

Solving Delay Differential Equations with Constant Lags using RKCeM Method

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ABSTRACT: This paper presents RK method based on Centroidal mean for solving Delay differential equations with constant delays. The delay term is approximated by using linear and Lagrange interpolation. The effectiveness of this method has been illustrated with examples. The numerical results are also compared with RK method based on Arithmetic mean.

KEYWORDS: Delay differential equations, constant delay, multiple delays, IVP, RKAM, RKCeM.

I. INTRODUCTION

Delay differential equations (DDEs) arise in population dynamics [1], control systems [2], chemical kinetics [3] and in many areas of Science and engineering. Recently there has been great interest in finding the numerical solutions of DDEs. Many numerical methods have been suggested for solving DDEs of first order as in [4-6]. Paul and Baker [7] discussed about the determination of stability region of RK method for DDEs. Hu et al. [8] considered the stability of RK methods for DDEs with multiple delays.

Evans and Yaakub [9] have proposed the fourth order RK method based on Centroidal mean (RKCeM) for solving IVPs. Murugesan et al. [10] have applied RKCeM to solve second order IVPs.

In this paper the fourth order RKCeM method has been discussed for obtaining the numerical solution of first order DDEs. Here we consider the first order DDEs in the following form:

$$y'(t) = f(t, y(t), y(t - \tau)), \quad t > t_0$$

$$y(t) = \Phi(t), \quad t \leq t_0$$

if it has one delay only, or

$$y'(t) = f(t, y(t), y(t - \tau_1), y(t - \tau_2), \dots, y(t - \tau_n)), \quad t > t_0$$

$$y(t) = \Phi(t), \quad t \leq t_0$$

if it has more than one delay term where $\Phi(t)$ is the initial function. Here the delay terms $\tau, \tau_1, \tau_2, \dots, \tau_n$ are positive constants. Three numerical examples have been considered to demonstrate the adaptability of RKCeM to DDEs.

This paper has been organized as follows:

In Section II, the fourth order RKAM and RKCeM formula for solving ODEs have been briefly mentioned. In Section III, the adaptability of RKCeM formula for solving DDEs has been explained. In Section IV, the convergence and

International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Vol. 5, Issue 4, April 2016

stability for RKCeM have been discussed. In Section V, the numerical examples have been provided to illustrate the effectiveness of RKCeM.

II. RKCEM METHOD FOR ORDINARY DIFFERENTIAL EQUATIONS

Consider the first order equation of the form

$$y' = f(x, y) \quad \text{with } y(x_0) = y_0.$$

The fourth order RKAM formula of the $(n+1)^{\text{th}}$ increment in y is computed as

$$y_{n+1} = y_n + \frac{h}{3} \left[\frac{k_1 + k_2}{2} + \frac{k_2 + k_3}{2} + \frac{k_3 + k_4}{2} \right]$$

$$= y_n + \frac{h}{3} \sum_{i=1}^3 \frac{k_i + k_{i+1}}{2} = y_n + \frac{h}{3} \sum_{i=1}^3 (AM)$$

where

$$k_1 = f(x_n, y_n)$$

$$k_2 = f\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_1\right)$$

$$k_3 = f\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_2\right)$$

$$k_4 = f(x_n + h, y_n + hk_3)$$

Evans and Yaakub have developed a new RK method of order four based on Centroidal mean in solving first order differential equations. The Centroidal mean of two points y_1 and y_2 is defined as

$$\text{CeM} = \frac{y_1^2 + y_1y_2 + y_2^2}{y_1 + y_2}.$$

By replacing the AM in eqn. (2.1) by CeM, the fourth order RKCeM formula is written as

$$y_{n+1} = y_n + \frac{2}{9}h \left[\frac{k_1^2 + k_1k_2 + k_2^2}{k_1 + k_2} + \frac{k_2^2 + k_2k_3 + k_3^2}{k_2 + k_3} + \frac{k_3^2 + k_3k_4 + k_4^2}{k_3 + k_4} \right]$$

$$i.e. \ y_{n+1} = y_n + \frac{h}{3} \left[\frac{2}{3} \sum_{i=1}^3 \left(\frac{k_i^2 + k_ik_{i+1} + k_{i+1}^2}{k_i + k_{i+1}} \right) \right]$$

International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Vol. 5, Issue 4, April 2016

where

$$\begin{aligned}
 k_1 &= f(x_n, y_n) \\
 k_2 &= f\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_1\right) \\
 k_3 &= f\left(x_n + \frac{1}{2}h, y_n + \frac{1}{24}hk_1 + \frac{11}{24}hk_2\right) \\
 k_4 &= f\left(x_n + h, y_n + \frac{11}{132}hk_1 - \frac{25}{132}hk_2 + \frac{73}{66}hk_3\right)
 \end{aligned}$$

The Butcher array for RKCeM is

0			
$\frac{1}{2}$	$\frac{1}{2}$		
$\frac{1}{2}$	$\frac{1}{24}$	$\frac{11}{24}$	
1	$\frac{11}{132}$	$-\frac{25}{132}$	$\frac{73}{66}$
$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	

III. RKCeM METHOD FOR DELAY DIFFERENTIAL EQUATIONS

Consider the first order DDE of the form

$$y'(t) = f(t, y(t), y(t - \tau)), \quad t > t_0$$

$$y(t) = \Phi(t), \quad t \leq t_0$$

where $\Phi(t)$ is the initial function and τ is the delay term.

When we adapt the fourth order RKCeM formulato DDE, we have

$$y_{n+1} = y_n + \frac{2}{9}h \left[\frac{k_1^2 + k_1k_2 + k_2^2}{k_1 + k_2} + \frac{k_2^2 + k_2k_3 + k_3^2}{k_2 + k_3} + \frac{k_3^2 + k_3k_4 + k_4^2}{k_3 + k_4} \right]$$

International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Vol. 5, Issue 4, April 2016

where

$$k_i = f \left(x_n + c_i h, y_n + h \sum_{j=1}^4 a_{ij} k_j, y(x_n + c_i h - \tau) \right), \quad (i = 1, \dots, 4).$$

The above RKCeM formula can also be extended to solve DDEs with multiple delays.

IV. CONVERGENCE AND STABILITY OF RKCeM

Order of Convergence of RKCeM:

When we use RKCeM formula to solve DDEs, the delay term $y(x_n + c_i h - \tau)$ need to be interpolated to approximate the value. Many techniques are available for the approximation. In this paper linear and Lagrange interpolation are used to approximate the delay term. The interpolation order have to be adapted to the order of the method.

For any given RK method, its adaptation to DDEs by means of interpolation procedure has an order of convergence equal to $\min \{ p, q \}$ where p denotes the order of consistency of the RK method and q is the number of support points of the interpolation procedure. (See [11])

For linear interpolation we are using two support points so that the order of convergence of RK method is two (See [12]) which is less than the order of the RK method. For Lagrange interpolation we are using five support points so that the order of convergence of RK method is four for DDEs also.

Linear stability for RKCeM:

There are many concepts of stability of numerical methods to DDE, which depends on the test equation as well as the delay term involved. Here our attention is to a linear test equation with a constant delay τ ,

$$y'(x) = \lambda y(x) + \mu y(x - \tau), \quad t \geq t_0$$

$$y(t) = \Phi(t), \quad t \leq t_0$$

where $\lambda, \mu \in \mathbb{R}$, $\tau > 0$ and Φ is continuous.

The Butcher array representation of RK method is of the form

$$\begin{array}{c|c} \mathbf{C} & \mathbf{A} \\ \hline & \mathbf{b}^T \end{array}$$

The stability function of s-stage RK method for ODE is given by

$$r(z) = 1 + z\mathbf{b}^T (I - z\mathbf{A})^{-1} \mathbf{e},$$

where $z = h\lambda$, $\mathbf{e} = (1, 1, \dots, 1)^T$, $\mathbf{b} = (b_1, b_2, \dots, b_s)^T$, $\mathbf{A} = (a_{ij})$.

International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Vol. 5, Issue 4, April 2016

In the case of DDEs, we consider $\tau = Nh$ where N is a positive integer. In this case when $u(nh + c_i h - \tau)$ is required a previous internal stage-value $Y_{n-N,i}$ is used. Here we consider the stability properties of a recurrence of the form

$$y_{n+1} = \left[1 + \lambda h b^T (I - \lambda h A)^{-1} e \right] y_n + \mu h b^T (I - \lambda h A)^{-1} u_{n-N}$$

where u_{n-N} is a vector consisting of 'back-values' $u(nh + c_i h - \tau)$.

This can be expressed in a convenient form as

$$y_{n+1} = r(\lambda h) y_n + \mu h b^T (I - \lambda h A)^{-1} u_{n-N}.$$

If we write $S \equiv S(\lambda h) = (I - \lambda h A)^{-1}$ and $\Phi_n = [Y_{n,1}, \dots, Y_{n,s}]^T$, the above can be written as

$$y_{n+1} = r(\lambda h) y_n + \mu h b^T S \phi_{n-N}$$

We can express this as the recurrence:

$$\Phi_n = X \Phi_{n-1} + Z \Phi_{n-N}$$

where $X = \begin{pmatrix} r(\lambda h) & | & 0 \\ \hline Se & | & 0 \end{pmatrix}$ and $Z = \begin{pmatrix} 0 & | & \mu h b^T S \\ \hline 0 & | & \mu h S A \end{pmatrix}$.

This recurrence is stable if the zeros ζ_i of the stability polynomial

$$S_h(\lambda, \mu; \zeta) = \det \left[\zeta^N I - \zeta^{N-1} X - Z \right].$$

The root condition for stability is the requirement that all the zeros ζ_i of $S_h(\lambda, \mu; \zeta)$ satisfy $|\zeta_i| \leq 1$, and if $|\zeta_i| = 1$ then ζ_i is semi-simple.

The stability polynomial of RKAM is obtained as

$$S(\lambda h, \mu h; \zeta) = \zeta^{5N} - \zeta^{5N-1} \left(1 + \lambda h + \frac{(\lambda h)^2}{2} + \frac{(\lambda h)^3}{6} + \frac{(\lambda h)^4}{24} \right) - \zeta^{4N-1} (\mu h) \left(1 + \lambda h + \frac{(\lambda h)^2}{2} + \frac{(\lambda h)^3}{6} \right) \\ - \zeta^{3N-1} \frac{(\mu h)^2}{2} \left(1 + \lambda h + \frac{(\lambda h)^2}{2} \right) - \zeta^{2N-1} \frac{(\mu h)^3}{6} (1 + \lambda h) - \zeta^{N-1} \frac{(\mu h)^4}{24}$$

and the stability polynomial of RKCeM is obtained as

$$S(\lambda h, \mu h; \zeta) = \zeta^{5N} - \zeta^{5N-1} \left(1 + \lambda h + \frac{(\lambda h)^2}{2} + \frac{(\lambda h)^3}{3} + \frac{(\lambda h)^4}{24} + \frac{37(\lambda h)^5}{5184} \right) - \zeta^{4N-1} (\mu h) \left(1 + \frac{2}{3} \lambda h + \frac{11}{48} (\lambda h)^2 \right) \\ - \zeta^{3N-1} (\mu h)^2 \left(\frac{1}{3} + \frac{11}{48} (\lambda h) \right) - \zeta^{2N-1} (\mu h)^3 \left(\frac{11}{144} \right).$$

International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Vol. 5, Issue 4, April 2016

V.NUMERICAL EXAMPLES

Example 1

Consider the first order DDE with single delay

$$y'(t) = y(t-1)$$

$$y(t) = \lambda e^t, \quad -1 \leq t \leq 0$$

with exact solution

$$y(t) = \begin{cases} \lambda + \frac{(-1+e^t)\lambda}{e} & \text{on } [0,1] \\ \lambda + \frac{\lambda}{e} + e^{(-2+t)}\lambda + t\lambda - \frac{t\lambda}{e} & \text{on } [1,2] \\ 2\lambda - \frac{2\lambda}{e} + e^{(-3+t)}\lambda + \frac{t^2\lambda}{2} - \frac{t^2\lambda}{2e} & \text{on } [2,3] \end{cases}$$

This problem is solved by RKAM and RKCeM by using linear interpolation and Lagrange interpolation for the delay term with $\lambda=1$. The absolute error results are shown in the following Tables 1 and 2.

Table 1 Results of Example 1 (Linear Interpolation)

Time	Absolute Error (RKAM)	Absolute Error (RKCeM)
0.50	1.99e-006	2.32e-006
1.00	5.27e-006	6.15e-006
1.50	7.71e-006	8.81e-006
2.00	1.27e-005	1.44e-005
2.50	1.79e-005	2.05e-005
3.00	2.61e-005	3.00e-005

Table 2 Results of Example 1 (Lagrange Interpolation)

Time	Absolute Error (RKAM)	Absolute Error (RKCeM)
0.50	3.89e-013	8.32e-012
1.00	5.27e-006	1.54e-010
1.50	5.27e-006	1.58e-010
2.00	5.27e-006	1.71e-010
2.50	7.90e-006	2.63e-010
3.00	1.05e-005	3.61e-010

International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Vol. 5, Issue 4, April 2016

Example 2:

Consider the system of first order DDE with single delay

$$\begin{aligned} y'_1(t) &= y_2(t) \\ y'_2(t) &= y_3(t) \\ y'_3(t) &= -y_1(t) - y_1(t-0.3) + e^{-(t+0.3)}, \quad 0 \leq t \leq 1 \end{aligned}$$

with initial functions

$$y_1(t) = e^{-t}; \quad y_2(t) = -e^{-t}; \quad y_3(t) = e^{-t}, \quad t \leq 0$$

with exact solutions

$$y_1(t) = e^{-t}; \quad y_2(t) = -e^{-t}; \quad y_3(t) = e^{-t}$$

This problem is solved by RKAM and RKCeM by using linear interpolation and Lagrange interpolation for the delay term. The absolute error results are shown in the following Tables 3 and 4.

Table 3 Results of Example 2 (Linear Interpolation)

Time	Absolute Error (RKAM)			Absolute Error (RKCeM)		
	y ₁	y ₂	y ₃	y ₁	y ₂	y ₃
0.20	3.57e-009	5.27e-008	5.10e-007	3.56e-009	5.26e-008	5.10e-007
0.40	2.72e-008	1.98e-007	9.24e-007	2.72e-008	1.97e-007	9.24e-007
0.60	8.75e-008	4.17e-007	1.25e-006	8.75e-008	4.17e-007	1.25e-006
0.80	1.98e-007	6.94e-007	1.50e-006	1.98e-007	6.93e-007	1.50e-006
1.00	3.68e-007	1.01e-006	1.66e-006	3.68e-007	1.01e-006	1.66e-006

Table 4 Results of Example 2 (Lagrange Interpolation)

Time	Absolute Error (RKAM)			Absolute Error (RKCeM)		
	y ₁	y ₂	y ₃	y ₁	y ₂	y ₃
0.20	8.57e-013	8.58e-013	8.43e-013	3.44e-015	3.44e-015	3.44e-015
0.40	1.40e-012	1.41e-012	1.35e-012	6.99e-015	6.99e-015	6.99e-015
0.60	1.72e-012	1.75e-012	1.50e-012	1.90e-014	1.89e-014	1.93e-014
0.80	1.86e-012	1.98e-012	1.37e-012	2.58e-014	2.57e-014	2.50e-014
1.00	1.87e-012	2.16e-012	1.07e-12	2.87e-014	2.90e-014	2.65e-014

Example 3:

Consider the system of first order DDE with multiple delay

$$\begin{aligned} y'_1(t) &= y_5(t-1) + y_3(t-1); \quad y'_2(t) = y_1(t-1) + y_2(t-\frac{1}{2}); \quad y'_3(t) = y_3(t-1) + y_1(t-\frac{1}{2}) \\ y'_4(t) &= y_5(t-1)y_4(t-1); \quad y'_5(t) = y_1(t-1) \quad \text{for } t \geq 0 \end{aligned}$$

with initial functions for $t \leq 0$

$$y_1(t) = e^{(t+1)}; \quad y_2(t) = e^{(t+\frac{1}{2})}; \quad y_3(t) = \sin(t+1); \quad y_4(t) = e^{(t+1)}; \quad y_5(t) = e^{(t+1)}$$

International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Vol. 5, Issue 4, April 2016

with analytical solutions for $0 \leq t \leq \frac{1}{2}$

$$y_1(t) = e^t - \cos t + e; \quad y_2(t) = 2e^t + e^{\frac{1}{2}} - 2; \quad y_3(t) = e^{(t+\frac{1}{2})} - \cos t + 1 - e^{\frac{1}{2}} + \sin(1);$$

$$y_4(t) = \frac{1}{2}e^{2t} - \frac{1}{2} + e; \quad y_5(t) = e^t + e - 1$$

This problem is solved by RKAM and RKCeM by using linear interpolation and Lagrange interpolation for the delay term. The absolute error results are shown in the following Tables 5 - 8.

Table 5 Results of Example 3 (Linear Interpolation)

Time	Absolute Error (RKAM)				
	y ₁	y ₂	y ₃	y ₄	y ₅
0.10	8.35e-007	1.75e-006	1.40e-006	1.85e-006	8.76e-007
0.20	1.68e-006	3.69e-006	2.88e-006	4.10e-006	1.85e-006
0.30	2.54e-006	5.83e-006	4.44e-006	6.85e-006	2.92e-006
0.40	3.44e-006	8.20e-006	6.10e-006	1.02e-006	4.10e-006
0.50	4.39e-006	1.08e-006	7.89e-006	1.43e-006	5.41e-006

Table 6 Results of Example 3 (Linear Interpolation)

Time	Absolute Error (RKCeM)				
	y ₁	y ₂	y ₃	y ₄	y ₅
0.10	1.37e-006	2.05e-006	1.99e-006	2.46e-006	1.02e-006
0.20	2.70e-006	4.31e-006	4.03e-006	5.47e-006	2.15e-006
0.30	4.03e-006	6.80e-006	6.14e-006	9.14e-006	3.40e-006
0.40	5.36e-006	9.56e-006	8.37e-006	1.36e-005	4.78e-006
0.50	6.73e-006	1.26e-005	1.07e-005	1.91e-005	6.31e-005

Table 7 Results of Example 3 (Lagrange Interpolation)

Time	Absolute Error (RKAM)				
	y ₁	y ₂	y ₃	y ₄	y ₅
0.10	2.25e-013	8.77e-012	3.07e-012	9.23e-012	4.38e-012
0.20	9.52e-013	1.85e-011	6.94e-012	2.05e-011	9.23e-012
0.30	2.27e-012	2.92e-011	1.17e-011	3.43e-011	1.46e-011
0.40	4.28e-012	4.10e-011	1.76e-011	5.11e-011	2.05e-011
0.50	7.07e-012	2.93e-008	1.43e-007	7.16e-011	2.70e-011

International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Vol. 5, Issue 4, April 2016

Table 8 Results of Example 3 (Lagrange Interpolation)

Time	Absolute Error (RKCeM)				
	y ₁	y ₂	y ₃	y ₄	y ₅
0.10	1.33e-011	7.31e-012	1.45e-011	1.54e-011	3.66e-12
0.20	2.56e-011	1.54e-011	2.88e-011	3.42e-011	7.70e-012
0.30	3.76e-011	2.44e-011	4.28e-011	5.71e-011	1.22e-011
0.40	4.81e-011	3.42e-011	5.67e-011	8.51e-011	1.71e-011
0.50	5.87e-011	2.80e-011	4.29e-010	1.19e-010	2.26e-011

VI. CONCLUSION

In this paper, RKCeM formula has been adopted to solve the delay differential equations with constant lags. The effectiveness of this approach has been illustrated via examples of DDE with single delay and multiple delays. To interpolate the delay term, both linear interpolation and Lagrange interpolation have been considered here. The numerical outcomes also have been compared with RK method based on arithmetic mean. From the numerical results, it is observed that the RKCeM method is well suitable for solving DDEs. It also suggests that the best results can be obtained when we use Lagrange interpolation with suitable number of support points for getting fourth order convergence.

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