

# Quasi-Quartic Trigonometric Bézier Curves and Surfaces with Shape Parameters

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**ABSTRACT:** In this paper a new kind of quasi-quartic trigonometric polynomial base functions with two shape parameters  $\lambda$  and  $\mu$  over the space  $\Omega = \text{span} \{1, \sin t, \cos t, \sin^2 t, \cos^2 t, \sin^3 t, \cos^3 t\}$  is presented and the corresponding quasi-quartic trigonometric Bézier curves and surfaces are defined by the introduced base functions. Each curve segment is generated by five consecutive control points. The shape of the curve can be adjusted by altering the values of shape parameters while the control polygon is kept unchanged. These curves inherit most properties of the usual quartic Bézier curves in the polynomial space and they can be used as an efficient new model for geometric design in the fields of CAGD.

**KEYWORDS:** Trigonometric Bézier basis function, Trigonometric Bézier curve, Trigonometric Bézier surface, Shape parameter.

## I. INTRODUCTION

Bézier curves and surfaces are the basic tools for shape modelling in Computer Aided Geometric Designing(CAGD) and Computer Graphics(CG). However, they have many shortcomings due to polynomial forms. In particular, they cannot represent exactly transcendental curves such as the helix and the cycloid etc. To overcome the shortcomings, many bases are presented using trigonometric functions or the blending of polynomial and trigonometric functions in [1], [2], [3], [4], [5], [6] and [7]. Some of these existing methods have no shape parameters; hence the shape of the curves or surfaces cannot be modified when once their control points are determined. Many authors have studied different kinds of spline for curve and surface through incorporating shape parameters into the classical basis functions, where the parameters can adjust the shape of the curves and surfaces without changing the control points [8], [9] and [10].

Recently, many papers investigate the trigonometric Bézier-like polynomial, trigonometric spline and their applications. Nikolis et al. [11] discussed the applications of trigonometric spline in dynamic systems. Dyllong et al. [12] and Su et al.[13] modified a trajectory for robot manipulators using trigonometric spline. Neamtu et al. [14] presented some methods for designing the CAM (Computer Aided Manufacturing) profile with trigonometric spline in CNC (Computer Numerical Control). Many kinds of methods based on trigonometric polynomials for free form curves and surfaces modeling were also established in [15], [16], [17], [18], [19] and [20]. In recent years, several new trigonometric splines have been studied in the literature; see [21], [22], [23], [24], [25] and [26].

In this paper a new kind of quasi-quartic trigonometric Bézier Curve with two shape parameters is presented. The paper is organized as follows. In section II, quasi-quartic trigonometric Bézier basis functions with two shape parameters are established and the properties of the basis functions are shown. In section III, quasi-quartic trigonometric Bézier curves are given and some properties are discussed. By using shape parameter, shape control of the quasi-quartic trigonometric Bézier curves is studied. In section IV, the representation of quasi-quartic trigonometric Bézier surface has been shown. In section V, the approximability of the quasi-quartic trigonometric Bézier curves and the quartic Bézier curves with Bernstein basis corresponding to their control polygon are studied. Conclusion is presented in Section VI.

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## II. QUASI-QUARTIC TRIGONOMETRIC BÉZIER BASIS FUNCTIONS

Firstly, the definition of quasi-quartic trigonometric Bézier basis functions is given as follows.

### The construction of the basis functions

**Definition 1.** For an arbitrarily selected real values of  $\lambda$  and  $\mu$ , where  $\lambda, \mu \in [0, 1]$ , the following five functions of  $t$  ( $t \in [0, \frac{\pi}{2}]$ ) are defined as Quasi-Quartic Trigonometric Bézier basis functions with two shape parameters  $\lambda$  and  $\mu$ :

$$\begin{cases} b_0(t) = (1 - \sin t)^2(1 - \lambda \sin t) \\ b_1(t) = \frac{3}{2} \sin t(1 - \sin t)\{1 + \lambda(1 - \sin t)\} \\ b_2(t) = \frac{1}{2}[1 - (1 - \sin t)\{1 + \lambda \sin t(1 - \sin t)\} - (1 - \cos t)\{1 + \mu \cos t(1 - \cos t)\}] \\ b_3(t) = \frac{3}{2} \cos t(1 - \cos t)\{1 + \mu(1 - \cos t)\} \\ b_4(t) = (1 - \cos t)^2(1 - \mu \cos t) \end{cases} \quad (1)$$

### The properties of the basis functions

**Theorem 1** The basis functions (1) have the following properties:

(a) **Non-negativity:**  $b_i(t) \geq 0$  for  $i = 0, 1, 2, 3, 4$ .

(b) **Partition of unity:**  $\sum_{i=0}^4 b_i(t) = 1$

(c) **Symmetry:**  $b_i(t; \lambda, \mu) = b_{4-i}(\frac{\pi}{2} - t; \mu, \lambda)$ , for  $i = 0, 1, 2, 3, 4$ .

(d) **Monotonicity:** For a given parameter  $t$ , as the shape parameter  $\lambda$  increases (or decreases),  $b_0(t)$ ,  $b_2(t)$  decreases (or increases) and  $b_1(t)$  increases (or decreases). Similarly as the shape parameter  $\mu$  increases (or decreases),  $b_4(t)$ ,  $b_2(t)$  decreases (or increases) and  $b_3(t)$  increases (or decreases).

**Proof** (a) For  $t \in [0, \frac{\pi}{2}]$  and  $\lambda, \mu \in [0, 1]$ , then

$$0 \leq (1 - \sin t) \leq 1, 0 \leq (1 - \cos t) \leq 1, 0 \leq \sin t \leq 1, 0 \leq \cos t \leq 1$$

It is obvious that  $b_i(t) \geq 0$  for  $i = 0, 1, 2, 3, 4$ .

$$(b) \sum_{i=0}^4 b_i(t) = (1 - \sin t)^2(1 - \lambda \sin t) + \frac{3}{2} \sin t(1 - \sin t)\{1 + \lambda(1 - \sin t)\} + \frac{1}{2}[1 - (1 - \sin t)\{1 + \lambda \sin t(1 - \sin t)\} - (1 - \cos t)\{1 + \mu \cos t(1 - \cos t)\}] + \frac{3}{2} \cos t(1 - \cos t)\{1 + \mu(1 - \cos t)\} + (1 - \cos t)^2(1 - \mu \cos t) = 1$$

The remaining cases follow obviously. The curves of the Quasi-Quartic Trigonometric Bézier basis functions for  $\lambda = 0$ ,  $\mu = 0$  are shown in Figure 1 and for  $\lambda = 1$ ,  $\mu = 1$  are shown in Figure 2.

## III. QUASI-QUARTIC TRIGONOMETRIC BÉZIER CURVE

### The construction of the Quasi-Quartic Trigonometric Bézier curve

**Definition 2.** Given control points  $P_i$  ( $i=0,1,2,3,4$ ) in  $R^2$  or  $R^3$ , then

$$r(t) = \sum_{i=0}^4 b_i(t)P_i, \quad t \in [0, 1], \lambda, \mu \in [0, 1] \quad (2)$$

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is called a Quasi-Quartic Trigonometric Bézier curve with two shape parameters  $\lambda$  and  $\mu$ .

From the definition of the basis function, some properties of the Quasi-Quartic Trigonometric Bézier curve can be obtained as follows:

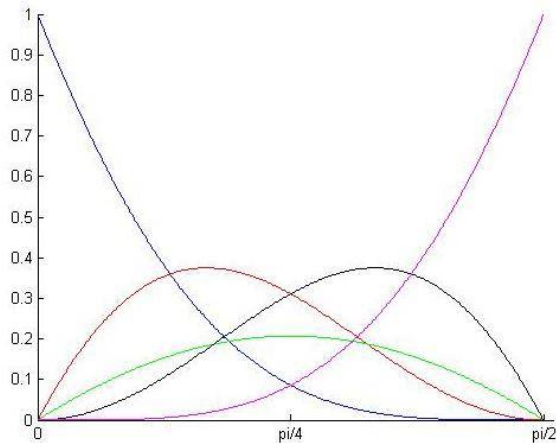


Fig. 1. Quasi-Quartic Trigonometric Bézier basis functions for  $\lambda = 0$  and  $\mu = 0$

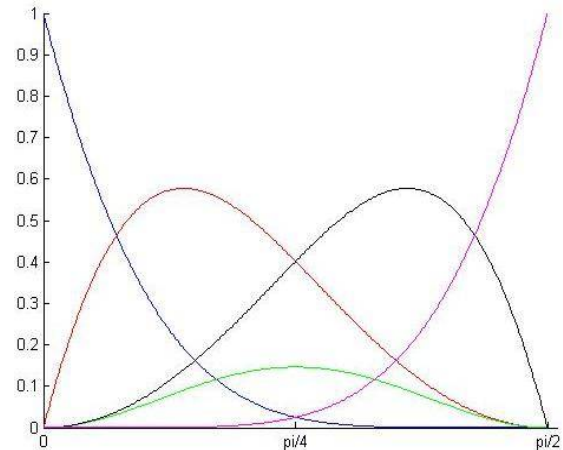


Fig. 2. Quasi-Quartic Trigonometric Bézier basis functions for  $\lambda = 1$  and  $\mu = 1$

**Theorem 2** The Quasi-Quartic Trigonometric Bézier curves (2) have the following properties:

(a) *Terminal Properties:* 
$$r(0) = P_0, \quad r\left(\frac{\pi}{2}\right) = P_4, \tag{3}$$

$$r'(0) = (2 + \lambda)(P_1 - P_0) + \left(\frac{1-\lambda}{2}\right)(P_2 - P_1),$$

$$r'\left(\frac{\pi}{2}\right) = \left(\frac{1-\mu}{2}\right)(P_3 - P_2) + (2 + \mu)(P_4 - P_3) \tag{4}$$

(b) *Symmetry:*  $P_0, P_1, P_2, P_3, P_4$  and  $P_4, P_3, P_2, P_1, P_0$  define the same Quasi-Quartic Trigonometric Bézier curve in different parametrizations, i.e.,

$$r\left(t; \lambda, \mu; P_0, P_1, P_2, P_3, P_4\right) = r\left(\frac{\pi}{2} - t; \mu, \lambda; P_4, P_3, P_2, P_1, P_0\right); \quad t \in \left[0, \frac{\pi}{2}\right], \lambda, \mu \in [0,1] \tag{5}$$

(c) *Geometric invariance:* The shape of a Quasi-Quartic Trigonometric Bézier curve is independent of the choice of coordinates, i.e. (2) satisfies the following two equations:

$$\begin{aligned} r(t; \lambda, \mu; P_0 + q, P_1 + q, P_2 + q, P_3 + q, P_4 + q) &= r(t; \lambda, \mu; P_0, P_1, P_2, P_3, P_4) + q \\ r(t; \lambda, \mu; P_0 * T, P_1 * T, P_2 * T, P_3 * T, P_4 * T) &= r(t; \lambda, \mu; P_0, P_1, P_2, P_3, P_4) * T \end{aligned} \tag{6}$$

$$t \in \left[0, \frac{\pi}{2}\right], \lambda, \mu \in [0,1]$$

where  $q$  is arbitrary vector in  $R^2$  or  $R^3$ , and  $T$  is an arbitrary  $d \times d$  matrix,  $d=2$  or  $3$ .

(d) *Convex hull property:* The entire Quasi-Quartic Trigonometric Bézier curve segment lies inside its control polygon spanned by  $P_0, P_1, P_2, P_3, P_4$ .

## Shape Control of the Quasi-Quartic Trigonometric Bézier Curve

For  $t \in \left[0, \frac{\pi}{2}\right]$ , we rewrite (2) as follows:

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$$r(t) = \sum_{i=0}^4 P_i c_i(t) + \frac{\lambda}{2} \sin t (1 - \sin t)^2 [2P_0 + 3P_1 - P_2] + \frac{\mu}{2} \cos t (1 - \cos t)^2 [2P_4 + 3P_3 - P_2] \quad (7)$$

where  $c_0(t) = (1 - \sin t)^2$ ,  $c_1(t) = \frac{3}{2} \sin t (1 - \sin t)$ ,  $c_2(t) = \frac{1}{2} (\sin t + \cos t - 1)$ ,  $c_3(t) = \frac{3}{2} \cos t (1 - \cos t)$ ,  
 $c_4(t) = (1 - \cos t)^2$ .

Obviously, shape parameter  $\lambda$  affects the curve on the control points  $P_0, P_1$  and  $P_2$  and  $\mu$  affects the curve on the control points  $P_2, P_3$  and  $P_4$ . The shape parameters  $\lambda$  and  $\mu$  serve to effect local control in the curves: as  $\lambda$  increases, the curve moves in the direction of control points  $P_0$  and  $P_1$  and as  $\lambda$  decreases, the curve moves in the opposite direction to the control points  $P_0$  and  $P_1$ . The same effect on the control points  $P_3$  and  $P_4$  are produced by the shape parameter  $\mu$ . As the shape parameter  $\lambda(=\mu)$  decreases or increases, the curve moves in the direction of point  $P_2$  or the opposite direction to the point  $P_2$ . Figure 3 shows one computed example of Quasi-Quartic Trigonometric Bézier with different values of shape parameters  $\lambda$  and  $\mu$ .

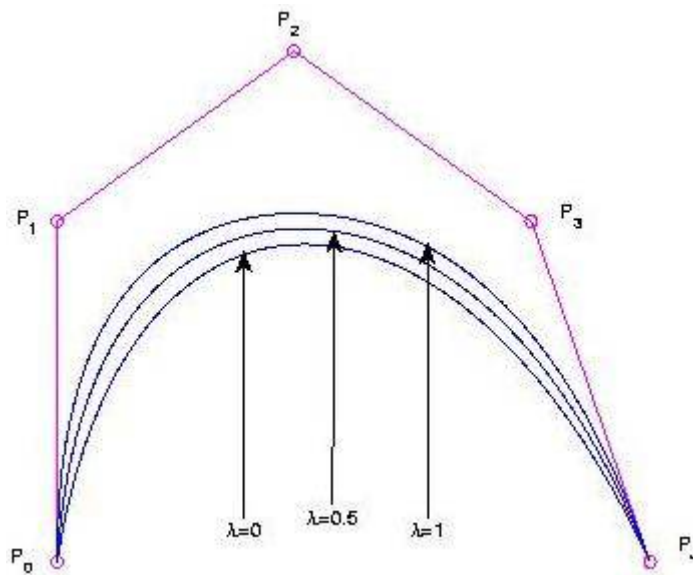


Fig. 3. Quasi-Quartic Trigonometric Bézier Curves with different values of shape parameters  $\lambda = \mu$

## IV. THE QUASI-QUARTIC TRIGONOMETRIC BÉZIER SURFACES

Given control points  $P_{rs}$  ( $r = i, \dots, i + 3; s = j, \dots, j + 3$ ), ( $i = 0, 1, \dots, n - 1; j = 0, 1, \dots, m - 1$ ) using the tensor product method, we can construct the Quasi-Quartic Trigonometric Bézier surface

$$T(u, v) = \sum_{r=i}^{i+3} \sum_{s=j}^{j+3} b_{r,4}(\lambda_1, \mu_1, u) b_{s,4}(\lambda_2, \mu_2, v) P_{rs}; \quad (u, v) \in [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$$

where  $b_{r,4}(\lambda_1, \mu_1, u)$  and  $b_{s,4}(\lambda_2, \mu_2, v)$  are quasi-quartic trigonometric polynomial base functions. Obviously these surfaces have properties similar to the corresponding quasi-quartic trigonometric Bézier curves.

## V. APPROXIMABILITY

Suppose  $P_0, P_1, P_2, P_3$  and  $P_4$  are not collinear; the relationship between Quasi-Quartic Trigonometric Bézier curve  $r(t)$  and the quartic Bézier curve  $B(t) = \sum_{i=0}^4 P_i \binom{4}{i} (1 - t)^{4-i} t^i$  with the same control points  $P_i$  is given by

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$$r\left(\frac{\pi}{4}\right) - P_2 = \frac{1}{2\sqrt{2}}(\sqrt{2}-1)^2\{(\sqrt{2}+\lambda)P_0 + (\sqrt{2}+\mu)P_4\} + \frac{3}{4\sqrt{2}}(\sqrt{2}-1)\{(\sqrt{2}+\lambda(\sqrt{2}-1))P_1 + \{(\sqrt{2}+\mu(\sqrt{2}-1))P_3\} - \frac{1}{2\sqrt{2}}[\sqrt{2} + (\sqrt{2}-1)\{2 + \lambda(\sqrt{2}-1)\} + (\sqrt{2}-1)\{2 + \mu(\sqrt{2}-1)\}]P_2$$

and

$$B\left(\frac{1}{2}\right) - P_2 = \frac{1}{16}[P_0 + 4P_1 - 10P_2 + 4P_3 + P_4] \quad (8)$$

These equations shows that quasi-quartic trigonometric Bézier curve can be made closer to the control polygon by altering the values of shape parameters  $\lambda$  and  $\mu$ .

### VI. CONCLUSION

As mentioned above Quasi-Quartic Trigonometric Bézier curve have all the properties that quartic Bézier curves have. Since there is nearly no difference in structure between a Quasi-Quartic Trigonometric Bézier curve and a quartic Bézier curve, it is not difficult to adapt a Quasi-Quartic Trigonometric Bézier curve to a CAD/CAM system that already uses the Bernstein form of quartic Bézier curves.

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