

Vertex Domination Number in S-valued Graphs

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ABSTRACT:In [5], we have introduced the notion of Vertex Domination on S- valued graphs, in which we have defined the terms weight dominating vertex set and the weight dominating vertex number $\gamma_V^S(G^S)$ of a S – valued graph G^S . In this paper, we study some results on the bounds of the weight dominating vertex number, $\gamma_V^S(G^S)$.

KEYWORDS:Semirings, Graphs, Domination number, S - valued graphs, Vertex domination on S – valued graphs.

I. INTRODUCTION

The theory of domination in graphs has been introduced by Berge [1], and continued by several authors. Haynes and Hedetniemi published a survey paper [2], that discussed the available results on domination in the year 1977. In [4], Jonathan Golan introduced the notion of semiring valued graphs. In [8], we studied the S- valued graphs in detail. In [5], we introduced the notion of weight dominating vertex set and weight dominating vertex number in the S-valued graph G^S . In this paper, we study some results on the Bounds of the weight dominating vertex number of S- valued graphs.

II. RELATED WORK

In crisp graph theory and fuzzy graph theory, bounds for vertex domination numbers have been obtained by several authors ([5], [9], [11]). In this, paper we obtain the bound for vertex domination number for S-valued graphs. The S-valued graph theory developed in [4] generalizes the above two theories of crisp graphs and fuzzy graphs. Thus, the bounds we obtain for vertex domination number will be a generalization of the bounds obtained for crisp as well as fuzzy graph theory available in the literature.

III. PRELIMINARIES

In this section, we recall some basic definitions that are needed for our work.

Definition 2.1 :[4] A semiring $(S, +, \cdot)$ is an algebraic system with a non-empty set S together with two binary operations + and \cdot such that

- (1) $(S, +, 0)$ is a monoid.
- (2) (S, \cdot) is a semigroup.
- (3) for all $a, b, c \in S$, $a \cdot (b+c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$.
- (4) $0 \cdot x = x \cdot 0 = 0 \forall x \in S$.

Definition 2.2 : Let $(S, +, \cdot)$ be a semiring. A canonical Pre-order \preceq in S is defined as follows : for $a, b \in S$, $a \preceq b$ if and only if, there exist $a', c \in S$ such that $a + c = b$.

International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Vol. 5, Issue 12, December 2016

Definition 2.3 : [8] Let $G = (V, E \subset V \times V)$ be a given graph with $V, E \neq \emptyset$. For any semiring $(S, +, \cdot)$, a semiring-valued graph (or a S -valued graph), G^S , is defined to be the graph $G^S = (V, E, \sigma, \psi)$ where $\sigma : V \rightarrow S$ and $\psi : E \rightarrow S$ is defined to be

$$\psi(x, y) = \begin{cases} \min\{\sigma(x), \sigma(y)\} & \text{if } \sigma(x) \lesssim \sigma(y) \text{ or } \sigma(y) \lesssim \sigma(x) \\ 0 & \text{otherwise} \end{cases}$$

for every unordered pair (x, y) of $E \subset V \times V$. We call σ , a S -vertex set and ψ , a S edge set of G^S .

Definition 2.4: A vertex $v \in D$ of G^S is said to be an S -isolate vertex if $N_S(v) \subseteq V - D$.

Definition 2.5 : [7] The Degree of the vertex v_i of the S -valued graph G^S is defined as $deg_S(v_i) = (\sum_{(v_i, v_j) \in E} \psi(v_i, v_j), \ell)$ where ℓ is the number of edges incident with v_i .

Definition 2.6 : [5] A vertex v in G^S is said to be a weight dominating vertex if $\sigma(u) \lesssim \sigma(v), \forall u \in N_S(v)$. A subset $D \subseteq V$ is said to be a weight dominating vertex set of G^S if

- (1) D is a dominating set in G .
- (2) for each $v \in D, \sigma(u) \lesssim \sigma(v), \forall u \in N_S(v)$.

Definition 2.7 : [5] If D is a weight dominating vertex set of G^S , then the scalar cardinality of D , denoted by $|D|_S$, is defined by $|D|_S = \sum_{v \in D} \sigma(v)$.

Definition 2.8 : [5] The cardinality of the minimal weight dominating vertex set $D \subseteq V$ is called the weight dominating vertex number of G^S . It is denoted by $\gamma_V^S(G^S)$. That is $\gamma_V^S(G^S) = (|D|_S, |D|)$.

Theorem 2.9 . [5] A weight dominating vertex set D of a graph G^S is a minimal weight dominating vertex set of G^S iff every vertex $v \in D$ satisfies atleast one of the following properties:

- (1) v is adjacent to no vertex of D .
- (2) there exist a vertex $u \in V - D$ such that $N_S(u) \cap (D \times S) = \{(v, \sigma(v))\}$.

IV. BOUNDS OF DOMINATION NUMBER

In this section, we obtain some results on weight dominating vertex number in G^S .

Definition 3.1 : Consider the S -valued graph $G^S = (V, E, \sigma, \psi)$. Let $u \in V$ be a vertex of G^S whose degree in the crisp graph G of G^S is equal to $\delta(G)$. That is $\delta(G) = \deg(u)$. Let $w \in V$ be a vertex of G^S whose degree in the crisp graph G of G^S is equal to $\Delta(G)$. That is $\Delta(G) = \deg(w)$. The Minimum Degree and the Maximum Degree of the S -valued graph G^S is defined as

$$\text{Mindeg}_S(u) = \left(\sum_{v \in N_S(u)} \psi(uv), \delta(G) \right) = \delta_S(G^S)$$

and

$$\text{maxdeg}_S(w) = \left(\sum_{v \in N_S(w)} \psi(wv), \Delta(G) \right) = \Delta_S(G^S).$$

International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Vol. 5, Issue 12, December 2016

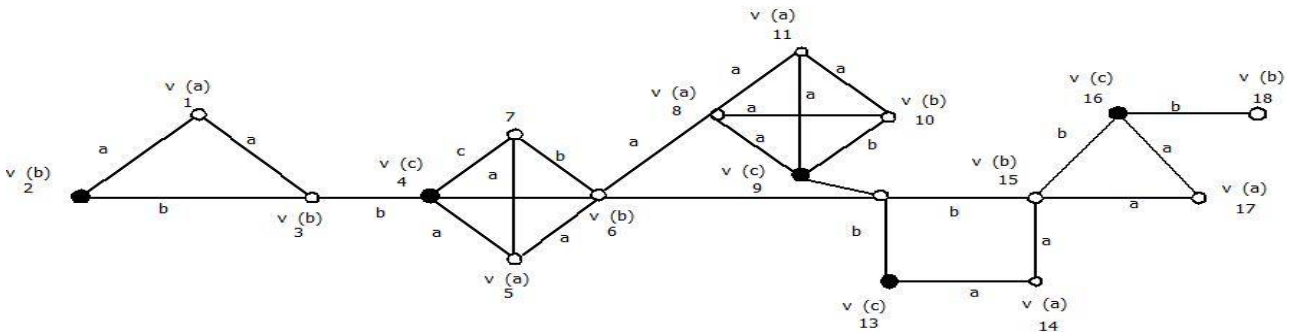
Example 3.2 : Let $(S = \{0, a, b, c\}, +, \cdot)$ be a semiring with the following cayley tables :

+	0	a	b	c
0	0	a	b	c
a	a	b	c	c
b	b	c	c	c
c	c	c	c	c

·	0	a	b	c
0	0	0	0	0
A	0	a	b	c
B	0	b	c	c
C	0	c	c	c

Let \preceq be a canonical pre-order in S, given by
 $0 \preceq 0, 0 \preceq a, 0 \preceq b, 0 \preceq c, a \preceq a, a \preceq b, a \preceq c, b \preceq b, b \preceq c, c \preceq c.$

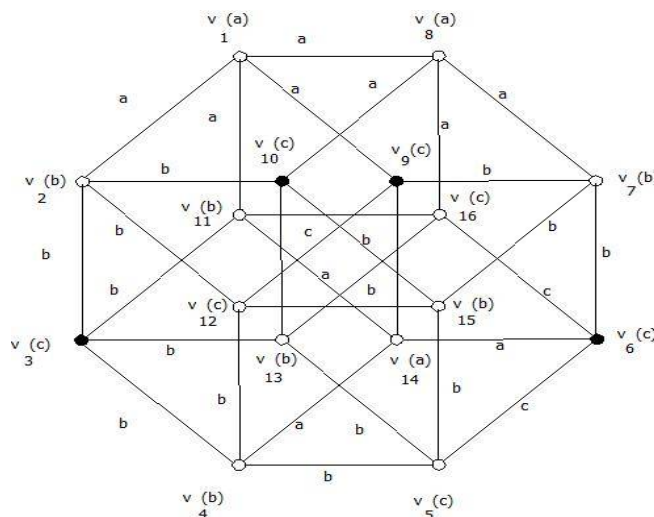
Consider the S- valued graph G^S ,



Here $D = \{v_2, v_4, v_9, v_{13}, v_{16}\}$ is a weight dominating vertex set of G^S .

$\delta_S(G^S) = (b, 1) = deg_S(v_{18})$; $\Delta_S(G^S) = (c, 5) = deg_S(v_6)$ and $\gamma_V^S(G^S) = (c, 5).$

Example 3.3 : Consider the semiring with the canonical pre-order \preceq given in Example 3.2. Consider the S-valued graph G^S given by :



International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Vol. 5, Issue 12, December 2016

Here $D = \{v_3, v_6, v_9, v_{10}\}$ is a weight dominating vertex set of G^S . For G^S , we have $\delta_S(G^S) = (c, 4) = \text{deg}_S(v_1)$; $\Delta_S(G^S) = (c, 4) = \text{deg}_S(v_4)$ and $\gamma_V^S(G^S) = (c, 4)$.

Lemma 3.4 :

Let $G^S = (V, E, \sigma, \psi)$ be a given S -valued graph, where $|V| = m$ and $|E| = n$. Then $(0, 1) \preceq \gamma_V^S(G^S) \preceq (\sum_{v \in V} \sigma(v), m)$.

Proof : Since S is a semiring, it has the zero element '0' and $0 \preceq a$ for every $a \in S$.

If D is a weight dominating vertex set of G^S , then D should have at least one vertex, which will be assigned at least the weight '0'.

Hence $(0, 1) \preceq \gamma_V^S(G^S)$.

If $|V| = m$ and $|E| = 0$, that is $E = \emptyset$, then V itself will be a minimal weight dominating vertex set.

Hence, $\gamma_V^S(G^S) \preceq (\sum_{v \in V} \sigma(v), m)$.

Thus $(0, 1) \preceq \gamma_V^S(G^S) \preceq (\sum_{v \in V} \sigma(v), m)$.

The second condition of theorem 2.9 gives the analogous of Ore's theorem for the graph G^S .

Let G^S be a given S -valued graph with $|V| = m$ and $|E| = n$. If D is a minimal weight dominating vertex set, then $V-D$ is also a weight dominating vertex set. Thus $|D|$ or $|V-D|$ is at most $\frac{m}{2}$.

Case (i) $|D| \leq \frac{m}{2}$

Then, $\sum_{v \in D} \sigma(v) \preceq \sum_{v \in V} \sigma(v) \Rightarrow \gamma_V^S(G^S) \preceq (\sum_{v \in V} \sigma(v), |V - D|)$.

Case (ii) $|V-D| \leq \frac{m}{2}$.

Then, $\sum_{v \in V-D} \sigma(v) \preceq \sum_{v \in V} \sigma(v) \Rightarrow \gamma_V^S(G^S) \preceq (\sum_{v \in V-D} \sigma(v), |D|)$.

Thus we have proved the following analogous of Ore's theorem for the S -valued graph G^S .

Theorem :3.5 Let G^S be a given S -valued graph with $|V| = m$ and $|E| = n$.

$$\text{Then } \gamma_V^S(G^S) \preceq \begin{cases} (\sum_{v \in D} \sigma(v), \frac{m}{2}) & \text{if } |D| \leq |V - D| \\ (\sum_{v \in V-D} \sigma(v), \frac{m}{2}) & \text{if } |V - D| \leq |D| \end{cases}$$

Theorem :3.6 Let G^S be a given S -valued graph and D be a weight dominating vertex set. Let $|V| = m$

(1) If G^S has no S -isolated vertices $\gamma_V^S(G^S) + \gamma_V^S(\overline{G^S}) \preceq (\sum_{v \in V} \sigma(v), m)$.

(2) If G^S has an S -isolated vertex, $\gamma_V^S(G^S) + \gamma_V^S(\overline{G^S}) \preceq (\sum_{v \in V} \sigma(v), m + 1)$.

Proof : If G^S has no S -isolated vertices then

$$\gamma_V^S(G^S) \preceq \begin{cases} (\sum_{v \in D} \sigma(v), \frac{m}{2}) & \text{if } |D| \leq |V - D| \\ (\sum_{v \in V-D} \sigma(v), \frac{m}{2}) & \text{if } |V - D| \leq |D| \end{cases}$$

If $\overline{G^S}$ has no S -isolated vertices then

$$\gamma_V^S(\overline{G^S}) \preceq \begin{cases} (\sum_{v \in D} \sigma(v), \frac{m}{2}) & \text{if } |D| \leq |V - D| \\ (\sum_{v \in V-D} \sigma(v), \frac{m}{2}) & \text{if } |V - D| \leq |D| \end{cases}$$

$\therefore \gamma_V^S(G^S) + \gamma_V^S(\overline{G^S}) \preceq (\sum_{v \in V} \sigma(v), m)$, thus proving 1.

If G^S has an S -isolated vertex and $\overline{G^S}$ has no S -isolated vertices then

$$\gamma_V^S(G^S) \preceq (\sum_{v \in V} \sigma(v), m) \text{ and } \gamma_V^S(\overline{G^S}) = (0, 1).$$

International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Vol. 5, Issue 12, December 2016

Then we have $\gamma_V^S(G^S) + \gamma_V^S(\overline{G^S}) \lesssim (\sum_{v \in V} \sigma(v), m + 1)$.

If $\overline{G^S}$ has an S-isolated vertex and G^S has no S-isolated vertices then

$\gamma_V^S(\overline{G^S}) \lesssim (\sum_{v \in V} \sigma(v), m)$ and $\gamma_V^S(G^S) = (0, 1)$

$\Rightarrow \gamma_V^S(G^S) + \gamma_V^S(\overline{G^S}) \lesssim (\sum_{v \in V} \sigma(v), m + 1)$, thus proving 2.

Theorem 3.7 :

Let G^S be a given S-valued graph and D be a weight dominating vertex set and $|V| = m$. Then

$\gamma_V^S(G^S) \cdot \gamma_V^S(\overline{G^S}) \lesssim (\sum_{v \in V} \sigma(v), m)$.

Proof :

For any subset X of $V(G^S)$, define the following sets

$D_0(X) = \{(u, \sigma(u)) / u \in V(G^S) - X \text{ and } \sigma(u), \psi(u, v) \in S \text{ for all } v \in X\}$

and $D_1(X) = \{(u, \sigma(u)) / u \in X \text{ and } \sigma(u), \psi(u, v) \in S \text{ for all } v \in X\}$

Let $V(G^S)$ be partitioned into subsets $V_1, V_2, \dots, V_{\gamma(G^S)}$, such that $(v_i, \sigma(v_i)) \in V_i \times S$

then $(v, \sigma(v)) = (v_i, \sigma(v_i))$ or $\{(v, \sigma(v)), (v_i, \sigma(v_i))\} \in E(G^S) \Rightarrow \psi(v, v_i) \in S$.

Choose this partition in such a way that $(\sum_{v_i \in V_i} \sigma(v_i), \sum_{i=1}^n |D_1(v_i)|_S)$ is a maximum.

Suppose $|D_0(v_j)|_S \geq (0, 1)$ for some j.

Then there is a vertex $(v, \sigma(v)) \in V_k \times S$ for $k \neq j$ such that

$\{(u, \sigma(u)), (v, \sigma(v))\} \in E(G^S) \times S$ for all $(u, \sigma(u)) \in V_j \times S$.

If $(v, \sigma(v)) \in D_0(v_k)$ then $(D - \{(v_j, \sigma(v_j)), (v_k, \sigma(v_k))\}) \cup (v, \sigma(v))$ is a weight dominating vertex set of G^S with cardinality smaller than $\gamma_V^S(G^S)$.

This is a contradiction and hence $(v, \sigma(v)) \in D_0(v_k)$.

We re-partition the vertices of G^S , so that $(v'_i, \sigma(v'_i)) = (v_i, \sigma(v_i))$ for $i \neq j, k$.

$(v'_j, \sigma(v'_j)) = (v_j, \sigma(v_j)) \cup \{(v, \sigma(v))\}$ and $(v'_k, \sigma(v'_k)) = (v_k, \sigma(v_k)) - \{(v, \sigma(v))\}$.

But then $|D_1(v'_j)|_S = |D_1(v_j)|_S$.

$$(3.1) \quad |D_1(v'_j)|_S = |D_1(v_j)|_S + (0, 1) \geq |D_1(v_j)|_S.$$

Similarly, we get $|D_1(v'_k)|_S \geq |D_1(v_k)|_S$ (by using 3.1), contradicting the choice of our original partition of G^S .

Hence we conclude that

$$|D_0(v_i)|_S = 0 \quad \forall i = 1, 2, \dots, \gamma(G^S).$$

As any set X with $|D_0(X)|_S = (0, 1)$ dominates $\overline{G^S}$, each set $(v_i, \sigma(v_i))$ dominates $\overline{G^S}$ and so $\gamma_V^S(G^S) \leq |v_i, \sigma(v_i)|_S \Rightarrow \gamma_V^S(G^S) \cdot \gamma_V^S(\overline{G^S}) \lesssim (\sum_{v \in V} \sigma(v), m)$.

Theorem 3.8 :

Consider the S-valued graph $G^S = (V, E, \sigma, \psi)$. Then $\delta_S(G^S) \lesssim \gamma_V^S(G^S) \lesssim \Delta_S(G^S)$.

International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Vol. 5, Issue 12, December 2016

Proof :

Let D be minimal weight dominating vertex set of G^S , $u \in V$ be such that $\delta(G) = \deg(u)$ and $w \in V$ be such that $\Delta(G) = \deg(w)$ in the crisp graph G of G^S .

Let $u \in V$ be a pendent vertex.

$$\therefore N_S[u] = \{ u, v / v \neq u \}$$

If $\sigma(u) = 0$ and $\sigma(v) = 0 \Rightarrow (uv) = 0$.

$$\therefore deg_S(u) = \{ (\sum_{v \in N_S[u]} \psi(uv), 1) \} = (0, 1) = \delta(G^S).$$

If $\sigma(u) = 0$ and $\sigma(v) \neq 0 \Rightarrow (uv) = \min_{v \in N_S[u]} \{ \sigma(u), \sigma(v) \} = 0$.

In this case also, $deg_S(u) = (0, 1)$.

If $\sigma(u) = \sigma(v) \neq 0 \Rightarrow (uv) = a$ for some $a \in S$.

$deg_S(u) = (a, 1) \preceq deg_S(v)$ even if v is a pendent vertex.

Let $v \in V$ be such that $\deg(u) = \delta(G)$, the minimum degree of G . By our definition,

$(a, m) \preceq (b, n) \Leftrightarrow a \preceq b$ and $m \preceq n$ for $a, b \in S$ and $m, n \in N$.

$deg_S(u) = \{ (\sum_{v \in N_S[u]} \psi(uv), \ell) \}$ where ℓ is the number of edges incident with u .

Since $\delta(G) = \deg(u)$, $\ell = \delta(G)$, Whatever be the value of $\sum_{v \in N_S[u]} \psi(uv)$ we get $deg_S(u) \preceq deg_S(v)$, $v \in N_S[u]$.

$$\Rightarrow \delta_S(G^S) = deg_S(u) = \{ (\sum_{v \in N_S[u]} \psi(uv), \delta(G)) \}.$$

Let D be a minimal weight dominating vertex set of G^S , $|D|_S = \sum_{v \in D} \sigma(v)$.

If $u \in D$, $|D|_S \geq \sum_{v \in N_S[u]} \psi(uv)$

If $u \notin D$, $w \in N_S[u]$ such that $\sigma(w) \geq \sigma(u) \Rightarrow |D|_S \geq \sum_{v \in N_S[u]} \psi(uv)$ and $|D| \geq \delta(G)$

Thus,

$$\gamma_V^S(G^S) = (|D|_S, |D|) \geq \{ (\sum_{v \in N_S[u]} \psi(uv), \delta(G)) \}$$

That is, $\delta_S(G^S) \preceq \gamma_V^S(G^S)$.

Any vertex in G^S can dominate atmost $\Delta(G)$ vertices where $\Delta(G)$ is the maximum degree in the crisp graph.

Hence

$$\gamma_V^S(G^S) = (|D|_S, |D|) \preceq (|D|_S, \Delta(G)) = \Delta_S(G^S).$$

Where D is a minimal weight dominating vertex set of G^S .

Thus $\delta_S(G^S) \preceq \gamma_V^S(G^S) \preceq \Delta_S(G^S)$.

V.CONCLUSION

In this paper we have obtained by Bounds of vertex domination numbers for S-valued graphs. In our future work we are going to obtain the same for connected domination number and the Bounds for it.

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