

Some Properties of Sub Semigroups of an M-Semigroup

L.Lakshmanan

Dean (Mathematical Sciences), Department of Mathematics, PSGR Krishnammalcollege for Women, Coimbatore, India

ABSTRACT: In this paper, we discuss the various properties of sub semi groups of an M-semigroup. It is clear that every M-semigroup contains at least one sub semigroup and at least one sub-M-semigroup .By discussing the properties of an M-semigroup through its sub semigroup or its sub-M-semigroup we get some interesting results.

KEYWORDS: Semi group, M- semi group, sub M-semi group, semi deflation, Fine Band.

I. INTRODUCTION

Tamura et al proved that an M – Groupoid S is the direct product of a right singular semigroup and a groupoid with a two sides identity. By an M-semigroup we mean a groupoid M in which

P₁: There is atleast one $e \in M$ such that $ex=x$ for all $x \in M$.

P₂: $x(yz)=(xy)z$ for all $x,y,z \in M$ (Assosiative law)

P₃: For each $x \in M$, there is unique left identity e such that $xe=x$.

In this paper we discuss some properties of M-semigroup through its sub M- semi group.

II. PRELIMINARIES

2.1 A binary operation on a non-empty set s is a mapping of $S \times S$ into S, where $S \times S$ is defined as $\{(x, y)/x, y \in S\}$; The image of (x, y) under this mapping is denoted by $x.y$ (or by xy) and is called the product of x and y; $x,y \in S$.

2.2 A semigroup S is a system consisting of s nonempty set S, together with an associative binary operation on S, so that

$$x(yz) = (xy)z \text{ for all } x,y,z \text{ in } S.$$

If the operation is denoted by ‘.’ Then the semigroup is denoted by (S,.) or sometimes by just S, when the operation is understood.

2.3 By a total semigroup S is meant one in which

$$S = S^2 = \{xy/x,y \in S\}.$$

2.4 By an idempotent element of a semigroup S, is meant an element x in S, such that $x = x^2$; a semigroup S is said to be an idempotent semigroup if every element of S is idempotent.

2.5 A band is an idempotent semigroup.

2.6 By an M - groupoid S we mean a groupoid which satisfies the following conditions:

(i) There is atleast one $e \in S$.

(ii) If y or z is a left identity of S then $x(yz) = (xy)z$ for all $x \in S$.

(iii) For any $x \in S$, there is a unique left identity e such that $xe = x$.

2.7 By an M – semigroup M we mean a semigroup that satisfies the following two conditions:

P₁ : There is at least one $e \in M$ such that $ex =x$ for all $x \in M$.

P₂ : For every $x \in M$ there is a unique left identity e such that $xe = x$.

2.8 An M – semigroup M is isomorphic to the direct product of a right zero semigroup and a semigroup with a two sided identity; and conversely.

International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Vol. 5, Issue 6, June 2016

That is $M \cong R \times S$ where R is a right zero semi – group and S is a semigroup with a two sided identity. The right zero semigroup R is the set of all left identities of M and $S \cong Me$, for a left identity e of M .

2.9 By a sub – M – semigroup M' of a semigroup S we mean a subsemigroup M' of S which is also an M – semigroup.

2.10 By an induced sub – M – semigroup M' of an M – semigroup M , we mean a sub – M – semigroup M' of M whose set of left identities is same as that of M .

2.11 By a normal sub – M – semigroup K of an M – semigroup we mean that K is a sub – M – semigroup of M satisfies:

- (i) $ka \subseteq ak$; for all a belong to M , and
- (ii) Every left identity of K is a left identity of M .

2.12 A band is an idempotent semigroup.

2.13 Let $S = \bigcup_{i \in A} S_i$ be a decomposition of a semigroup S_i into subsemigroups over an index A . If for each (i,j) in $A \times A$, there exists an element k in A such that $S_i S_j \subseteq S_k^2$ then A becomes thereby a band. Then S is called as the union of the fine band A of semigroups $S_i (i \in A)$ or S is a fine band A of semigroups S_i .

III. PROPERTIES OF AN M-SEMIGROUP

M-Semigroup and Sub-Semigroup Of a Semigroup

3.1 Lemma:

Every semi-deflation of an M -semigroup to its sub semigroup is an M -semigroup.

Proof :

Let $M \equiv R \times S$ be an M -semigroup such that M is semi-deflatable to M' , a sub semigroup of M . If φ is the semi-deflation map then there is a homomorphism, $\varphi : M \rightarrow M'$ such that $x\varphi = x$ for all $x \in M'$.

Let $e \in R$. Obviously any element x of M' is also an element of M . Therefore $e.x = x$ for all $x \in M'$. That is, $e\varphi$ is a left identities of M' . Hence M' contains left identities.

For any $x \in M'$, x being an element of M , there exists a unique e such that $x.e = x$

Now, $(X.e)\varphi = x\varphi .e\varphi = x.e\varphi$ and $(X.e)\varphi = x.\varphi = x$

Therefore, $x = x.(e\varphi)$ and hence for each $x \in M'$ there exists a unique $e\varphi$ such that

$x.e\varphi = x$. This proves that M' is an M -semigroup and hence the lemma.

3.2 Lemma:

In a M -semigroup $M \equiv R \times S$, R is a maxima rectangular band.

Proof:

If R' is a rectangular band such that $R' \supset R$, let $x \in R' \setminus R$. Then X being an element of M , there exists an $e \in R'$ such that $xe = x$.

Now, $e.xe.e = e$ since $e, xe = x \in R'$ and R' is a rectangular band.

Also $e.xe.e = xe = x$. Therefore $x = e \in R$.

This is, $R' \subset R$ that is $R' = R$. Therefore R is a maximal rectangular band.

3.3 Lemma :

In an M -semigroup $M \equiv R \times S$, we can always define a congruence relation.

Proof:

Define a relation R^* on M such that for any two a, b of M , $aR^* b$ if and only if $ae = be$ for all $e \in R$. Clearly R^* is an equivalence relation. Let $aR^* b$, and let $c = cf$ be any element of M . Since

International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Vol. 5, Issue 6, June 2016

$aR^*b, ae = be$ for all $e \in R$.

i.e., $ae.cf = be.cf$.

i.e., $acf = bef$.

i.e., $acfe = befe$; for any $e \in R$

i.e., $ace = bce$; for all $e \in R$.

i.e., acR^*bc .

i.e., R^* is right compatible.

Let aR^*b and let $c = cf$ be any element of M .

$aR^*b \Rightarrow ae = be$; for all $e \in R$

$\Rightarrow cae = cbe$; for all $e \in R$

$\Rightarrow (ca)e = (cb)e$; for all $e \in R$

$\Rightarrow caR^*cb$

i.e., R^* is an left compatible.

Therefore R^* is a congruence relation on M . Hence the lemma.

Sub-M-semigroup of any semigroup, in particular that of an M-semigroup, plays an important role in the study of the semigroup. The following lemmas are the characterizations for a semigroup S to have a sub-M-semigroup.

3.4 Lemma:

The necessary and sufficient condition for a semigroup S to have a sub-M-semigroup is that it contains a right zero semigroup.

Proof:

Let S contain a sub-M-semigroup M . The set of left identities of M forms a right zero semigroup and hence S contains a right zero semigroup.

Conversely, if S contains a right zero semigroup R , then R itself is a sub-M-semigroup of S .

3.5 Lemma:

The necessary and sufficient condition for a semigroup S to have a sub-M-semigroup is that S contains an idempotent element.

Proof:

If S contains an idempotent element 'a' then $\{a\}$ is an M-semigroup and hence S contains a sub-M-semigroup M then every left identity of M is idempotent element of S . This proves the lemma.

3.6 Lemma:

A semigroup S with R as its set of left identities will have a sub-M-semigroup M which consists of the elements of S that commute with any one of $e \in R$, if and only if S satisfies the condition that any $x \in S$ can commute with atmost one $e \in R$.

Proof:

Let S contains a sub-M-semigroup M , consists of the elements of S which commute with any one of $e \in R$. Since e commutes with itself for all $e \in R$, $R \subseteq M$, and R is the set of left identities of M . M being an M-semi-group $Me \cap cMF = \phi$ for all $e, f \in R$. Let any

$x \in S$, commute with two left identities e and f of R .

Therefore, $xe = ex = x$ and $xf = fx = x$. That is, $x \in Me \cap Mf$, which is a contradiction.

Therefore, x can commute with atmost one $e \in R$.

Conversely, let S satisfied the condition that any $x \in S$ can commute with atmost one $e \in R$. Denote the set of element of S which commute with an e belong to R as A_e . since $e \in A_e$, A_e is not empty for all $e \in R$. For any x, y belong to A_e ,

$xye = xey = exy$.

Therefore xy is an element of A_e and hence A_e is a subsemigroup.

Consider $\bigcup_{e \in R} A_e = M$

International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Vol. 5, Issue 6, June 2016

For any $x \in Ae$ and $y \in Af$, $e, f \in R$,

$$(xy)f = x(fy) = (xf)y = fxy = xy.$$

Therefore $xy \in M$, which proves that M is a subsemigroup of S .

Since $e.xf = xf$, e is a left identity of M .

Similarly, each element of R is a left identity of M . Obviously, by the condition on S , for each x belong to M , there exist a unique e belong to R such that

$$x.e = e.x = x.$$

Therefore M is a sub-M-semigroup of S consists of the element of S which commute with any one of e belongs to R . This proves the lemma.

The following theorem gives a necessary condition for a semigroup S to have a sub-M-semigroup.

3.7 Theorem:

If a semigroup S satisfied the condition that for any $x \in S$, $xe \neq xf$; $e, f \in R$, where R is the set of left identities of S , then S contains an M -semi-group, $M = \bigcup_{e \in R} Se$, With R as its set of left identities.

Proof:

Consider $M = \bigcup_{e \in R} Se$

$Se \cap Sf = \emptyset$ because $xe \neq xf$; for all $x \in S$ and $e, f \in R$

Clearly $R \subset M$

For any $xe, yf \in M$, $xe.yf = xyf \in Sf \subset M$, proves that M is a subsemigroup.

If M contains a left identity $xf \in M \setminus R$, then xf being an element of $M = \bigcup_{e \in R} Se$

$$Xf.f = x(ff) = xf \text{ and } xf.f = f$$

i.e., $f = xf \in R$, Therefore, R is the set of left identity of M

For any $xe \in M$, $xe.e = xe = e.xe$

And if $xe.f = f(xe)$ then $xf = xe$, which is not so. Therefore for any $xe \in M$, there exists a unique $e \in R$ such that $Xe.e = e.xe = xe$.

This proves that M is an M - semigroup, with R as its set of left identities.

Hence the theorem.

The following lemma, gives a condition on a semigroup S to be a semi-inflation of its sub-M-semigroup.

3.8 Lemma:

Me is a sub semigroup of M .

3.9 Theorem:

An M -semigroup M is a disjoint union of its sub-M-semigroups which are left ideals of M containing exactly one identity and each such sub-M-semigroup is a semi deflation of M .

3.10 Lemma:

If a semigroup S with R as its set of left identities satisfies the conditions,

(i) For any $x \in S$, $Xe \neq xf$; $e, f \in R$, $e \neq f$ and

(ii). $xy.f = f.xy$ implies $yf = fy$ for all $f \in R$ then S is a semi-inflation of its sub-M-semigroup $M = \bigcup_{e \in R} Se$

Proof:

From 3.7 $M = \bigcup_{e \in R} Se$ is a sub-M-semigroup of S .

Define $\varphi: S \rightarrow M$ such that $x\varphi = x$; for all $x \in M$ and $a\varphi = ae$, for a fixed $e \in R$; for all $a \in S \setminus M$. For any two $xe, yf \in M$, $(xe.yf)\varphi = (xyf)\varphi = xyf$.

$$xe\varphi.yf\varphi = xe.yf = xyf.$$

For any $xf \in M$ and $a \in S \setminus M$; $xf.a \notin M$ because if $xf.a \in M$, M being an M -semigroup there exists a unique $g \in R$ such that $xf.a = xfa.g$

$$\text{i.e., } g[(xf)(a)] = [(xf)(a)]g$$

International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Vol. 5, Issue 6, June 2016

i.e., $ag = gaby(ii)$

i.e., $a = ag \in m$ a contradiction .

therefore, $xf \in S \setminus M$.

$$(xf)a\varphi = xfae = xae$$

and $(xf)\varphi a\varphi = xf.ae = xae$.

For any $a \in S \setminus M$ and $xf \in M$, $axf \in M$, because axf commute with f .

Therefore

$$(axf)\varphi = axf \text{ and } a\varphi.(xf)\varphi = ae.xf = axf.$$

For any two $a, b \in S \setminus M$, $ab \notin M$ because if $ab \in M$, then there exists a $g \in R$ such that

$$abg = ab = g.ab$$

i.e., $gb = bg$ by(ii).

i.e., $b = bg \in M$, a contradiction.

Therefore, $ab \in S \setminus M$,

$$(ab)\varphi = abe \text{ and } a\varphi.b\varphi = ae.be = abe.$$

Therefore φ is a homomorphism fixing the elements of M . Hence S is a semi-inflation of M .

From 3.8 and 3.9, M is a fine band R of sub- M -semigroups. Combining this with 3.10, we have the following theorem.

3.11 Theorem:

A semigroup S which satisfies the conditions (i) and (ii) of 3.10 is a semi-inflation of its sub- m -semigroup which is a fine band R of sub- M -semigroups.

The following theorem gives a structure of a semigroup S to be an inflation of its set of factorizable elements which is a sub- M -semigroup of S .

3.12 Theorem:

Any semigroup S which satisfies the following four conditions:

- i. S is a disjoint union of its subsets U_i ($i \in \Lambda$) where (a) and U_i is a subsemigroup satisfies

$$U_i^2 = \{\text{singleton}\}$$

$$\text{And (b) } S^2 = \bigcup_{i \in \Lambda} U_i^2$$

- ii. There exists at least one U_i ($i \in \Lambda$) such that each $x \in U_i$ implies $xy \in U_j$; where $y \in U_j$.
- iii. For each $x \in U_j$; there exists a unique U_i satisfies (ii) such that $x(U_i^2) = x$ and
- iv. If $xy \in U_k$ for any $x \in U_i, y \in U_j$ then $xy \in U_k$; for all $x \in U_i, y \in U_j$
is an inflation of its set of factorizable elements which is an M -semigroup.

Proof: Consider S^2 . Denote the factorizable elements of the subsets of S satisfying (ii) as e, f, \dots since there is at least one such set, there exists at least one e such that $e.y = y$, for all $y \in S^2$. That is e is a left identity of S^2 . By (iii) for any $x \in S^2$, there exists a unique $e \in S^2$ such that $xe = x$. therefore S^2 is an M -semigroup.

Define $\varphi: S \rightarrow S^2$ such that $x\varphi = \{U_i\}^2$ where $x \in U_i$

φ is well defined.

If $x \in U_i, y \in U_j$ and $xy \in U_k$ then

$$(xy)\varphi = (U_k)^2$$

$$X\varphi.y\varphi = (U_i)^2.(U_j)^2 = (U_k)^2 \text{ by(iv)}$$

International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Vol. 5, Issue 6, June 2016

i.e., φ is a homomorphism. Since $S^2 = \bigcup_{i \in \Lambda} U_i^2$; φ fixes the elements of S^2 . Hence φ is an inflation of its set of factorizable elements which is an M-semigroup.

IV. CONCLUSION

Thus we have structured of an M- Semi group through the semi deflation of its sub M- Semi group.

REFERENCES

1. A. Anjanayalu, "On primary semi groups" Czhe. Mathematical Journal 30(105) 1980, 382-386.
2. Ananth K. Atre, "On certain faithful elements of the S-set of a subset of a semi group(Abstract)" Math.Student XLI (1973) 9.
3. Ananth K. Atre, "On the set of factorizable elements of an inne semi group" Math.Edn.VIII(3) (1974 Sep).
4. Ault Janet E., "Semi groups with mid units" Trans. Am. Math. Society 190 (1974) 375-384.
5. Clifford A.H., "Bands of semi groups" Proc. Am. Math. Soc. 5 (1954) 499-504.
6. Clifford A.H. and Miller D.D., "Semi groups having zeroid elements" Am. J. Maths. 70 (1948) 117-125.
7. Clifford A.H. and Preston, G.B., "The algebraic theory of semi groups" Vol.II. Mathematical Surveys, No.7, 1961, American Mathematical Society, Rhode Island.