , where  $\Delta x = (\Delta x)_k = (x_k - x_{k+1})_{k=1}^\infty$  and showed that these are Banach space with norm

$\|x\| = |x_1| + \|\Delta x\|_\infty$ . Later on Et and Colak [10] generalized the notion as follows :Let  $m \in \mathbb{N}$ ,  $Z(\Delta^m) = \{x = (x_k) : \Delta^m x \in Z\}$  for  $Z = l_\infty, c, c_0$  where  $m \in \mathbb{N}$ ,

$$\Delta^0 x = (x_k), (\Delta x)_k = (x_k - x_{k+1}), \Delta^m x = (\Delta^m x)_k = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1})_{k=1}^\infty$$

The generalized difference has the following binomial representation :

$$\Delta^m x_k = \sum_{\gamma=0}^m (-1)^\gamma \binom{m}{\gamma} x_{k+\gamma} \gamma, \text{ They proved that these are Banach spaces with the norm } \|x\|_\Delta = \sum_{i=1}^m |x_i| + \|\Delta^m x\|_\infty \text{ Orlicz}$$

[4] used the idea of Orlicz function to construct the space  $(L^M)$ . Lindenstrauss and Tzafriri [1] investigated Orlicz sequence space in more detail, and they proved that every Orlicz sequence space  $l_M$  contains a subspace isomorphic to  $l_p (1 \leq p < \infty)$ . Sufebsequently ,the different classes of sequence spaces were de fined by Parashar and Choudhry [2] , Mursaleen etel [3] , Bektas and Altin [4] , Tripathy et al. [5], Rao and subramanian [6], and many others. The Orlicz sequence spaces are the special cases of Orlicz spaces studied in [7].

An Orlicz function is a function  $M : [0, \infty) \rightarrow [0, \infty)$  which is continuous, non-decreasing and convex with  $M(0) = 0$ ,

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$M(x) > 0, x > 0$  and  $M(x) \rightarrow \infty$  as  $x \rightarrow \infty$ .

If the convexity of Orlicz function  $M$  is replaced by  $M(x+y) \leq M(x) + M(y)$ . then this function is called modulus function, introduced by Nakano [18] and further discussed by Ruckle [8] and Maddox[9], and many others.

An Orlicz function  $M$  is said to satisfy  $\Delta_2$  condition for all values of  $u$ , if there exists a constant  $K > 0$ , such that  $M(2u) \leq KM(u)$  ( $u \geq 0$ ). The condition  $\Delta_2$  is equivalent to  $M(lu) \leq K l M(u)$ , for all values  $u$  and for  $l > 1$ . Lindstrauss and Tzafriri [1] used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\} \dots (1)$$

Where  $w = \{\text{all complex sequence}\}$ . The space  $\ell_M$  with the norm

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\} \dots (2)$$

becomes a Banach space which is called an Orlicz sequence space. For  $M(t) = t^p, (1 \leq p < \infty)$  the space  $\ell_M$  coincide with the classical sequence space  $\ell_p$ . Given sequence  $x = \{x_k\}$  its  $n^{\text{th}}$  section is the sequence  $x^n = \{x_1, x_2, \dots, x_n, 0, 0, \dots\}$

$\delta^n = (0, 0, \dots, 1, 0, 0, \dots)$ , 1 in the  $n^{\text{th}}$  place and zero's else where; and  $\delta^{n+1} = (0, 0, \dots, 1, -1, 0, 0, \dots)$ , 1 in the  $n^{\text{th}}$  place, -1 in the  $(n+1)^{\text{th}}$  place and zero's else where.

If  $X$  is a sequence space, we define

i.  $X'$  = continuous dual of  $X$ .

ii.  $X^\alpha = \left\{ a = (a_k) : \sum_{k=1}^{\infty} |a_k x_k| < \infty, \text{ for each } x \in X \right\}$

iii.  $X^\beta = \left\{ a = (a_k) : \sum_{k=1}^{\infty} a_k x_k \text{ is convergent, for each } x \in X \right\}$

iv.  $X^\gamma = \left\{ a = (a_k) : \sup_n \sum_{k=1}^{\infty} |a_k x_k| < \infty, \text{ for each } x \in X \right\}$

v. Let  $X$  be an FK space  $\supset \phi$ . Then  $X^f = \{f(\delta^n) : f \in X'\}$

$X^\alpha, X^\beta, X^\gamma$  are called the  $\alpha$ - (or  $K\ddot{o}$  the  $-T\ddot{o}$ plitz) dual of  $X$ ,  $\beta$ - (or generalized  $K\ddot{o}$  the  $-T\ddot{o}$ plitz) dual of  $X$ ,  $\gamma$ - dual of  $X$ . Note that  $X^\alpha \subset X^\beta \subset X^\gamma$ . If  $X \subset Y$  then  $Y^\mu \subset X^\mu$ , for  $\mu = \alpha, \beta, \text{ or } \gamma$

An FK space (Frechet coordinate space) is a Frechet space which is made up of numerical sequences and has the property that the coordinate functional  $sp_k(x) = x_k (k=1,2,3,\dots)$  are continuous.

An FK-space is a locally convex Frechet space which is made up of sequences and has the property that coordinate projections are continuous. An metric space  $(X, d)$  is said to have AK (or sectional convergence) if and only if  $d(x^n, x) \rightarrow 0$  as  $n \rightarrow \infty$ . [10] The space is said to have AD (or) be an AD space if  $\phi$  is dense in  $X$ . We note that implies AD by [4].

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**II.DEFINITION AND PRELIMINARIES**

**II.1DEFINITION**

Let  $w$ ,  $\chi_M^\pi$  and  $\Lambda_M$  denote the spaces of all, Orlicz space of gai sequences and Orlicz space of bounded sequence respectively. Let  $w$  denote the set of all complex sequence  $x = (x_k)_{k=1}^\infty$  and  $M : [0,\infty) \rightarrow [0,\infty)$  be an orlicz function. Let  $t$  denote the sequence with  $t_k = |x_k|^{1/k}$  for all  $k \in \mathbb{N}$  denote  $k \in \mathbb{N}$ . Define the sets

$$\chi_M^\pi = \left\{ x \in w : M \left( \frac{K! t_k}{\pi_k^k \rho} \right)^{\frac{1}{k}} \rightarrow 0 \text{ as } k \rightarrow \infty, \text{ for some } \rho > 0 \right\}.$$

$$\Lambda_M^\pi = \left\{ x \in w : \sup_k M \left( \frac{t_k}{\pi_k^k \rho} \right)^{\frac{1}{k}} < \infty, \text{ for some } \rho > 0 \right\}.$$

The space  $\chi_M^\pi$  is a metric space with the metric

$$d(x, y) = \inf \left\{ \rho > 0 : \sup_k M \left( \frac{K! |x_k - y_k|}{(\pi_k)^k \rho} \right)^{\frac{1}{k}} \leq 1 \right\}.$$

The space  $\Lambda_M$  is a metric space with the metric

$$d(x, y) = \inf \left\{ \rho > 0 : \sup_k M \left( \frac{(|x_k - y_k|)^{\frac{1}{k}}}{(\pi_k)^k \rho} \right) \leq 1 \right\}.$$

Let  $(\chi_M^\pi)_s = \{x = x_k : \xi = \xi_k \in \chi_M^\pi\}$  where is a  $\xi_k = 1!x_1 + 2!x_2 + \dots + k!x_k$  for  $k=1,2,3,\dots$  and  $(\Lambda_M)_s = \{y = y_k : \eta = \eta_k \in \Lambda_M\}$  where  $\eta_k = y_1 + y_2 + \dots + y_k$  for  $k=1,2,3,\dots$ . Then  $(\chi_M^\pi)_s$  is a metric space with the metric.

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$$d(x, y) = \inf \left\{ \rho > 0 : \sup_k \left( M \left( \frac{(K! |\xi_k - \eta_k|)^{\frac{1}{k}}}{(\pi_k)^{\frac{1}{k}} \rho} \right) : k = 1, 2, 3, \dots \right) \leq 1 \right\}$$

And  $(\Lambda_M)_S$  is a metric space with the metric

$$d(x, y) = \inf \left\{ \rho > 0 : \sup_k \left( M \left( \frac{(|\xi_k - \eta_k|)^{\frac{1}{k}}}{(\pi_k)^{\frac{1}{k}} \rho} \right) : k = 1, 2, 3, \dots \right) \leq 1 \right\}$$

Let  $\sigma(\chi_M^\pi)$  denote the vector space of all sequence  $x = (x_k)$  that  $\left(\frac{\xi_k}{k}\right)$  is an orlicz space of gai sequence . We recall that  $cs_0$  denotes the vector space of all sequences  $x = (x_k)$  such that  $(\xi_k)$  is a null sequence.

**II.2.LEMMA**

Let X be an FK-space  $\supset \phi$ . Then

- (i)  $X^\gamma \subset X^f$
- (ii) If X has AK .  $X^\beta \subset X^f$
- (iii) If X has AD,  $X^\beta \subset X^\gamma$

We note that

$$\chi_M^\alpha = \chi_M^\beta = \chi_M^\gamma = \chi_M^f = \Lambda \text{ for all } x = x_k \text{ and } y = y_k \text{ in } \chi_M^\pi$$

**II.3.REMARK**

$$x \{ x_k \} \in \sigma(\chi_M^\pi) \Leftrightarrow \left(\frac{\xi_k}{k}\right) \in \chi_M^\pi \Leftrightarrow \left( M \left( \frac{(|\xi_k|)^{\frac{1}{k}}}{\pi_k^{\frac{1}{k}} \rho^k} \right) \rightarrow 0 \text{ as } k \rightarrow \infty \right) \Leftrightarrow \left( M \left( \frac{(|\xi_k|)^{\frac{1}{k}}}{\pi_k^{\frac{1}{k}} \rho} \right) \rightarrow 0 \text{ as } k \rightarrow \infty \right)$$

,because  $k^{\frac{1}{k}} \rightarrow 1$  as  $k \rightarrow \infty \Leftrightarrow x = (x_k) \in (\chi_M^\pi)_S$

Hence  $(\chi_M^\pi)_S = \sigma(\chi_M^\pi)$  the cesaro space of order 1.

**II.4PROPOSITION**

$$(\chi_M^\pi)_S \subset (\chi_M^\pi)$$

Proof:

$$x \in (\chi_M^\pi)_S \Rightarrow \xi \in (\chi_M^\pi)$$

$$\left( M \left( \frac{(|\xi_k|)^{\frac{1}{k}}}{\pi_k^k \rho} \right) \right) \rightarrow 0 \text{ as } k \rightarrow \infty \quad \text{But } x_k = \xi_k - \xi_{k-1}$$

$$\begin{aligned} \left( M \left( \frac{(k!|\xi_k|)^{\frac{1}{k}}}{\pi_k^k \rho} \right) \right) &\leq \left( M \left( \frac{(|\xi_k|)^{\frac{1}{k}}}{\pi_k^k \rho} \right) \right) + \left( M \left( \frac{(|\xi_{k-1}|)^{\frac{1}{k}}}{\rho} \right) \right) \\ &\leq \left( M \left( \frac{k!(|\xi_k|)^{\frac{1}{k}}}{\pi_k^k \rho} \right) \right) + \left( M \left( \frac{(k!|\xi_{k-1}|)^{\frac{1}{k}}}{\pi_k^k \rho} \right) \right) \rightarrow 0 \text{ as } k \rightarrow \infty \end{aligned}$$

$$\text{Therefore } \left( M \left( \frac{(k!|x_k|)^{\frac{1}{k}}}{\pi_k^k \rho} \right) \right) \rightarrow 0 \text{ as } k \rightarrow \infty \Rightarrow x \in \chi_M^\pi$$

Hence  $(\chi_M^\pi)_S \subset (\chi_M^\pi)$

Note: The above inclusion is strict. Take the sequence  $\delta^{(1)} \in \chi_M^\pi$

$$\text{We have } M \left( \frac{\xi_1}{\pi_k^k \rho} \right) = 1$$

$$M \left( \frac{\xi_2}{\pi_k^k \rho} \right) = 1 + 0 = 1$$

$$M \left( \frac{\xi_3}{\pi_k^k \rho} \right) = 1 + 0 + 0 = 1$$

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$$M \left( \frac{\xi_k}{\pi_k^k \rho} \right) = 1 + 0 + 0 + 0 + \dots + 1 = 1 \rightarrow k \text{ terms and so on.}$$

Now

$$\left( M \left( \frac{\left( \frac{\xi_k}{\pi_k^k \rho} \right)^{\frac{1}{k}}}{\pi_k^k \rho} \right) \right) = 1 \text{ for all } k. \text{ Hence } \left( M \left( \frac{\left( \frac{\xi_k}{\pi_k^k \rho} \right)^{\frac{1}{k}}}{\pi_k^k \rho} \right) \right) \text{ does not tend to zero as } k \rightarrow \infty.$$

So  $\delta^{(1)} \in (\mathcal{X}_M^\pi)_S$ . Thus the inclusion  $(\mathcal{X}_M^\pi)_S \subset (\mathcal{X}_M^\pi)$  is strict. This completes the proof.

**II.5. PROPOSITION**

$(\mathcal{X}_M^\pi)_S$  has AK – property Proof:

Let  $x = (x_k) \in (\mathcal{X}_M^\pi)_S$  and take  $x^{[n]} = (x_1, x_2, \dots, x_n, 0, \dots)$  for  $n = 1, 2, 3, \dots$

$$\begin{aligned} \text{Hence } d(x, x^{[n]}) &= \inf \left\{ \rho > 0 : \sup_k M \left( \frac{\left( \left| \xi_k - \xi_k^{(n)} \right| \right)^{\frac{1}{k}}}{(\pi_k)^k \rho} \right) \leq 1 \right\} \\ &= \inf \left\{ \rho > 0 : \sup_k M \left( \frac{\left( \left| \xi_{n+1} - \xi_n \right| \right)^{\frac{1}{n+1}}}{(\pi_k)^k \rho} \right) \right\}, \left\{ M \left( \frac{\left( \left| \xi_k - \xi_k^{(n)} \right| \right)^{\frac{1}{k}}}{(\pi_k)^k \rho} \right) \dots \right\} \leq 1 \end{aligned}$$

Therefore  $x^{[n]} \rightarrow x$  as  $n \rightarrow \infty$  in  $(\mathcal{X}_M^\pi)_S$ . Thus  $(\mathcal{X}_M^\pi)_S$  has AK. This completes the proof.

**II.6. PROPOSITION**

$(\mathcal{X}_M^\pi)_S$  is a linear space over the field complex numbers.

Proof: It is easy. Therefore omit the proof.

**II.7. PROPOSITION**

$(\mathcal{X}_M^\pi)_S$  is solid

Proof: Let  $|x_k| \leq |y_k|$  with  $y = (y_k) \in (\mathcal{X}_M^\pi)_S$ . So,  $|\xi_k| \leq |\eta_k|$  with  $\eta = (\eta_k) \in \mathcal{X}_M^\pi$ . But  $(\mathcal{X}_M^\pi)$  is solid.

Hence  $\xi = (\xi_k) \in \mathcal{X}_M$ . Therefore  $x = (x_k) \in (\mathcal{X}_M^\pi)$ . Hence

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$(\chi_M^\pi)_S$  is solid. This completes this proof.

**II.8 PROPOSITION**

$$\Lambda \subset [(\chi_M^\pi)_S]^\beta \subset \Lambda_M(\Delta)$$

Proof: step 1: By proposition II.4., we have  $(\chi_M^\pi)_S \subset (\chi_M^\pi)$ . Hence.  $(\chi_M^\pi)_S \subset [(\chi_M^\pi)_S]^\beta$

But  $(\chi_M^\pi)^\beta = \Lambda$ . Therefore is  $\Lambda \subset [(\chi_M^\pi)_S]^\beta \dots(4)$

Step 2: Let  $y = y_k \in [(\chi_M^\pi)_S]^\beta$  consider  $f(x) = \sum_{k=1}^\infty x_k y_k$  with  $x \in (\chi_M^\pi)_S$

Take  $x = \delta^n - \delta^{n+1} = (0, 0, \dots, 1, -1, 0, 0, \dots)$ , 1 in the n th place and zero's elsewhere .

Then  $f(\delta^n - \delta^{n+1}) = y_n - y_{n+1}$ .

$$\text{Hence } \left\{ M \left( \frac{|y_n - y_{n+1}|}{\pi_k^k \rho} \right) \right\} = \left\{ M \left( \frac{|f(\delta^n - \delta^{n+1})|}{\pi_k^k \rho} \right) \right\} \leq 1 \|f\| d((\delta^n - \delta^{n+1}), 0) \leq \|f\| \cdot 1 .$$

$$\text{So, } \left\{ M \left( \frac{|y_n - y_{n+1}|}{\pi_k^k \rho} \right) \right\} \text{ is bounded. Consequently } \left\{ M \left( \frac{|y_n - y_{n+1}|}{\pi_k^k \rho} \right) \right\} \in \Lambda_M$$

$$\text{That is } \left\{ M \left( \frac{y_n}{\pi_k^k \rho} \right) \right\} \in \Lambda_M(\Delta) . \text{ But } y = (y_n) \text{ is originally in } [(\chi_M^\pi)_S]^\beta$$

Therefore  $[(\chi_M^\pi)_S]^\beta \subset \Lambda_M(\Delta) \dots(5)$

From (4) and (5) We conclude that

$$\Lambda \subset [(\chi_M^\pi)_S]^\beta \subset \Lambda_M(\Delta)$$

This completes proof.

**II.9 PROPOSITION**

The  $\beta$  - dual space of  $(\chi_M^\pi)_S$  is  $\Lambda$

Proof: Let  $y = (y_k)$  be an arbitrary point in  $[(\chi_M^\pi)_S]^\beta$ . If y is not in  $\Lambda$ , then for each natural number n, we can find an index k(n) such that

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$$M \left( \frac{k(n)^{\frac{1}{k(n)}}}{\pi_k^{\frac{1}{k}} \rho} \right) > \frac{n!}{n}, (n = 1, 2, 3, \dots)$$

Define  $x = (x_k)$  by

$$M \left( \frac{(k!) |x_k|}{\pi_k^{\frac{1}{k}} \rho} \right) = \frac{1}{n^k} \text{ for } k=k(n) \text{ and}$$

$$M \left( \frac{(k!) |x_k|}{\pi_k^{\frac{1}{k}} \rho} \right) = 0 \text{ otherwise Then } x \text{ is in } (\mathcal{X}_M^\pi), \text{ but for infinitely many } k,$$

$$M \left( \frac{(k!) |y_k x_k|}{\pi_k^{\frac{1}{k}} \rho} \right) > 1$$

Consider the sequence  $z = \{z_k\}$  where  $z_1 = 1! x_1$  with  $s = k! \sum x_k$ ; and  $z_k = k! x_k (k = 2, 3, \dots)$

Then  $z$  is a point of  $(\mathcal{X}_M^\pi)_S$ . But by the equation  $\sum M \left( \frac{(k!) z_k x_k}{\pi_k^{\frac{1}{k}} \rho} \right)_S$  does not converge. Thus the sequence  $y$  would not

be in  $[(\mathcal{X}_M^\pi)_S]^\beta$ . This contradiction proves that

$$[(\mathcal{X}_M^\pi)_S]^\beta \subset \Lambda \dots\dots(7)$$

Step2: By proposition 1.8 we have  $\Lambda \subset [(\mathcal{X}_M^\pi)_S]^\beta \dots\dots(8)$

From (7) and (8) it follows that the  $\beta$ -dualspace of  $[(\mathcal{X}_M^\pi)_S]^\beta$  is  $\Lambda$ . This completes the proof.

**II.10. PROPOSITION**

$$[(\mathcal{X}_M^\pi)_S]^\mu = \Lambda \text{ for } \mu = \alpha, \beta, \gamma, f$$

Proof:

$(\mathcal{X}_M)_s$  has AK by Proposition 4.5. Hence by Lemma 1.2(ii)

We get  $[(\mathcal{X}_M^\pi)_S]^\beta = [(\mathcal{X}_M^\pi)_S]^f$ . But  $[(\mathcal{X}_M^\pi)_S]^\beta = \Lambda$



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Hence  $[(\chi_M^\pi)_S]^\beta = \Lambda \dots\dots(9)$

Step 2: Since AK implies AD.Hence by Lemma 1.2(iii) we get

$$[(\chi_M^\pi)_S]^\beta = [(\chi_M^\pi)_S]^\gamma$$

Therefore

$$[(\chi_M^\pi)_S]^\gamma = \Lambda \dots\dots(10)$$

Step 3:

$(\chi_M^\pi)_S$  is normal by Proposition 1.7.Hence , by[15 proposition 1.7],we get

$$[(\chi_M^\pi)_S]^\alpha = [(\chi_M^\pi)_S]^\gamma = \Lambda \dots\dots(11)$$

From (9) , (10) and (11) we have

$$[(\chi_M^\pi)_S]^\alpha = [(\chi_M^\pi)_S]^\beta = [(\chi_M^\pi)_S]^\gamma = [(\chi_M^\pi)_S]^f = \Lambda$$

**II.11.PROPOSITION**

The dual space of  $[(\chi_M^\pi)_S]$  is  $\Lambda$  . In other words  $[(\Gamma_M)_S]^* = \Lambda$

Proof:We recall that  $S^k$  has 1 in the k th place , -1 in the (k+1)<sup>th</sup> place and zero's else where with

$$\xi = s^k \left\{ M \left( \frac{\xi_k}{\pi_k^k \rho} \right) \right\} = \left\{ \frac{M(0)^{\frac{1}{k}}}{\pi_k^k \rho}, \frac{M(0)^{\frac{1}{2}}}{\pi_k^k \rho}, \dots, \frac{M(0)^{\frac{1}{k}}}{\pi_k^k \rho}, \frac{M(0)^{\frac{1}{k+1}}}{\pi_k^k \rho} \right\}$$

$$= \left\{ 0,0,0,\dots, \frac{M(0)^{\frac{1}{k}}}{\pi_k^k \rho}, 0,\dots \right\}$$

Exists. Hence  $s^k \in (\chi_M^\pi)_S$  .We have  $f(x) = \sum_{k=1}^\infty \xi_k y_k$  with

$\xi \in (\chi_M)_S$  and  $f \in [(\chi_M^\pi)_S]^*$  ,where  $[(\chi_M^\pi)_S]^*$  is the dual space of  $(\chi_M^\pi)_S$  .Take  $\xi \in (\chi_M^\pi)_S$  and  $f \in [(\chi_M^\pi)_S]^*$  ,where  $[(\chi_M^\pi)_S]^*$  , is the dual of  $(\chi_M^\pi)_S$  .Take  $\xi = s^K \in (\chi_M^\pi)_S$

Then  $|y_k| \leq \|f\| d(s^k, 0) < \infty$  for all k.....(12)

Thus  $(y_k)$  is a bounded sequence and hence an analytic sequence .In other words ,  $(y \in \Lambda)$  .Therefore

$$[(\chi_M)_S]^* = \Lambda$$

This completes the proof.

Lemma 2.12

$Y \supset X \Leftrightarrow Y^f \subset X^f$  where X is an AD space and Y an FK space.

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### II.12.PROPOSITION

Let  $Y$  be any FK-space  $\supset \phi$  Then  $Y \supset (\chi_M^\pi)_S$  if and only if the sequence  $(\delta^k)$  weakly analytic.

Proof: The following implications establish the result

$Y \supset (\chi_M^\pi)_S \Leftrightarrow Y^f \subset [(\chi_M^\pi)_S]^f$ , Since  $(\chi_M^\pi)_S$  has AD and by lemma  
 $\Leftrightarrow Y^f \subset \Lambda$ . Since  $[(\chi_M^\pi)_S]^f = \Lambda \Leftrightarrow \text{foreach } f \in y'$ , the topological dual of  $Y \Leftrightarrow f(\delta^k)$  is analytic  $\Leftrightarrow (\delta^k)$  is weakly analytic. This completes the proof.

### II.13.PROPOSITION

In  $(\chi_M^\pi)$  weak converges does not imply strong convergence..

Proof: Assume that weak convergence implies strong convergence in  $(\chi_M^\pi)_S$ . Then we would have

$[(\chi_M^\pi)_S]^{\beta\beta} = (\chi_M^\pi)_S$ . But  $[(\chi_M^\pi)_S]^{\beta\beta} = \Lambda^\beta = \Gamma$ . By proposition 1.4  $(\chi_M^\pi)_S$  is a proper subspace of  $(\chi_M^\pi)$

Thus  $[(\chi_M^\pi)_S]^{\beta\beta} \neq (\chi_M^\pi)_S$ . Hence weak convergence does not imply strong convergence  $(\chi_M^\pi)_S$ . This completes the proof.

### II.14.DEFINITION

Fix  $k=0,1,2,\dots$  given a sequence  $(x)_K$ . put

$$M\left(\frac{\xi_{k,p}}{\rho}\right) = M\left(\frac{(1+k)!x_{1+k} + (2+k)!x_{2+k} + (3+k)!x_{3+k} + \dots + (p+k)!x_{p+k}}{\pi_k^{\frac{1}{k}} \rho}\right) \text{ for } p = 1,2,3,\dots$$

Let  $(\xi_{k,p} : p = 1,2,3,\dots) \in (\chi_M^\pi)$  uniformly in  $k=0,1,2,3,\dots$ . Then we call  $(x_k)$  an "almost orlicz space of gai sequence". The set of all almost orlicz space gai sequence is denoted by  $(\Delta_M)$

### II.15.PROPOSITION

$(\chi_M^\pi) \cap \sigma^\alpha \alpha (\chi_M^\pi) = \Delta_M$  where  $(\Delta_M)$  is the set of all almost orlicz space of gai sequences.

Proof: Put  $k=0$ . Then  $(\xi_{0,p}) \in (\chi_M^\pi)$

$$\Leftrightarrow \left(\frac{1!x_1 + 2!x_2 + 3!x_3 + \dots + p!x_p}{\pi_k^{\frac{1}{k}} \rho}\right) \in (\chi_M^\pi)$$

$$\Leftrightarrow M\left(\frac{|1!x_1 + 2!x_2 + 3!x_3 + \dots + p!x_p|}{\rho\rho^{\frac{1}{p}}}\right) \rightarrow 0 \text{ as } p \rightarrow \infty$$

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$$\Leftrightarrow M \left( \frac{|1!x_1 + 2!x_2 + 3!x_3 + \dots + p!x_p|^{1/p}}{\rho} \right) \rightarrow 0 \text{ as } p \rightarrow \infty \dots (13)$$

$$\Leftrightarrow (x_k) \in cs_0$$

$$\Leftrightarrow (\Delta_M) \in cs_0 \text{ put } k=1$$

Then  $\Leftrightarrow (\xi_{1,p}) \in \chi_M^\pi \Leftrightarrow \left( \frac{2!x_1 + 3!x_3 + 4!x_4 + \dots + p!x_p}{\rho} \right) \in (\chi_M^\pi)$

$$\Leftrightarrow M \left( \frac{|2!x_1 + 3!x_3 + 4!x_4 + \dots + p!x_p|^{1/p}}{\rho\rho} \right) \rightarrow 0 \text{ as } p \rightarrow \infty$$

$$\Leftrightarrow M \left( \frac{|2!x_1 + 3!x_3 + 4!x_4 + \dots + p!x_p|^{1/p}}{\pi_k^k \rho} \right) \rightarrow 0 \text{ as } p \rightarrow \infty \dots (14)$$

$$\Leftrightarrow M \left( \frac{|3!x_3 + 4!x_4 + \dots + p!x_p|^{1/p}}{\pi_k^k \rho} \right) \rightarrow 0 \dots (15)$$

$$\Leftrightarrow M \left( \frac{|4!x_4 + \dots + p!x_p|^{1/p}}{\pi_k^k \rho} \right) \rightarrow 0 \dots (16) \text{ and so on.}$$

From (13) and (14) it follows that  $M \left( \frac{1!x_1}{\pi_k^k \rho} \right)$

$$= M \left( \frac{1!x_1 + 2!x_2 + 3!x_3 + 4!x_4 + \dots + p!x_p}{\pi_k^k \rho} \right) = 0$$

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$$= M \left( \frac{2!x_1 + 3!x_3 + 4!x_4 + \dots + p!x_p}{\pi_k^{\frac{1}{k}} \rho} \right) = 0$$

Similarly we obtain  $M \left( \frac{2!x_2}{\rho} \right) = 0$  ,  $M \left( \frac{3!x_3}{\rho} \right) = 0$  and so on.

Hence  $(\Delta_M) = \theta$  denote the sequence  $(0,0,\dots)$

Thus we have proved that  $(\chi_M^\pi) \cap \sigma^\alpha (\chi_M^\pi) = \theta$  and  $\Delta_M = \theta$

In other words  $(\chi_M^\pi) \cap \sigma^\alpha (\chi_M^\pi) = \Delta_M$

This completes this proof

**II.16.PROPOSITION**

$$(\chi_M^\pi)_S = (\chi_M^\pi) \cap CS_0$$

Proof:By proposition 1.4  $(\chi_M^\pi)_S \subset (\chi_M^\pi)$

Also, since every orlicz space of gai sequence  $M \left( \frac{\xi_K}{\rho} \right)$  exists,it follows that  $M \left( \frac{\xi_K}{\pi_k^{\frac{1}{k}} \rho} \right)$  exists. In other words

$$M \left( \frac{\xi_K}{\pi_k^{\frac{1}{k}} \rho} \right) \in CS_0 . \text{ Thus } (\chi_M^\pi)_S \subset CS_0 . \text{ Consequently , } (\chi_M^\pi)_S \subset (\chi_M^\pi) \cap CS_0 \dots(17)$$

On the otherhand, if  $M \left( \frac{k!x_K}{\pi_k^{\frac{1}{k}} \rho} \right) \in \chi_M^\pi \cap CS_0$  .then  $f(z) = \sum_{k=1}^{\infty} M \left( \frac{k!x_k}{\rho} \right) z^{k-1}$  is an Orlicz space of an gai of rate

function. But  $M \left( \frac{k!x_K}{\pi_k^{\frac{1}{k}} \rho} \right) \in CS_0$  .So,

$$f(1) = M \left( \frac{1!x_1 + 2!x_1 + 3!x_3 + 4!x_4 + \dots + p!x_p}{\pi_k^{\frac{1}{k}} \rho} \right) = 0$$

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Hence  $\frac{f(z)}{1-z} = \sum_{k=1}^{\infty} M \left( \frac{\xi_k}{\pi_k^k \rho} \right) z^{k-1}$  is also orlicz space of an gai function .

Hence  $M \left( \frac{\xi_k}{\pi_k^k \rho} \right) \in \mathcal{X}_M^\pi$  . So  $x = ((x_k) \in (\mathcal{X}_M^\pi))_s$  but  $(x_k)$  is arbitrary in  $\mathcal{X}_M^\pi \cap cs_0$  .

Therefore  $(\mathcal{X}_M^\pi) \cap cs_0 \subset (\mathcal{X}_M^\pi)_s \dots(18)$

From (17) and (18) we get  $(\mathcal{X}_M^\pi)_s = (\mathcal{X}_M^\pi) \cap cs_0$  .

This completes the proof.

**II.17.DEFINITION**

Let  $\alpha > 0$  be not an integer Write  $S_n^{(\alpha)} = \sum_{\gamma=1}^n A_{n-\gamma}^{\alpha-1} x_\gamma$  ,

Where  $(A_\mu^{(\alpha)})$  denote the binomial coefficient

$\left( \frac{(\mu + \alpha)(\mu + \alpha - 1) \dots (\mu + \alpha + 1)}{\mu!} \right)$  Then  $(x_n) \in \sigma^\alpha(\mathcal{X}_M^\pi)$  means that

$$\left( \frac{S_n^\alpha}{A_n^{(\alpha-1)}} \right) \in (\mathcal{X}_M^\pi)$$

**II.18.PROPOSITION**

Let  $\alpha > 0$  be a number which is not an integer.

Then  $(\mathcal{X}_M^\pi) \cap \sigma^\alpha(\mathcal{X}_M^\pi) = \theta$  where  $\theta$  denotes the sequence  $(0,0,\dots,0)$

Proof:Since  $(x_n) \in \sigma^\alpha(\mathcal{X}_M^\pi)$  We have  $\left( \frac{S_n^\alpha}{A_n^{(\alpha-1)}} \right) \in (\mathcal{X}_M^\pi)$  .

This is equivalent to  $S_n^\alpha \in (\mathcal{X}_M^\pi)$  . This, in turn, is equivalent to the assertion that

$$f_\alpha(z) = \sum_{n=1}^{\infty} S_n^\alpha z^{n-1} \text{ is orlicz space of an gai function.}$$

Now  $f_\alpha(z) = \frac{f(z)}{(1-z)^\alpha}$  since  $\alpha$  is not an integer,  $f(z)$  and  $f_\alpha(z)$  can not both be integral functions, for if one is an integral function, the other has a branch at  $z=1$ . Hence the assertion holds good. So, the sequence  $0=(0,0,0,\dots)$  belongs to both

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$(\chi_M^\pi)$  and  $\sigma^\alpha(\chi_M^\pi)$ . But this is the only sequence common to both these spaces.  
Then  $(\chi_M^\pi) \cap \sigma^\alpha(\chi_M^\pi) = \theta$ .

This completes the proof.

### III.CONCLUSION

In this paper we have study some properties of orlicz space of  $\chi^\pi$  sequences

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