

# Some generalization on Fixed Point Theorems Related to Fuzzy Metric Spaces

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**ABSTRACT:** In the present paper some generalization on fixed point and common fixed point theorems in complete Fuzzy 4-metric spaces are established which are motivated by Gahler [13-15], Sharma, Sharma and Isekey [30], Sharma, S.[31], Shrivastava R., Singhal J.[36].

**KEYWORDS:** Fuzzy metric spaces, fuzzy 2- metric spaces, fuzzy 3- metric spaces, fuzzy 4- metric spaces fixed point, Common fixed point.

## I. INTRODUCTION

In 1965, the concept of fuzzy sets was introduced by Zadeh [36]. After that many authors have expansively developed the theory of fuzzy sets and applications. Especially, Deng [8], Erceg [10], Kaleva and Seikhala [23], Karamosil and Michalek [25], have introduced the concept of fuzzy metric spaces in different ways. Recently, many authors [1,6,11,17,20,21,22,27,28,31,32 ] have also studied the fixed point theory in the fuzzy metric spaces and [2,3,4,5,19,26,33] have studied for fuzzy mappings which opened an avenue for further development of analysis in such spaces and such mappings. Consequently in due course of time some metric fixed point results were generalized to fuzzy metric spaces by various authors.

Gahler in a series of papers [13, 14, and 15] investigated 2-metric spaces. Sharma, Sharma and Iseki [30] studied for the first time contraction type mappings in 2-metric space. We [34, 35] have also worked on 2-Metric spaces and 2- Banach spaces for rational expressions.

We know that 2 and 3-metric space is a real valued function of a point triples on a set X, which abstract properties were suggested by the area function in Euclidean spaces. Now it is natural to expect 4-Metric space, which is suggested by the volume function.

## II. SOME FIXED POINT THEOREMS IN FUZZY 2-METRIC SPACE

**Definition (3 A):** A binary operation  $*$ :  $[0, 1] \times [0,1] \times [0,1] \rightarrow [0,1]$  is called a continuous t-norm if  $([0,1],*)$  is an abelian topological monoid with unit 1 such that  $a_1 * b_1 * c_1 \geq a_2 * b_2 * c_2$  whenever  $a_1 \geq a_2, b_1 \geq b_2, c_1 \geq c_2$ , for all  $a_1, a_2, b_1, b_2,$  and  $c_1, c_2$  are in  $[0,1]$ .

**Definition (3 B):** The 3-tuple  $(X, M, *)$  is called a fuzzy 2-metric space if X is an arbitrary set,  $*$  is continuous t-norm and M is fuzzy set in  $X^3 \times [0, \infty)$  satisfying the followings

$$(FM' - 1): M(x, y, z, 0) = 0$$

$$(FM' - 2): M(x, y, z, t) = 1, \forall t > 0, \Leftrightarrow x = y$$

$$(FM' - 3): M(x, y, t) = M(x, z, y, t) = M(y, z, x, t), \text{ Summary about three variable}$$

$$(FM' - 4): M(x, y, z, t_1, t_2, t_3) \geq M(x, y, u, t_1) * M(x, u, z, t_2) * M(x, y, z, t_3)$$

$$(FM' - 5): M(x, y, z): [0,1] \rightarrow [0,1] \text{ is left continuous, } \forall x, y, z, u \in X, t_1, t_2, t_3 > 0$$

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**Definition (3 C):** Let  $(X, M, *)$  be a fuzzy 2-metric space. A sequence  $\{x_n\}$  in fuzzy 2-metric space  $X$  is said to be convergent to a point  $x \in X$ ,

$$\lim_{n \rightarrow \infty} M(x_n, x, a, t) = 1, \text{ for all } a \in X \text{ and } t > 0$$

A sequence  $\{x_n\}$  in fuzzy 2-metric space  $X$  is called a Cauchy sequence, if

$$\lim_{n \rightarrow \infty} M(x_{n+p}, x_n, a, t) = 1, \text{ for all } a \in X \text{ and } t, p > 0$$

A fuzzy 2-metric space in which every Cauchy sequence is convergent is said to be complete.

**Definition (3 D):** A function  $M$  is continuous in fuzzy 2-metric space, iff whenever for all  $a \in X$  and  $t > 0$ .

$$x_n \rightarrow x, y_n \rightarrow y, \text{ then } \lim_{n \rightarrow \infty} M(x_n, y_n, a, t) = M(x, y, a, t), \forall a \in X \text{ and } t > 0$$

**Definition (3 E):** Two mappings  $A$  and  $S$  on fuzzy 2-metric space  $X$  are weakly commuting iff  $M(ASu, SAu, a, t) \geq M(Au, Su, a, t), \forall u, a \in X \text{ and } t > 0$

**Theorem 3.1.** Let  $(X, M, *)$  be a complete fuzzy 2-metric space. Let  $f$  and  $g$  be weakly compatible self maps of  $X$  satisfying

$$M(gx, gy, a, kt) \geq M(fx, fy, a, t) \text{ where } 0 < k < 1, a > 0$$

$$g(X) \subseteq f(X)$$

If one of  $g(X)$  or  $f(X)$  is complete then  $f$  and  $g$  have a unique common fixed point.

**Proof.** Let  $x_0 \in X$ . Since  $g(X) \subseteq f(X)$ . Choose  $x_1 \in X$  such that  $g(x_0) = f(x_1)$ . In general, choose  $x_{n+1}$  such that  $y_n = f x_{n+1} = g x_n$ . Then by (3.1), we have

$$\begin{aligned} M(fx_n, fx_{n+1}, a, t) &= M(gx_{n-1}, gx_n, a, t) \geq M\left(fx_{n-1}, fx_n, a, \frac{t}{k}\right) \\ &= M\left(gx_{n-2}, gx_{n-1}, a, \frac{t}{k}\right) \geq \dots \geq M\left(fx_0, fx_1, a, \frac{t}{k^n}\right). \end{aligned}$$

Therefore, for any  $p$ ,

$$\begin{aligned} M(fx_n, fx_{n+p}, a, t) &\geq M\left(fx_n, fx_{n+1}, a, \frac{t}{p}\right) \geq \dots \geq M\left(fx_{n+p-1}, fx_{n+p}, a, \frac{t}{p}\right) \geq M\left(fx_0, fx_1, a, \frac{t}{pk^n}\right) \geq \dots \\ &\geq M\left(fx_0, fx_1, a, \frac{t}{pk^{n+p-1}}\right) \end{aligned}$$

As  $n \rightarrow \infty$ ,  $\{fx_n\} = \{y_n\}$  is a Cauchy sequence in fuzzy 2-metric space and so, by completeness of  $X$ ,  $\{y_n\} = \{fx_n\}$  is convergent. We call the limit  $z$ , then  $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = z$ . As  $f(X)$  is complete, so there exist a point  $p$  in  $X$  such that  $f_p = z$ . Now, from (3.1),

$$\text{As } n \rightarrow \infty, M(gp, gx_n, a, kt) \geq M(fp, fx_n, a, t),$$

$$M(gp, z, a, kt) \geq M(fp, z, a, t),$$

$$M(gp, z, a, kt) \geq M(z, z, a, t),$$

$$M(gp, z, a, kt) \geq 1,$$

$$M(gp, z, a, kt) = 1,$$

$$gp = z = fp.$$

As  $f$  and  $g$  are weakly compatible. Therefore  $fgp = gfp$  i.e.  $fz = gz$ . Now, we show that  $z$  is fixed point of  $f$  and  $g$ . From (3.1),

$$\text{As } n \rightarrow \infty, M(gz, gx_n, a, kt) \geq M(fz, fx_n, a, t),$$

$$M(gz, z, a, kt) \geq M(fz, z, a, t),$$

$$M(gz, z, a, kt) \geq M(gz, z, a, t),$$

$$gz = z = fz.$$

Hence  $z$  is a common fixed point of  $f$  and  $g$ . For uniqueness, let  $w$  be another fixed point of  $f$  and  $g$ . Then by (3.1),

$$M(gz, gw, a, kt) \geq M(fz, fw, a, t), M(z, w, a, kt) \geq M(z, w, a, t) \text{ and } z=w.$$

Therefore  $z$  is unique common fixed point of  $f$  and  $g$ .

**Theorem 3.2.** Let  $(X, M, *)$  be a fuzzy 2-metric space. Let  $f$  and  $g$  weakly compatible self maps of  $X$  satisfying condition (3.1) and (3.2). If one of  $g(X)$  or  $f(X)$  is complete then  $f$  and  $g$  have a unique common fixed point.

**Proof.** From the proof of above theorem. We conclude that  $\{fx_n\} = \{y_n\}$  is a Cauchy sequence in  $X$ . Now suppose that  $f(X)$  is a complete subspace of  $X$ . Then the subsequence of  $\{y_n\}$  must get a limit in  $f(X)$ . Call it be  $u$  and  $f(v) = u$ . As  $\{y_n\}$  is a Cauchy sequence containing a convergent subsequence, therefore the sequence  $\{y_n\}$  also converges implying thereby the convergence of subsequence of the convergent sequence. Now, from (3.1),

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$$\begin{aligned} \text{As } n \rightarrow \infty. M(gv, gx_n, a, kt) &\geq M(fv, fx_n, a, t), \\ M(gv, u, a, kt) &\geq M(fv, u, a, t), \\ M(gv, u, a, kt) &\geq M(u, u, a, t), \\ M(gv, u, a, kt) &\geq 1, \\ M(gv, u, a, kt) &= 1, \\ gv &= u = fv. \end{aligned}$$

Which shows that pair (f,g) has a point of coincidence. Since, f and g are weakly compatible,  $fgv = gfv$ , i.e.  $fu = gu$ . Now, we show that u is a fixed point of f and g. From (3.1).

$$\begin{aligned} \text{As } n \rightarrow \infty. M(gu, gx_n, a, kt) &\geq M(fu, fx_n, a, t), \\ M(gu, u, a, kt) &\geq M(fu, u, a, t), \\ M(gu, u, a, kt) &\geq M(fu, u, a, t), \\ gu &= u = fu. \end{aligned}$$

Hence u is a fixed point of f and g. For uniqueness, let w be another fixed point of f and g. Then by (3.1),  $M(gz, gw, a, kt) \geq M(fz, fw, a, t)$ ,  $M(z, w, a, kt) \geq M(z, w, a, t)$  and  $z=w$ . Therefore z is unique common fixed point of f and g.

### III. SOME FIXED POINT THEOREMS IN FUZZY 3-METRIC SPACES

**Definition (4 A):** A binary operation  $*$ :  $[0, 1] \times [0, 1] \times [0, 1] \times [0, 1] \rightarrow [0, 1]$  is called a continuous t-norm if  $([0, 1], *)$  is an abelian topological monoid with unit 1 such that  $a_1 * b_1 * c_1 * d \geq a_2 * b_2 * c_2 * d_2$  whenever  $a_1 \geq a_2$ ,  $b_1 \geq b_2$ ,  $c_1 \geq c_2$  and  $d_1 \geq d_2$  for all  $a_1, a_2, b_1, b_2, c_1, c_2$  and  $d_1, d_2$  are in  $[0, 1]$ .

**Definition (4 B):** The 3-tuple  $(X, M, *)$  is called a fuzzy 3-metric space if X is an arbitrary set,  $*$  is continuous t-norm and M is fuzzy set in  $X^4 \times [0, \infty)$  satisfying the followings

$$\begin{aligned} (FM'' - 1): M(x, y, z, w, 0) &= 0 \\ (FM'' - 2): M(x, y, z, w, t) &= 1, \forall t > 0 \\ (FM'' - 3): M(x, y, z, w, t) &= M(x, w, z, y, t) = M(z, w, x, y, t) = \dots \\ (FM'' - 4): M(x, y, z, w, t_1 + t_2 + t_3) &\geq M(x, y, z, u, t_1) * M(x, y, u, w, t_2) * M(x, u, z, w, t_3) * M(x, y, z, w, t_4) \\ (FM'' - 5): M(x, y, z, w): [0, 1] &\rightarrow [0, 1] \text{ is left continuous, } \forall x, y, z, u \in X, t_1, t_2, t_3, t_4 > 0 \end{aligned}$$

**Definition (4 C):** Let  $(X, M, *)$  be a fuzzy 3-metric space. A sequence  $\{x_n\}$  in fuzzy 3-metric space X is said to be convergent to a point  $x \in X$ ,

$$\lim_{n \rightarrow \infty} M(x_n, x, a, b, t) = 1, \text{ for all } a, b \in X \text{ and } t > 0$$

A sequence  $\{x_n\}$  in fuzzy 3-metric space X is called a Cauchy sequence, if

$$\lim_{n \rightarrow \infty} M(x_{n+p}, x_n, a, b, t) = 1, \text{ for all } a, b \in X \text{ and } t, p > 0$$

A fuzzy 3-metric space in which every Cauchy sequence is convergent is said to be complete.

**Definition (4 D):** A function M is continuous in fuzzy 3-metric space, iff whenever for all  $a \in X$  and  $t > 0$ .

$$x_n \rightarrow x, y_n \rightarrow y, \text{ then } \lim_{n \rightarrow \infty} M(x_n, y_n, a, b, t) = M(x, y, a, b, t), \forall a, b \in X \text{ and } t > 0$$

**Definition (4 E):** Two mappings A and S on fuzzy 3-metric space X are weakly commuting iff  $M(ASu, SAu, a, b, t) \geq M(Au, Su, a, t), \forall u, a, b \in X \text{ and } t > 0$ .

**Theorem 4.1.** Let  $(X, M, *)$  be a complete fuzzy 3-metric space. Let f and g be weakly compatible self maps of X satisfying

$$\begin{aligned} M(gx, gy, a, b, kt) &\geq M(fx, fy, a, t) \text{ where } 0 < k < 1, a, b > 0 \\ g(X) &\subseteq f(X) \end{aligned}$$

If one of  $g(X)$  or  $f(X)$  is complete then f and g have a unique common fixed point.

**Proof.** Let  $x_0 \in X$ . Since  $g(X) \subseteq f(X)$ . Choose  $x_1 \in X$  such that  $g(x_0) = f(x_1)$ . In general, choose  $x_{n+1}$  such that  $y_n = fx_{n+1} = gx_n$ . Then by (4.1), we have

$$\begin{aligned} M(fx_n, fx_{n+1}, a, b, t) &= M(gx_{n-1}, gx_n, a, b, t) \geq M\left(fx_{n-1}, fx_n, a, b, \frac{t}{k}\right) \\ &= M\left(gx_{n-2}, gx_{n-1}, a, b, \frac{t}{k}\right) \geq \dots \geq M\left(fx_0, fx_1, a, b, \frac{t}{k^n}\right). \end{aligned}$$

Therefore, for any p,

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$$M(fx_n, fx_{n+p}, a, b, t) \geq M\left(fx_n, fx_{n+1}, a, b, \frac{t}{p}\right) \geq \dots \geq M\left(fx_{n+p-1}, fx_{n+p}, a, b, \frac{t}{p}\right) \geq M\left(fx_0, fx_1, a, b, \frac{t}{pk^n}\right) \geq \dots$$

$$\geq M\left(fx_0, fx_1, a, b, \frac{t}{pk^{n+p-1}}\right)$$

As  $n \rightarrow \infty$ ,  $\{fx_n\} = \{y_n\}$  is a Cauchy sequence in fuzzy 3-metric space and so, by completeness of X,  $\{y_n\} = \{fx_n\}$  is convergent. We call the limit z, then  $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = z$ . As f(X) is complete, so there exist a point p in X such that  $fp = z$ . Now, from (3.1),

$$\begin{aligned} \text{As } n \rightarrow \infty. M(gp, gx_n, a, b, kt) &\geq M(fp, fx_n, a, b, t), \\ M(gp, z, a, b, kt) &\geq M(fp, z, a, b, t), \\ M(gp, z, a, b, kt) &\geq M(z, z, a, b, t), \\ M(gp, z, a, b, kt) &\geq 1, \\ M(gp, z, a, b, kt) &= 1, \\ gp = z = fp. \end{aligned}$$

As f and g are weakly compatible. Therefore  $fgp = gfp$  i.e.  $gz = fz$ . Now, we show that z is fixed point of f and g. From (4.1),

$$\begin{aligned} \text{As } n \rightarrow \infty. M(gz, gx_n, a, b, kt) &\geq M(fz, fx_n, a, b, t), \\ M(gz, z, a, b, kt) &\geq M(fz, z, a, b, t), \\ M(gz, z, a, b, kt) &\geq M(gz, z, a, b, t), \\ gz = z = fz. \end{aligned}$$

Hence z is a common fixed point of f and g. For uniqueness, let w be another fixed point of f and g. Then by (3.1),  $M(gz, gw, a, b, kt) \geq M(fz, fw, a, b, t)$ ,  $M(z, w, a, b, kt) \geq M(z, w, a, b, t)$  and  $z=w$ .

Therefore z is unique common fixed point of f and g.

**Theorem 4.2.** Let  $(X, M, *)$  be a fuzzy 3-metric space. Let f and g weakly compatible self maps of X satisfying condition (4.1) and (4.2). If one of  $g(X)$  or  $f(X)$  is complete then f and g have a unique common fixed point.

**Proof** From the proof of above theorem. We conclude that  $\{fx_n\} = \{y_n\}$  is a Cauchy sequence in X. Now suppose that  $f(X)$  is a complete subspace of X. Then the subsequence of  $\{y_n\}$  must get a limit in  $f(X)$ . Call it be u and  $f(v) = u$ . As  $\{y_n\}$  is a Cauchy sequence containing a convergent subsequence, therefore the sequence  $\{y_n\}$  also converges implying thereby the convergence of subsequence of the convergent sequence. Now, from (4.1),

$$\begin{aligned} \text{As } n \rightarrow \infty. M(gv, gx_n, a, b, kt) &\geq M(fv, fx_n, a, b, t), \\ M(gv, u, a, b, kt) &\geq M(fv, u, a, b, t), \\ M(gv, u, a, b, kt) &\geq M(u, u, a, b, t), \\ M(gv, u, a, b, kt) &\geq 1, \\ M(gv, u, a, b, kt) &= 1, \\ gv = u = fv. \end{aligned}$$

Which shows that pair (f,g) has a point of coincidence. Since, f and g are weakly compatible,  $fgv = gfv$ , i.e.  $fu = gu$ .

Now, we show that u is a fixed point of f and g. From (4.1).

$$\begin{aligned} \text{As } n \rightarrow \infty. M(gu, gx_n, a, b, kt) &\geq M(fu, fx_n, a, b, t), \\ M(gu, u, a, b, kt) &\geq M(fu, u, a, b, t), \\ M(gu, u, a, b, kt) &\geq M(fu, u, a, b, t), \\ gu = u = fu. \end{aligned}$$

Hence u is a fixed point of f and g. For uniqueness, let w be another fixed point of f and g. Then by (4.1),

$$M(gz, gw, a, b, kt) \geq M(fz, fw, a, b, t), M(z, w, a, b, kt) \geq M(z, w, a, b, t) \text{ and } z=w.$$

Therefore z is unique common fixed point of f and g.

### IV. SOME FIXED POINT THEOREMS IN FUZZY 4-METRIC SPACES

**Definition (5 A):** A binary operation  $*$ :  $[0,1]^5 \rightarrow [0,1]$  is called a continuous t-norm if  $\{[0,1], *\}$  is an abelian topological monoid with unit 1 such that  $a_1 * b_1 * c_1 * d_1 * e_1 \geq a_2 * b_2 * c_2 * d_2 * e_2$  whenever  $a_1 \geq a_2, b_1 \geq b_2, c_1 \geq c_2, d_1 \geq d_2$  and  $e_1 \geq e_2$  for all  $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2$  and  $e_1, e_2$  are in  $[0,1]$ .

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**Definition (5 B):** The 3-tuple  $(X, M, *)$  is called a fuzzy 4-metric space if  $X$  is an arbitrary set,  $*$  is continuous t-norm monoid and  $M$  is a fuzzy set in  $X^4 \times [0, \infty]$  satisfying the following conditions:

- (1)  $M(x, y, u, v, w, 0) = 0$
- (2)  $M(x, y, u, v, w, t) = 1 \forall t > 0$
- (3)  $M(x, y, u, v, w, t) = M(x, v, u, y, w, t) = M(u, y, x, v, w, t) = \dots$
- (4)  $M(x, y, u, v, w, t_1 + t_2 + t_3 + t_4 + t_5) \geq M(x, y, u, v, w, t_1) + M(x, y, u, v, w, t_2) + M(x, y, u, v, w, t_3) + M(x, y, u, v, w, t_4) + M(x, y, u, v, w, t_5)$
- (5)  $M(x, y, u, v, w): [0, 1] \rightarrow [0, 1]$  is left continuous,  $\forall x, y, u, v, w \in X, t_1, t_2, t_3, t_4, t_5 > 0$

**Definition (5 C):** Let  $(X, M, *)$  be a fuzzy 4 metric space:

- (1) A sequence  $\{x_n\}$  in fuzzy 4 metric space  $X$  is called convergent to appoint  $x \in X$ , if  $\lim_{n \rightarrow \infty} M(x_n, x, a, b, c, t) = 1$ , for all  $a, b \in X$  and  $t > 0$
- (2) A sequence  $\{x_n\}$  in fuzzy 4 metric space  $X$  is called Cauchy sequence, if  $\lim_{n \rightarrow \infty} M(x_{n+p}, x, a, b, c, t) = 1$ , for all  $a, b \in X$  and  $t, p > 0$
- (3) A fuzzy 4 metric space in which every Cauchy sequence is convergent is said to be complete.

**Definition (5 D):** A function  $M$  is continuous in fuzzy 3-metric space if  $x_n \rightarrow x, y_n \rightarrow y$ , then  $\lim_{n \rightarrow \infty} M(x_n, y_n, a, b, c, t) = M(x, y, a, b, c, t), \forall a, b \in X$  and  $t > 0$

**Definition (5 E):** Two mappings  $A$  and  $S$  on fuzzy 4-metric space  $X$  are weakly commuting iff,

$$M(ASu, SAu, a, b, c, t) \geq M(Au, Su, a, b, c, t) \forall u, a, b, c \in X \text{ and } t > 0$$

**Theorem (5.1)** Let  $(X, M, *)$  be a complete fuzzy 4-metric space. Let  $f$  and  $g$  be weakly compatible self maps of  $X$  satisfying

$$(5.1) M(gx, gy, a, b, c, kt) \geq M(fx, fy, a, b, c, t) \text{ where } 0 < k < 1, a, b > 0$$

$$(5.2) g(X) \subseteq f(X)$$

If one of  $g(X)$  or  $f(X)$  is complete then  $f$  and  $g$  have a unique common fixed point.

**Proof.** Let  $x_0 \in X$ . Since  $g(X) \subseteq f(X)$ . Choose  $x_1 \in X$  such that  $g(x_0) = f(x_1)$ . In general, choose  $x_{n+1}$  such that  $y_n = fx_{n+1} = gx_n$ . Then, we have

$$\begin{aligned} M(fx_n, fx_{n+1}, a, b, c, t) &= M(gx_{n-1}, gx_n, a, b, c, t) \geq M\left(fx_{n-1}, fx_n, a, b, c, \frac{t}{k}\right) = M\left(gx_{n-2}, gx_{n-1}, a, b, c, \frac{t}{k}\right) \\ &\geq \dots \geq M\left(fx_0, fx_1, a, b, c, \frac{t}{k^n}\right) \end{aligned}$$

Therefore for any  $p$ ,

$$\begin{aligned} M(fx_n, fx_{n+p}, a, b, c, t) &\geq M\left(fx_n, fx_{n+1}, a, b, c, \frac{t}{p}\right) \dots \geq M\left(fx_{n+p-1}, fx_{n+p}, a, b, c, \frac{t}{p}\right) \\ &\geq M\left(fx_0, fx_1, a, b, c, \frac{t}{pk^n}\right) \geq \dots \geq M\left(fx_0, fx_1, a, b, c, \frac{t}{pk^{n+p-1}}\right) \end{aligned}$$

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As  $n \rightarrow \infty$ ,  $\{fx_n\} = \{y_n\}$  is a Cauchy sequence in fuzzy 4-metric space and so, by completeness of  $X$ ,  $\{y_n\} = \{fx_n\}$  is convergent. We call the limit  $z$ , then  $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = z$ . As  $f(X)$  is complete, so there exist a point in  $X$  such that  $fp = z$ . Now,

$$\text{As } n \rightarrow \infty, M(gp, gx_n, a, b, c, kt) \geq M(fp, fx_n, a, b, c, t)$$

$$M(gp, z, a, b, c, kt) \geq M(fp, z, a, b, c, t)$$

$$M(gp, z, a, b, c, kt) \geq M(z, z, a, b, c, t)$$

$$M(gp, z, a, b, c, kt) \geq 1$$

$$M(gp, z, a, b, c, kt) = 1$$

$$g = z = fp.$$

As  $f$  and  $g$  are weakly compatible. Therefore  $fgp=gfp$  i.e.  $gz=gz$ . Now, we show that  $z$  is fixed point of  $f$  and  $g$ .

$$\text{As } n \rightarrow \infty M(gz, gx_n, a, b, c, kt) \geq M(fz, fx_n, a, b, c, t)$$

$$M(gz, z, a, b, c, kt) \geq M(fz, z, a, b, c, t)$$

$$M(gz, z, a, b, c, kt) \geq M(gz, z, a, b, c, t)$$

$$gz = z = fz.$$

Hence  $z$  is a common fixed point of  $f$  and  $g$ . For uniqueness, let  $q$  be another fixed point of  $f$  and  $g$ . Then

$$M(gz, gq, a, b, c, kt) \geq M(fz, fq, a, b, c, t), M(z, q, a, b, c, kt) \geq M(z, q, a, b, c, t) \text{ and } z=q.$$

Therefore  $z$  is unique common fixed point of  $f$  and  $g$ .

**Theorem (5.2)** Let  $(X, M, *)$  be a fuzzy 4-metric space. Let  $f$  and  $g$  are weakly compatible self maps of  $X$  satisfying conditions (5.1) and (5.2).

**Proof.** From the proof of above theorem. We conclude that  $\{fx_n\} = \{y_n\}$  is a Cauchy sequence in  $X$ . Now suppose that  $f(X)$  is complete subspace of  $X$ . Then the subsequence of  $\{y_n\}$  must get a limit in  $f(X)$ . Call it be  $r$  and  $f(s)=r$ . Then  $\{y_n\}$  is a Cauchy sequence containing a subsequence, therefore the sequence also convergent implying thereby the convergence of the convergent sequence. Now, from (5.1),

$$\text{As } n \rightarrow \infty, M(gs, gx_n, a, b, c, kt) \geq M(fs, fx_n, a, b, c, t),$$

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$$M(gs, r, a, b, c, kt) \geq M(fs, r, a, b, c, t),$$

$$M(gs, r, a, b, c, kt) \geq M(r, r, a, b, c, t),$$

$$M(gs, r, a, b, c, kt) \geq 1,$$

$$M(gs, r, a, b, c, kt) = 1,$$

$$gs = r = fs$$

Which shows that pair (f,g) have a point of coincidence. Since, f and g are weakly compatible,  $fgs = gfs$ , i. e.  $fr = gr$ .

Now, we show that r is a fixed point of f and g. From (5.1).

$$M(gr, gx_n, a, b, c, kt) \geq M(fr, fx_n, a, b, c, t),$$

$$M(gr, r, a, b, c, kt) \geq M(fr, u, a, b, c, t),$$

$$M(gr, gx_n, a, b, c, kt) \geq M(gr, fx_n, a, b, c, t),$$

$$gr = r = fr$$

Hence r is a fixed point of f and g. For uniqueness, let d be another fixed point of f and g. Then by (5.1).

$$M(gr, gd, a, b, c, kt) \geq M(fr, fd, a, b, c, t),$$

$$M(r, d, a, b, c, kt) \geq M(r, d, a, b, c, t), \text{ and}$$

$$r = d.$$

Therefore r is unique common fixed point of f and g.

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