

Control chart for number of customers in the system of $M^{[X]} / M / 1$ Queueing system

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Abstract: Queueing models characterized by bulk arrival or bulk service of either fixed or variable sizes are commonly found in modelling of traffic and transportation systems, complex computer and telecommunication systems, inventory replenishment system and other real-life applications. Control chart technique enables to monitor the performance of these queues. Average queue length and average waiting time are the main observable performance characteristics of any queueing system. In this paper control limits are established for $M^{[X]} / M / 1$ queueing model when the batch size follows geometric distribution. Numerical results are added to highlight its applications.

Keywords: bulk arrival, queue length, geometric distribution, Poisson arrival, exponential service, control limits.

I. INTRODUCTION

In general, in queueing models, it is assumed that the customers arrive singly at service facility. But this assumption is violated in many real life queueing situations.

Letters arriving at a post office, ships arriving at a port in a convoy, people attending a wedding reception etc., are some of the examples of queueing situations in which customers arrive in groups. Queueing model consisting of bulk arrival has been discussed by Gross and Harris (1998), Carmen Armero and David Conesa (2000) and several others.

Control chart is a quality control technique evolved initially to monitor production processes. Montgomery (2005) proposed a number of applications of Shewhart control charts in assuring quality in manufacturing industries. Shore (2000) developed control chart for random queue length of $M / M / S$ queueing model by considering the first three moments and also Shore (2006) developed Shewhart-like general control charts for $G / G / S$ queueing system using skewness. Khaparde and Dhabe (2010) constructed the control chart for random queue length of $M / M / 1$ queueing model using method of weighted variance. Poongodi and Muthulakshmi (2012) analysed number of customers in system of $M / E_k / 1$ queueing model using control chart technique.

In this paper Shewhart control chart for number of customers in the system of bulk arrival queueing model $M^{[X]} / M / 1$ is proposed. Section 2 relates to the queueing model description. Mean and variance of number of customers in the queueing system are derived in section 3. Parameters of control chart for number of customers are established in section 4. Numerical results are obtained in section 5 to analyse the effect of parameters.

II. $M^{[X]} / M / 1$ MODEL DESCRIPTION

Consider a single server queueing model in which the arrivals occur in batches according to Poisson process with rate $\lambda > 0$. The batch size X is a random variable with $P(X = k) = C_k$, $k = 1, 2, 3, \dots$. Customers are served one by one and the service time distribution is exponential with rate μ .

Let P_n be the probability that there are n customers in the system.

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The steady state equations governing this model are

$$0 = -(\lambda + \mu) P_n + \mu P_{n+1} + \lambda \sum_{k=1}^n P_{n-k} c_k \quad (n \geq 1)$$

$$0 = -\lambda P_0 + \mu P_1 \quad (1)$$

The system of equations (1) may be solved using generating function approach.

Define the generating functions of the steady state probabilities $\{P_n\}$ and the batch size distribution $\{c_n\}$ respectively as

$$P(z) = \sum_{n=0}^{\infty} P_n z^n, \quad |z| \leq 1$$

$$C(z) = \sum_{n=1}^{\infty} c_n z^n, \quad |z| \leq 1$$

Multiplying equation (1) by appropriate powers of z and summing, we get

$$0 = -\lambda \sum_{n=0}^{\infty} P_n z^n - \mu \sum_{n=1}^{\infty} P_n z^n + \frac{\mu}{z} \sum_{n=1}^{\infty} P_n z^n + \lambda \sum_{n=1}^{\infty} \sum_{k=1}^n P_{n-k} c_k z^n \quad (2)$$

Consider $\sum_{n=1}^{\infty} \sum_{k=1}^n P_{n-k} c_k z^n = \sum_{k=1}^{\infty} c_k z^k \sum_{n=k}^{\infty} P_{n-k} z^{n-k} = C(z)P(z)$. (3)

Using equation (3), equation (2) becomes

$$0 = -\lambda P(z) - \mu(P(z) - P_0) + \frac{\mu}{z}(P(z) - P_0) + \lambda C(z)P(z)$$

Solving for $P(z)$, we get

$$P(z) = \frac{\mu P_0 (1-z)}{\mu(1-z) - \lambda z(1-C(z))}, \quad |z| \leq 1 \quad (4)$$

The generating function of the complementary batch size probabilities $\Pr(X > x) = 1 - C_x = \tilde{C}_x$ is given by

$$\tilde{C}(z) = \sum_{n=1}^{\infty} \tilde{c}_n z^n = \frac{1-C(z)}{1-z}$$

Taking $r = \lambda/\mu$, equation (4) yields

$$P(z) = \frac{P_0}{1 - rz\tilde{C}(z)}$$

Clearly, $\tilde{C}(1) = E(X)$ and $\tilde{C}'(1) = \frac{E(X(X-1))}{2}$

Using the normalizing conditions, we obtain $P_0 = 1 - \rho$, where $\rho = r E(X)$.

If N_s and N_q are number of customers in the system and number of customers in the queue respectively then

$$N_s = \frac{\rho + rE(X^2)}{2(1-\rho)}$$

and $N_q = N_s - \rho$.

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Assume that the number of customers in any arriving batch follows geometric distribution with parameter α . Then the probability mass function of the batch size is

$$c_x = (1-\alpha) \alpha^{x-1}, \quad 0 < \alpha < 1$$

$$\text{Then } C(z) = \frac{z(1-\alpha)}{1-\alpha z} \tag{5}$$

$$\text{and } E(X) = \frac{1}{1-\alpha} \text{ with } \rho = \frac{r}{1-\alpha}$$

From equations (4) and (5), we obtain

$$P(z) = (1-\rho) \left[\sum_{n=0}^{\infty} ((\alpha + (1-\alpha)\rho)z)^n - \sum_{n=0}^{\infty} \alpha(\alpha + (1-\alpha)\rho)^n z^{n+1} \right]$$

Comparison of like powers of z on both sides gives

$$P_n = \rho(1-\rho)(1-\alpha)(\alpha + (1-\alpha)\rho)^{n-1}, \quad n > 0.$$

III. MEAN AND VARIANCE

Let N_s denote the number of customers in the system (both in queue and in service). Then the expected number of customers in the system is

$$E(N_s) = \frac{\rho}{(1-\rho)(1-\alpha)} \tag{6}$$

and the variance of the number of customers in the system is

$$\text{Var}(N_s) = \frac{\rho(1+\alpha(1-\rho))}{(1-\rho)^2(1-\alpha)^2} \tag{7}$$

IV. PARAMETERS OF THE CONTROL CHART

The parameters of Shewhart control chart, under the assumption that the number of customers in the system follows normal distribution, are given by

$$\left. \begin{aligned} \text{UCL} &= E(N_s) + 3\sqrt{\text{Var}(N_s)} \\ \text{CL} &= E(N_s) \\ \text{LCL} &= E(N_s) - 3\sqrt{\text{Var}(N_s)} \end{aligned} \right\} \tag{8}$$

The parameters of the control chart for $M^{[X]}/M/1$ queueing model, using (6) and (7) in (8), are obtained as

$$\text{CL} = \frac{\rho}{(1-\rho)(1-\alpha)}$$

$$\text{UCL} = \frac{\rho + 3\sqrt{\rho(1+\alpha(1-\rho))}}{(1-\alpha)(1-\rho)}$$

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$$LCL = \frac{\rho - 3\sqrt{\rho(1 + \alpha(1 - \rho))}}{(1 - \alpha)(1 - \rho)}$$

V. NUMERICAL ANALYSIS

Numerical analysis is carried out to analyze the performance of queueing system with reference to the parameters λ , μ and α . As LCL values are negative for the selected values of the parameters, they are considered as zero and therefore not shown in the table as a separate column.

Table gives the traffic intensity and the control chart parameters for number of customers in the queueing system for various values of λ , μ and α .

TABLE
CONTROL CHART PARAMETERS FOR THE SYSTEM SIZE OF M^(X)/M/1 MODEL

λ	μ	α	ρ	CL	UCL
4	10	0.15	0.471	1.046	5.797
		0.16	0.476	1.082	5.980
		0.17	0.482	1.121	6.172
		0.18	0.488	1.161	6.375
	15	0.15	0.314	0.538	3.563
		0.16	0.317	0.554	3.659
		0.17	0.321	0.570	3.758
		0.18	0.325	0.588	3.862
	20	0.15	0.235	0.362	2.726
		0.16	0.238	0.372	2.795
		0.17	0.241	0.382	2.866
		0.18	0.244	0.393	2.940
6	10	0.15	0.706	2.824	13.126
		0.16	0.714	2.976	13.779
		0.17	0.723	3.143	14.491
		0.18	0.732	3.326	15.269
	15	0.15	0.471	1.046	5.797
		0.16	0.476	1.082	5.980
		0.17	0.482	1.121	6.173
		0.18	0.488	1.161	6.375
	20	0.15	0.353	0.642	4.0358
		0.16	0.357	0.661	4.148
		0.17	0.361	0.682	4.265
		0.18	0.366	0.704	4.387

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8	10	0.15	0.941	18.823	77.288
		0.16	0.952	23.810	97.280
		0.17	0.964	32.129	130.606
		0.18	0.976	48.780	197.265
	15	0.15	0.627	1.981	9.693
		0.16	0.635	2.070	10.089
		0.17	0.643	2.166	10.515
		0.18	0.650	2.269	10.970
	20	0.15	0.471	1.046	5.797
		0.16	0.476	1.082	5.980
		0.17	0.482	1.121	6.173
		0.18	0.488	1.161	6.375

Numerical values in the table reveal the following features:

- (a) increase in α increases the parameters ρ , CL and UCL for fixed values of λ and μ .
- (b) increase in μ decreases the parameters ρ , CL and UCL for fixed values of λ and α .
- (c) increase in λ increases the parameters ρ , CL and UCL for fixed values of μ and α .

VI. CONCLUSION

The presented model has potential applications in practical systems such as manufacturing systems, telecommunication systems and computer networks.

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