

# Estimation of Time of a Step Change with VSS $\bar{X}$ Charts

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**ABSTRACT:** The conventional  $\bar{X}$  chart the time of a shift arise in the location parameter could be obtained. But many times, the signal may not be obtained at the time of actual change in mean. Only at a later stage it may be detected or false alarm might be obtained. Samuel et al. (1998) have proposed an estimator for  $\bar{X}$  chart with Fixed Sample Size to estimate the time of a change in the average. Pandurangan (2002) has established an M – Estimator for  $\bar{X}$  chart with VSS to detect the time at which a change has occurred in the mean.

In this paper a new methodology is proposed with an M – Estimator to detect the time of a step change using  $\bar{X}$  control chart with Variable Sample Size(VSS). This methodology can be easily implemented in real life problems.

## I. INTRODUCTION

Various control charts are used to monitor a production process for changes by distinguishing between chance (common) causes of variability (Montgomery, 1996). The control chart is an on – line process filter widely used in production process to detect the variation[7], if any, due to assignable causes. Shewhart (1926) developed the general theory of control charts for measurable as well as non-measurable quality characteristics. His methodology consists of a sampling from a production process over time, such that, successive samples or subgroups selected are independent of each other[8]. It is referred to as conventional sampling scheme (CSS).

The conventional  $\bar{X}$  Chart suggested by Shewhart may detect a moderate-to-large shifts in the process average[6]. Mostly, subgroups of size  $n = 4, 5$  or  $6$  are used to construct these charts. But, larger sample sizes are to be used to detect small change in the average. At the same time, larger sample sizes involve more of the cost of inspection and time of inspection[9]. When a control chart signals that a special cause is present, department of Quality Assurance must find out the cause of the process disturbance[5]. The conventional  $\bar{X}$  control chart plots individual subgroup means against upper and lower control limits[10]. This chart can issue a signal for a shift in a process mean after the shift in the process mean actually occurred. Samuel et al.(1998) have proposed a Maximum Likelihood Estimator for the time of change in the process mean once the Shewhart  $\bar{X}$  control chart issues a signal[4,3,2].

In this paper a model for the time of a step change in the average (mean) of a process is developed and two[1] ML estimators are proposed one for VSS. This method is applied when an  $\bar{X}$  control chart signals that a special cause is present[11].

## II. MODEL FOR PROCESS STEP-CHANGE WITH VSS

To develop the model, the following assumptions are made.

1. The process is initially in control, with each observation is from  $X \sim N(\mu_0, \sigma^2)$ , where  $\mu_0$  is the known mean and  $\sigma^2$  is the known variance.
2.  $\tau$  is the unknown point in time at which the process mean shifts from  $\mu_0$  to  $\mu_1 = \mu_0 + \delta\sigma_0$  where  $\delta$  is the known magnitude of the process change.

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3. Once this step change occurs in the process mean, the process remains at the level of  $\mu_1$ , until the special cause has been detected and removed.
4.  $\bar{X}_\tau$  be the first subgroup average (at  $T^{th}$  sample) to exceed a control limit and that this signal is not a false alarm.

The UCL and LCL for different sample size  $n_1, n_2, \dots$  are

$$UCL = \mu_0 + \frac{3\sigma_0}{\sqrt{n_i}} \text{ and } LCL = \mu_0 - \frac{3\sigma_0}{\sqrt{n_i}} \text{ for } i = 1, 2, 3, \dots$$

Thus  $\bar{X}_1, \bar{X}_2, \dots, \bar{X}_\tau$  are the subgroup averages that came from the in-control process where as  $\bar{X}_{\tau+1}, \bar{X}_{\tau+2}, \dots, \bar{X}_T$  are from the changed process.

i.e., with mean  $\mu_1$ . When two different sample sizes are considered then it is now the problem of estimating the unknown  $\delta$  and  $\mu_1$  is

$$\mu_1 = \mu_0 + \delta\sigma_0 \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \text{ and there by getting the MLE for } \tau$$

**III. DERIVATION OF ML ESTIMATOR FOR THE TIME OF A STEP CHANGE WITH VSS**

In this section the ML Estimator of  $\tau$  with variable sample sizes, the process location change point is derived. For more details refer Casella(1990). Since the random observations come from Normal distribution with mean either  $\mu_0$  or  $\mu_1$ , we have

$$P[X \sim N(\mu_0, \sigma_0^2) \text{ or } Y \sim N(\mu_1, \sigma_0^2)] = P[(X - \mu_0) \sim N(0, \sigma_0^2) \text{ or } (Y - \mu_1) \sim N(0, \sigma_0^2)]$$

Then the Likelihood function will be

$$L = C_1 \exp \left\{ - \left( \frac{X - \mu_0}{\sqrt{2\sigma_0}} \right)^2 \right\} x C_2 \exp \left\{ - \left( \frac{X - \mu_1}{\sqrt{2\sigma_0}} \right)^2 \right\}$$

here  $C_1$ , and  $C_2$  are constants independent of the parameter  $\mu$ .

Assuming a shift had taken place at time  $\tau$  and alarm for process shift occurred at time T, the log-likelihood function for T samples becomes[12]

$$\begin{aligned} \text{In } L(\tau, \mu_1, \bar{X}) &= \text{const} - \left\{ \sum_{i=1}^{\tau} \left( \frac{(\bar{X}_i - \mu_0)^2}{\frac{2\sigma_0^2}{n_i}} \right) + \sum_{i=\tau+1}^T \left( \frac{(\bar{X}_i - \mu_1)^2}{\frac{2\sigma_0^2}{n_i}} \right) \right\} \\ &= - \frac{1}{2\sigma_0^2} \left\{ \sum_{i=1}^{\tau} n_i (\bar{X}_i - \mu_0)^2 + \sum_{i=\tau+1}^T n_i (\bar{X}_i - \mu_1)^2 \right\} \\ &= - \frac{1}{2\sigma_0^2} \left\{ \left[ \sum_{i=1}^{\tau} n_i \bar{X}_i^2 - 2\mu_0 \sum_{i=1}^{\tau} n_i \bar{X}_i + \sum_{i=1}^{\tau} n_i \mu_0^2 \right] + \left[ \sum_{i=\tau+1}^T n_i \bar{X}_i^2 - 2\mu_1 \sum_{i=\tau+1}^T n_i \bar{X}_i + \sum_{i=\tau+1}^T n_i \mu_1^2 \right] \right\} \end{aligned}$$

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$$= -\frac{1}{2\sigma_0^2} \left\{ \left[ \sum_{i=1}^T n_i \bar{X}_i^2 - 2\mu_0 \sum_{i=1}^{\tau} n_i \bar{X}_i + \sum_{i=1}^{\tau} n_i \mu_0^2 - 2\mu_1 \sum_{i=\tau+1}^T n_i \bar{X}_i + \sum_{i=\tau+1}^T n_i \mu_1^2 \right] \right\} \dots\dots\dots(1)$$

The M – estimator of  $\mu_1$  is the solution of  $\frac{\delta \text{Log}(\tau, \mu_1, \bar{X})}{\delta \mu_1} = 0$

$$\Rightarrow -\frac{1}{2\sigma_0^2} \left\{ -2 \sum_{i=\tau+1}^T n_i \bar{X}_i + 2\mu_1 \sum_{i=\tau+1}^T n_i \right\} = 0$$

$$\Rightarrow \hat{\mu}_1 = \frac{\sum_{i=\tau+1}^T n_i \bar{X}_i}{\sum_{i=\tau+1}^T n_i}; \tau \text{ is unknown.} \quad \text{Substituting } \hat{\mu}_1 \text{ in (1)}$$

$$\begin{aligned} \text{In } L(\tau, \mu_1 / \bar{X}) &= -\frac{1}{2\sigma_0^2} \left\{ \left[ \sum_{i=1}^T n_i \bar{X}_i^2 - 2\mu_0 \sum_{i=1}^{\tau} n_i \bar{X}_i + \sum_{i=1}^{\tau} n_i \mu_0^2 - 2\hat{\mu}_1 \sum_{i=\tau+1}^T n_i \bar{X}_i + \sum_{i=\tau+1}^T n_i \hat{\mu}_1^2 \right] \right\} \\ &= -\frac{1}{2\sigma_0^2} \left\{ \left[ \sum_{i=1}^T n_i \bar{X}_i^2 - 2\mu_0 \sum_{i=1}^{\tau} n_i \bar{X}_i + \sum_{i=1}^{\tau} n_i \mu_0^2 + 2\mu_0 \sum_{i=\tau+1}^T n_i \bar{X}_i - \sum_{i=\tau+1}^T n_i \mu_0^2 - 2\hat{\mu}_1 \sum_{i=\tau+1}^T n_i \bar{X}_i + \sum_{i=\tau+1}^T n_i \hat{\mu}_1^2 \right] \right\} \\ &= -\frac{1}{2\sigma_0^2} \left\{ cont - 2 \sum_{i=\tau+1}^T n_i \bar{X}_i (\hat{\mu}_1 - \mu_0) + \sum_{i=\tau+1}^T n_i (\hat{\mu}_1^2 - \mu_0^2) \right\} \\ &= -\frac{1}{2\sigma_0^2} \left\{ cont - \sum_{i=\tau+1}^T n_i [\hat{\mu}_1^2 - 2\mu_0 \hat{\mu}_1 + \mu_0^2] \right\} \\ &= -\frac{1}{2\sigma_0^2} \left\{ cont - \sum_{i=\tau+1}^T n_i (\hat{\mu}_1 - \mu_0)^2 \right\} \end{aligned}$$

ML Estimator of  $\tau$  is the value of  $\tau$  that maximizes (2)

$$\begin{aligned} \hat{\tau} &= \arg. \max_t \left\{ \frac{1}{2\sigma_0^2} \sum_{i=t+1}^T n_i (\hat{\mu}_1 - \mu_0) \right\} \\ &= \arg. \max_t \left\{ \sum_{i=t+1}^T n_i \left[ \frac{\sum_{i=t+1}^T n_i \bar{X}_i}{\sum_{i=t+1}^T n_i} - \mu_0 \right]^2 \right\} \\ &= \arg. \max_t \left\{ \sum_{i=t+1}^T n_i (\bar{X}_{T,t} - \mu_0)^2 \right\}, \text{ Where } \bar{X}_{T,t} = \left\{ \frac{\sum_{i=t+1}^T n_i \bar{X}_i}{\sum_{i=t+1}^T n_i} \right\} \text{ is the reverse cumulative average.} \\ &= \arg. \max_t \{C_t\} \dots\dots\dots(**) \end{aligned}$$

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$$\text{here } C_t = \left\{ \sum_{i=t+1}^T n_i (\bar{X}_{T,t} - \mu_0)^2 \right\}$$

This estimator becomes the same as the ML estimator proposed by Samuel et al. (1998) when  $n_i = n$  [i.e for fixed sample size][13].

## IV. CONCLUSION

The ML estimator derived for VSS to identify the time of a step change in the process can be used as an important tool in the Statistical Process Control. In the illustrative examples it has been shown that a change in the process mean occurred at a time earlier to the time at which a signal has obtained.

It is an economical and elegant model in the sense that the cost of inspection is very much less. It is quicker than the FSS since the total number of samples drawn is very less.

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