

Attractive Results for the Nonlinear Volterra Random Integral Equation in Separable Banach Space

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ABSTRACT: in this paper, we prove an the existence and attractivity results for a nonlinear Volterra type random integral equation in separable Banach space under mixed generalized compactness, contraction and caratheodory conditions, and also the existence of the Extremal solutions is proved under some certain monotonicity conditions.

KEYWORDS: Banach space, nonlinear random integral equation, existence result, locally attractive solution, Extremal solution.

I. INTRODUCTION

In the last six to seven decades scientist, mathematicians have become more increasingly aware about the fact that the most natural phenomena that does not allow strictly deterministic lows but rather oscillate randomly as an average behaviour. This type of natural phenomena is especially true in the life science where environmental influences should be taken as random. The random equation arises in many branches of physics, life science and engineering, as well as in other branches of technology and science. In the few last decades many specific types of random equations have been studied by engineers, physicists and mathematicians. In order to understand how random equations arise and why they are important to study. We have looked at the experimental procedures used to investigate various physical phenomena, models, mathematical equations are used to describe these phenomena. To obtain a random solution of the mathematical equation and then study it's properties. Thus the study of the natural phenomena with the help of the random equations forms a very important branch of the nonlinear analysis and some important details about the random equations and their analysis appears in the monographs of Bharucha-Ried [3].

Random fixed point theorems are the stochastic generalizations of the classical or deterministic fixed point theorems in abstract spaces and are useful in the study of nonlinear random equations for proving the existence theorems [6, 7, 8, 14 and 16]. The study of random fixed point theorems was initiated by the Prague school of probabilists in 1950's, Since then there have been several interesting results and a lot of activity in this area has appeared. In this paper we study the existence and attractivity results for a nonlinear Volterra type random integral equation in separable Banach space and also the existence of the Extremal solutions is proved under some certain monotonicity conditions by using the Dhage's fixed point theorem which is a random version of the generalization of Schaefer's nonlinear alternative due to Martelli [1].

Let \mathbb{R} denotes the real numbers and \mathbb{R}_+ be a set of nonnegative real numbers, that is $\mathbb{R}_+ = (0, \infty) \subset \mathbb{R}$ and let Y is a separable Banach space with the norm $\|\cdot\|$. Now consider the nonlinear random integral equation of Volterra type is in the form

$$x(t, \omega) = g(t, \omega) + \int_0^t k(s, x(s, \omega); \omega) ds \quad (1.1)$$

for all $t \geq 0$, where the function $k : \mathbb{R}_+ \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ is an random kernel defined for $0 \leq s \leq t \leq \infty$ and $\omega \in \Omega$, $x : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ is a unknown random function and the function $g : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ is a known function. This type of a nonlinear Volterra type random integral equation (1.1) have many applications in physical phenomena in life sciences, technology, numbered of physical problems, engineering [3 and 9] and also these are used to developed

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deterministically two models for chemotherapy by Bellman, Jacquez, and Kalaba [15], the random version of these chemotherapy models that describe the distribution of a highly toxic drug flow in one organ and two organ biological systems. these models of the chemotherapy are related in these properties of a mixing of blood in the heart, re-circulation of blood in the heart, presence of an organ such as one organ or two organ and the randomness of the diffusion of highly toxic drugs in the organ tissues. This gives a random integral equation of the above form which describes in Padgett and Toskos [9 and 10] and which has been studied by Bharucha-ried, Mukherjea, and Tserpes [2].

II. PRELIMINARIES

In this section, we present here some notations, definitions and preliminary facts that will be used in the proofs of our main results.

Let the set (Ω, \mathcal{A}) be a measurable space and Y be the Banach space with σ -algebra β_Y .

Definition 2.1: A mapping $T : \Omega \times Y \rightarrow Y$ is called a random operator, if $T(\omega, x)$ is measurable for each $x \in Y$ and this random operator is generally denoted as $T(\omega)x$.

Definition 2.2: A random variable $\eta : \Omega \rightarrow Y$ is called a random fixed point of a random operator $T(\omega) : \Omega \times Y \rightarrow Y$, if $T(\omega)\eta(\omega) = \eta(\omega)$ for every $\omega \in \Omega$.

Definition 2.3: A random operator $T(\omega) : \Omega \times Y \rightarrow Y$ is called a totally bounded operator, if for any bounded subset D of Y , $T(\omega)(D)$ is totally bounded subset of Y for every $\omega \in \Omega$.

Definition 2.4: A random operator $T(\omega)$ is called a completely continuous operator on Banach space Y , if random operator $T(\omega)$ is continuous and totally bounded random operator on Y .

Definition 2.5: A random operator $T(\omega) : \Omega \times Y \rightarrow Y$ is called compact random operator, if $\overline{T(\omega)(Y)}$ is compact subset of Y and also note that the every compact operator is totally bounded but reverse is not true.

A Kuratowski measure α of noncompactness of a bounded subset D of Y is a nonnegative, nonzero real number $\alpha(D)$ which is defined by

$$\alpha(D) = \inf \left\{ r > 0 : D = \bigcup_{n=1}^m D_n : \dim(D_n) \leq r \text{ for all } n \right\}$$

A random operator $T(\omega)$ is generally known as α -condensing random operator, if for any bounded subset D of Y , $T(\omega)(D)$ is bounded and it satisfy the $\alpha(T(\omega)(D)) < \alpha(D)$ for $\alpha(D) > 0$ for each $\omega \in \Omega$, and here we note that any compact random operators and contraction operators are α -condensing and converse of this is statement is not true. Now we obtain a random version of the generalization of fixed point theorem of Schaefer's nonlinear alternative due to A. Martelli [1].

Theorem 2.1: [5] Let Y be a Banach space and $T : Y \rightarrow Y$ be a α -condensing and continuous map, then either

- i) T has a fixed point or
- ii) The set $\rho = \{x \in Y \setminus x = \lambda Tx, \text{ for } \lambda \in (0,1)\}$ is an unbounded set.

Definition 2.6: A random operator $T(\omega) : \Omega \times Y \rightarrow Y$ is called a \mathcal{D} -Lipschitzian, if there exist a nondecreasing continuous function $\varphi : \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that for every $\omega \in \Omega$

$$\|T(\omega)x - T(\omega)y\| \leq \varphi_\omega(\|x - y\|) \tag{2.1}$$

for all $x, y \in Y$, where $\varphi_\omega(r) = \varphi(\omega, r)$ with $\varphi(\omega, 0) = 0$, here the some special case $\varphi_\omega(r) = \alpha(\omega)r$, for $\alpha(\omega) > 0$ for every $\omega \in \Omega$ then the random operator $T(\omega)$ is called a Lipschitzian with Lipschitz constant $\alpha(\omega)$, for all $\omega \in \Omega$. In particular for each $\omega \in \Omega$, $\alpha(\omega) < 1$ then random operator $T(\omega)$ is known as a contraction with contraction constant $\alpha(\omega)$, and if $\varphi_\omega(r) < r$, for $r > 0$ and for every $\omega \in \Omega$ then random operator $T(\omega)$ is known as a nonlinear contraction.

Definition 2.7: (Dugundji and Granas [11]) An operator A on a Banach space Y into itself is called Compact, if for any bounded subset S of Y , $A(S)$ is a relatively compact subset of Y . If A is continuous and compact, then it is called completely continuous on Y .

Theorem 2.2: (Arzela-Ascoli theorem) If every uniformly bounded and equi-continuous sequence $\{f_n\}$ of functions in $C(\mathbb{R}_+, \mathbb{R})$, then it has a convergent subsequence.

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Theorem 2.3: (Dominated convergence theorem) Let $\{f_n\}$ be a sequence of increasing nonnegative measurable functions on a measurable subset M of \mathbb{R} , if for all $x \in M$, $|f_n(x)| \leq h(x)$ for some measurable function h with $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ for almost every $x \in M$, then

$$\lim_{n \rightarrow \infty} \int_M f_n d\mu = \int_M \lim_{n \rightarrow \infty} f_n d\mu = \int_M f d\mu$$

Theorem 2.4: (Monotone convergence theorem) Let $\{f_n\}$ be a sequence of increasing nonnegative measurable functions with $\lim_{n \rightarrow \infty} f_n(x) = f(x)$, then the function f is a measurable function and

$$\int_M \lim_{n \rightarrow \infty} f_n d\mu = \int_M f d\mu$$

Now we shall seek the existence of the solution to the nonlinear Volterra type random integral equation of the type in the space $BM(\mathbb{R}_+, \mathbb{R})$ be the space of all bounded and measurable real valued functions on \mathbb{R}_+ with the norm is defined as

$$\|x\|_{BM} = \max_{t \in \mathbb{R}_+} |x(t)|$$

under some suitable conditions.

Let $Y = BM(\mathbb{R}_+, \mathbb{R})$ be the space of all bounded and measurable real valued functions on \mathbb{R}_+ , the space $CB(\mathbb{R}_+, \mathbb{R})$ be the space of all continuous and bounded real valued functions defined on \mathbb{R}_+ , and Ω be a nonempty subset of Y . Let $\varphi : \Omega \times Y \rightarrow Y$ be a mapping and consider the following random equation

$$\varphi(\omega)x(t) = x(t, \omega) \tag{2.2}$$

for all $t \geq 0$, and $\omega \in \Omega$.

Below we give different characterizations of the solutions for operator equation (2.2) on \mathbb{R}_+ , we need the following definitions.

Definition 2.8: B. C. Dhage [4] The random equation (2.2) has said to be have locally attractive random solution on \mathbb{R}_+ , if there exists a closed ball $\mathfrak{B}_r(x_1)$ in the space $CB(\mathbb{R}_+, \mathbb{R})$ for some $x_1 \in CB(\mathbb{R}_+, \mathbb{R})$ and for any real number $r \geq 0$ such that for any random solution $x(t, \omega)$ and $y(t, \omega)$ of the random equation (2.2) belonging to $\mathfrak{B}_r(x_1) \cap C$, we have

$$\lim_{n \rightarrow \infty} (x(t, \omega) - y(t, \omega)) = 0 \tag{2.3}$$

for every $\omega \in \Omega$. Here in this case when the limit of equation (2.3) is uniform with respect to the set $\overline{\mathfrak{B}_r(x_1)} \cap C$. i.e. every $\varepsilon > 0$ then there exists a $T > 0$ such that for every $t \geq T$,

$$|x(t, \omega) - y(t, \omega)| \leq \varepsilon \tag{2.4}$$

for every $\omega \in \Omega$ and $x, y \in \overline{\mathfrak{B}_r(x_1)} \cap C$ being the random solution of equation (2.2), then we will say that the random solution is uniformly locally attractive on \mathbb{R}_+ .

Concepts of the locally attractivity of solutions were introduced in [4, 7, and 8].

Now we give the Dhage's fixed point theorem which is a random version of the generalization of Schaefer's nonlinear alternative due to A. Martelli [1], which is the main tool in the existence theorem of solutions to the nonlinear Volterra type random integral equation (1.1).

Theorem 2.5: [5] Let Y be any separable banach space and let a function $C : \Omega \times Y \rightarrow Y$ be the random operator for every $\omega \in \Omega$, if it satisfying all the following conditions,

- 1) $C(\omega)$ is the continuous random operator and also α -condensing,
 - 2) The set $\eta = \{x \in Y \mid x = \lambda(\omega)C(\omega)x\}$ is an bounded set for any measurable function $\lambda : \Omega \rightarrow \mathbb{R}$ with $0 < \lambda(\omega) < 1$.
- Then the random operator $C(\omega)$ has a random fixed point.

An interesting corollary to Theorem 2.4 in its applicable form is

Corollary 2.1: [5] Let any separable banach space Y , and let the mapping $C(\omega) : \Omega \times Y \rightarrow Y$ be any random operator, for every $\omega \in \Omega$, if it satisfying all the following described conditions,

- 1) $C(\omega)$ is a completely continuous random operator,
- 2) The set $\eta = \{x \in Y \mid C(\omega)x = \alpha(\omega)x\}$ is bounded set for any measurable functions $\alpha : \Omega \rightarrow \mathbb{R}_+$ with the $\alpha(\omega) > 1$.

Then the random operator $C(\omega)$ has a random fixed point.

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III. EXISTENCE RESULT

Let $B(\mathbb{R}_+, \mathbb{R})$ and $M(\mathbb{R}_+, \mathbb{R})$ denotes the space of all bounded and measurable real valued functions defined on \mathbb{R}_+ respectively, and the space $BM(\mathbb{R}_+, \mathbb{R})$ be the space of all bounded and measurable real valued functions on \mathbb{R}_+ with the norm is defined as

$$\|y\| = \max_{t \in \mathbb{R}_+} |y(t)|$$

Then it is clear that, the defined space $BM(\mathbb{R}_+, \mathbb{R})$ is Banach space with respect to this standard supremum norm defined as above and also the space $BM(\mathbb{R}_+, \mathbb{R})$ is a separable Banach space with respect to this standard supremum norm defined as above. The space $C(\mathbb{R}_+, \mathbb{R})$ denotes the space of all continuous real valued functions on \mathbb{R}_+ and we get the relation between them as $C(\mathbb{R}_+, \mathbb{R}) \subset BM(\mathbb{R}_+, \mathbb{R}) \subset M(\mathbb{R}_+, \mathbb{R})$. The space $L^1(\mathbb{R}_+, \mathbb{R})$ denotes the space of all Lebesgue integral real valued functions defined on \mathbb{R}_+ with respect to the norm $\|\cdot\|_{L^1}$ defined by

$$\|y\|_{L^1} = \int_{t_1}^{t_2} |y(s)| ds$$

Now we need some following definitions,

Definition 3.1: [5] A function $\beta : \mathbb{R}_+ \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ is said to be L^1_ω -Carathéodory function, if

- i) $t \rightarrow \beta(t, x; \omega)$ is measurable for all $x \in \mathbb{R}$,
- ii) $x \rightarrow \beta(t, x; \omega)$ is continuous for almost every $t \in \mathbb{R}_+$,
- iii) For every $r > 0$, there exist $q_r \in L^1(\mathbb{R}_+, \mathbb{R})$ with $|\beta(t, x; \omega)| \leq q_r(t, \omega)$ for almost every $t \in \mathbb{R}_+$ and for all $x \in \mathbb{R}$ with $|x| \leq r$.

Now we consider the following some hypothesis,

(H₁) The function $g : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ is a measurable and bounded with $K_1 = \sup |g(t, \omega)|$ and also the $\lim_{t \rightarrow \infty} g(t, \omega) = 0$,

(H₂) The function $\omega \rightarrow k(s, x(s, \omega); \omega)$ is measurable for all $s \in \mathbb{R}_+$,

(H₃) The function $k(s, x; \omega)$ is L^1_ω -Carathéodory,

(H₄) There exists a continuous function $\varphi : \Omega \rightarrow L^1(\mathbb{R}_+, \mathbb{R})$ such that
 $|k(s, x; \omega)| \leq \varphi(s, \omega)$

for almost every $s \in \mathbb{R}_+$ and for all $x \in \mathbb{R}$ with $\omega \in \Omega$,

(H₅) The continuous function $v : \mathbb{R}_+ \rightarrow \mathbb{R}$ defined by as $v(t, \omega) = \int_0^t q_r(s, \omega) ds$ bounded on \mathbb{R}_+ and vanish at infinity, i.e. $\lim_{t \rightarrow \infty} v(t, \omega) = 0$.

Remark 3.1: Here note that the hypothesis (H₁), (H₃), (H₄), and (H₅) holds, then there exists a constant terms K_1 and K_2 such that $K_1 > 0, K_2 > 0$ with $K_1 = \sup\{g(t, \omega) \mid t \in \mathbb{R}_+ \text{ and } \omega \in \Omega\}$ and $K_2 = \sup\{v(t, \omega) \mid t \in \mathbb{R}_+ \text{ and } \omega \in \Omega\}$.

Theorem 3.1: Suppose that the hypothesis (H₁) to (H₅) holds, further if there exist a real number $r > 0$ such that

$$K_1 + K_2 = r$$

Then the nonlinear Volterra type integral equation (1.1) has a random solution. And moreover, random solutions are locally attractive on \mathbb{R}_+ .

Proof: Let $Y = BM(\mathbb{R}_+, \mathbb{R})$ be a separable Banach space, now we define the operator $C(\omega)$ on $BM(\mathbb{R}_+, \mathbb{R})$, by

$$C(\omega)x(t) = g(t, \omega) + \int_0^t k(s, x(s, \omega); \omega) ds \tag{3.1}$$

then the Volterra type random nonlinear integral equation (1.1) is equivalent to the operator equation

$$C(\omega)x(t) = x(t, \omega) \tag{3.2}$$

for all $t \in \mathbb{R}_+$.

Step I: First, we show that $C(\omega)$ are random operators on Y .

By hypothesis (H₁) and (H₂) as $\omega \rightarrow g(t, \omega)$ and the function $\omega \rightarrow k(s, x(s, \omega); \omega)$ are measurable for every $\omega \in \Omega$, and for all $t \in \mathbb{R}_+$ on Y , and the space Y is a separable Banach space. Now the integral

$$\int_0^t k(s, x(s, \omega); \omega) ds$$

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is the limit of a finite sum of the measurable functions, so that

$$\omega \rightarrow \int_0^t k(s, x(s, \omega); \omega) ds$$

is measurable, again as we know that the sum of any two measurable functions are measurable functions. Therefore the random operator $C(\omega)$ is measurable.

Hence the operator $C(\omega)$ is random operators on Y .

Now we will show that the random operators $C(\omega)$ satisfy all the conditions of corollary 2.1.

Step II: Now we show that the random operator $C(\omega)$ is a continuous random operator on Y .

Let B be a bounded subset of Y , then there is real number $r > 0$ such that $\|x\| \leq M$ for all $x \in B$. Let $\{x_n\}$ be a convergent sequence of points in B converging to the point $x \in B$, then it is enough to prove that $\lim_{n \rightarrow \infty} C(\omega)x_n(t) = C(\omega)x(t), \forall t \in \mathbb{R}_+$, then by the Lebesgue dominated converging theorem, we obtain,

$$\begin{aligned} \lim_{n \rightarrow \infty} C(\omega)x_n(t) &= \lim_{n \rightarrow \infty} \left[g(t, \omega) + \int_0^t k(s, x_n(s, \omega); \omega) ds \right] \\ &= \lim_{n \rightarrow \infty} g(t, \omega) + \lim_{n \rightarrow \infty} \int_0^t k(s, x_n(s, \omega); \omega) ds \\ &\leq \lim_{n \rightarrow \infty} g(t, \omega) + \int_0^t \lim_{n \rightarrow \infty} k(s, x_n(s, \omega); \omega) ds \\ &\leq g(t, \omega) + \int_0^t k(s, x(s, \omega); \omega) ds \\ &= C(\omega)x(t) \end{aligned} \tag{3.3}$$

for every $\omega \in \Omega$ and for all $t \in \mathbb{R}_+$.

This proves that the random operator $C(\omega)$ is continuous operator on Y .

Step III: The operator $C(\omega)$ is compact operator on Y .

Since by the hypothesis (H_3) the function $k(s, x; \omega)$ is L^1_ω -Carathéodory, and let B be any bounded subset of Y with bound M , then for every $\omega \in \Omega$, by (H_4)

$$\begin{aligned} |C(\omega)x(t)| &= \left| g(t, \omega) + \int_0^t k(s, x(s, \omega); \omega) ds \right| \\ |C(\omega)x(t)| &= |g(t, \omega)| + \left| \int_0^t k(s, x(s, \omega); \omega) ds \right| \\ |C(\omega)x(t)| &\leq |g(t, \omega)| + \int_0^t |k(s, x(s, \omega); \omega)| ds \\ &\leq |g(t, \omega)| + \int_0^t q_r(s, \omega) ds \\ &\leq |g(t, \omega)| + \|q_r(\omega)\|_{L^1} \end{aligned}$$

taking maximum for all over $t \in \mathbb{R}_+$, we get,

$$\|C(\omega)x(t)\| \leq \|g(t, \omega)\| + \|q_r(\omega)\|_{L^1}$$

for every $x \in B$, this shows that the set $\{C(\omega)B\}$ is uniformly bounded set in Y , for every $\omega \in \Omega$.

Secondly, we show that $C(\omega)B$ is an equicontinuous set in Y .

Now let $t, \tau \in \mathbb{R}_+$, and for every $x \in B$,

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$$\begin{aligned}
 |C(\omega)x(t) - C(\omega)x(\tau)| &= \left| g(t, \omega) + \int_0^t k(s, x(s, \omega); \omega) ds - g(\tau, \omega) + \int_0^\tau k(s, x(s, \omega); \omega) ds \right| \\
 &= \left| g(t, \omega) - g(\tau, \omega) + \int_0^t k(s, x(s, \omega); \omega) ds - \int_0^\tau k(s, x(s, \omega); \omega) ds \right| \\
 &= |g(t, \omega) - g(\tau, \omega)| + \left| \int_0^t k(s, x(s, \omega); \omega) ds - \int_0^\tau k(s, x(s, \omega); \omega) ds \right| \\
 &\leq |g(t, \omega) - g(\tau, \omega)| + \int_0^t |k(s, x(s, \omega); \omega)| ds - \int_0^\tau |k(s, x(s, \omega); \omega)| ds \\
 &\leq |g(t, \omega) - g(\tau, \omega)| + \int_0^t q_r(s, \omega) ds - \int_0^\tau q_r(s, \omega) ds \\
 &\leq |g(t, \omega) - g(\tau, \omega)| + |v(t, \omega) - v(\tau, \omega)| \tag{3.4}
 \end{aligned}$$

for every $\omega \in \Omega$, since $t \rightarrow g(t, \omega)$ is continuous on \mathbb{R}_+ , and hence it is uniformly continuous, and Since v is uniformly continuous function, it follows that $|C(\omega)x(t) - C(\omega)x(\tau)| \rightarrow 0$ as $t \rightarrow \tau$ for all $x \in B$ and $\omega \in \Omega$. Therefore $\{C(\omega)B\}$ is an equi-Continuous set in Y for all $t, \tau \in \mathbb{R}_+$.

As $\{C(\omega)B\}$ is uniformly bounded and equi-continuous subset in Y , and it follows that from equation (3.4) that the set $\{C(\omega)B\}$ is compact set in Y , then by the Aezela-Ascoli theorem for every $\omega \in \Omega$, $C(\omega)$ is continuous and compact random operator on Y .

$\therefore C(\omega)$ is a completely continuous random operator.

Step IV: Now for any solution x to $C(\omega)x = \alpha(\omega)x$, with $\alpha(\omega) > 1$. One has

$$\begin{aligned}
 |x(t, \omega)| &= \left| g(t, \omega) + \int_0^t k(s, x(s, \omega); \omega) ds \right| \\
 &= |g(t, \omega)| + \left| \int_0^t k(s, x(s, \omega); \omega) ds \right| \\
 &\leq |g(t, \omega)| + \int_0^t |k(s, x(s, \omega); \omega)| ds \\
 &\leq |g(t, \omega)| + \int_0^t \varphi(s, \omega) ds \\
 &\leq |g(t, \omega)| + v(t, \omega)
 \end{aligned}$$

taking suprimum all over t , we get,

$$\begin{aligned}
 \sup |x(t, \omega)| &\leq \sup |g(t, \omega)| + \sup v(t, \omega) \\
 \|x(t, \omega)\| &\leq K_1 + K_2
 \end{aligned}$$

But $K_1 + K_2 = r$

$\Rightarrow \|x(t, \omega)\| \leq r$ for each $\omega \in \Omega$.

Thus all the condition of corollary 2.1 are satisfied and hence the application this corollary 2.1 yields that the random operator equation (3.2) has a random solution in Y and consequently the nonlinear Volterra type random integral equation (1.1) has a random solution.

Step V: Next, we show that the solutions of the nonlinear Volterra type random integral equation (1.1) are locally attractive.

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Let the p and q be any arbitrary two solutions of the nonlinear Volterra type random integral equation (1.1) in the closed ball $\overline{\mathfrak{B}}_r(0)$ defined on \mathbb{R}_+ , then we have

$$\begin{aligned} |p(t, \omega) - q(t, \omega)| &= \left| g(t, \omega) + \int_0^t k(s, p(s, \omega); \omega) ds - g(t, \omega) + \int_0^t k(s, q(s, \omega); \omega) ds \right| \\ &= \left| g(t, \omega) - g(t, \omega) + \int_0^t k(s, p(s, \omega); \omega) ds - \int_0^t k(s, q(s, \omega); \omega) ds \right| \\ &\leq \left| \int_0^t k(s, p(s, \omega); \omega) ds - \int_0^t k(s, q(s, \omega); \omega) ds \right| \\ &\leq \left| \int_0^t k(s, p(s, \omega); \omega) ds \right| + \left| \int_0^t k(s, q(s, \omega); \omega) ds \right| \\ &\leq \left| \int_0^t \varphi(s, \omega) ds \right| + \left| \int_0^t \varphi(s, \omega) ds \right| \\ &\leq v(t, \omega) + v(t, \omega) \\ &\leq 2v(t, \omega) \end{aligned}$$

Then taking limit supremum as $n \rightarrow \infty$, we get,

$$\begin{aligned} \lim_{n \rightarrow \infty} \sup |p(t, \omega) - q(t, \omega)| &= \lim_{n \rightarrow \infty} \sup 2v(t, \omega) \\ &= 2 \lim_{n \rightarrow \infty} \sup v(t, \omega) \end{aligned}$$

but from the hypothesis (H_5) , we know that the $\lim_{n \rightarrow \infty} v(t, \omega) = 0$, this gives that the

$$\lim_{n \rightarrow \infty} \sup |p(t, \omega) - q(t, \omega)| = 0$$

Thus the nonlinear Volterra type random integral equation (1.1) has a random solution and all the random solutions are locally attractive on \mathbb{R}_+ .

This completes the proof.

IV. EXISTENCE OF EXTREMAL SOLUTION

Definition 4.1: Let a closed subset \mathbb{K} of a Banach Space \mathbb{X} , then the set \mathbb{K} is called a cone, if

- i) $\mathbb{K} + \mathbb{K} \subseteq \mathbb{K}$
- ii) $\delta \mathbb{K} \subset \mathbb{K}$ for all $\delta \in \mathbb{R}_+$,
- iii) $\{-\mathbb{K}\} \cap \mathbb{K} = \{0\}$

where the element 0 is a zero element of a Banach space \mathbb{X} .

Now we introduce an ordered relation " \leq " in a Banach space \mathbb{X} with the help of a cone \mathbb{K} is defined as above in a Banach space \mathbb{X} as follows,

Let for any arbitrary elements $x, y \in \mathbb{X}$, then we define the relation

$$x \leq y \quad \text{iff} \quad y - x \in \mathbb{K}.$$

Definition 4.2: A cone \mathbb{K} in a Banach space \mathbb{X} is known as normal, if the norm $\|\cdot\|$ is defined on \mathbb{X} is an semi-monotone on a cone \mathbb{K} . i.e. for every $x, y \in \mathbb{K}$ then $\|x + y\| \leq \|x\| + \|y\|$.

Definition 4.3: A cone \mathbb{K} in a Banach space \mathbb{X} is known as a regular, if every increasing, bounded sequence in \mathbb{X} is convergent in norm, and also similarly the cone \mathbb{K} is called a fully regular, if every increasing normed bounded sequence converges in that Banach space \mathbb{X} .

The all details about the different types of cones and some interesting properties of a cone are appeared in Zeidler [12] and G. S. Ladde and V. Lakshmikantham [13].

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The space $C(\mathbb{R}_+, \mathbb{R})$ be the space of all real valued continuous function defined on \mathbb{R}_+ , and the space $C(\mathbb{R}_+, \mathbb{R})$ be the Banach space with the suprimum norm defined as

$$\|x\| = \sup_{t \in \mathbb{R}_+} |x(t)|$$

and also this space $C(\mathbb{R}_+, \mathbb{R})$ is a separable Banach space.

Definition 4.4: A random operator $A(\omega) : \Omega \times X \rightarrow X$ is called a increasing operator, if $\forall x, y \in X$ with $x \leq y$ then $A(\omega)x \leq A(\omega)y$ for every $\omega \in \Omega$.

Now we introduce an ordered relation " \leq " on $C(\mathbb{R}_+, \mathbb{R})$ with the help of a cone \mathbb{K} , which is defined as follows

$$\mathbb{K} = \{x \in C(\mathbb{R}_+, \mathbb{R}) \mid x(t) \geq 0, \forall t \in \mathbb{R}_+\}$$

Thus we have, $x \leq y \Rightarrow x(t) \leq y(t) \quad \forall t \in \mathbb{R}_+$

It is known as the cone \mathbb{K} is a normal in $C(\mathbb{R}_+, \mathbb{R})$

For any functions $p, q : \Omega \rightarrow C(\mathbb{R}_+, \mathbb{R})$ we define a random interval $[p, q]$ in $C(\mathbb{R}_+, \mathbb{R})$ by

$$[p, q] = \{x \in C(\mathbb{R}_+, \mathbb{R}) \mid p(\omega) \leq x \leq q(\omega), \forall t \in \mathbb{R}_+\}$$

$$[p, q] = \bigcap_{\omega \in \Omega} [p(\omega), q(\omega)]$$

We use the random fixed point theorem of B. C. Dhage which is as follows,

Theorem 4.1: Let the space (Ω, \mathcal{A}) be a measurable space, and let $[p, q]$ be the random ordered interval in the separable Banach space X and let an operator $\Gamma : \Omega \times [p, q] \rightarrow [p, q]$ be a increasing and a completely continuous random operator, then the random operator Γ has a least fixed point $x_{\#}$ and greatest fixed point $x^{\#}$ in $[p, q]$, moreover a sequence $\{\Gamma(\omega)x_n\}$ with $x_0 = p$ and $\{\Gamma(\omega)y_n\}$ with $y_0 = q$ converges to $x_{\#}$ and $y^{\#}$ respectively.

Definition 4.5: A random solution $x_M(t, \omega)$ of the nonlinear Volterra type random integral equation (1.1) is called a maximal random solution, if for all random solutions to the Volterra type nonlinear random integral equation (1.1), one has $x(t, \omega) \leq x_M(t, \omega)$ for all $\omega \in \Omega$, and $\forall t \in \mathbb{R}_+$. Similarly a minimal random solution to the nonlinear Volterra type random integral equation (1.1) on \mathbb{R}_+ is defined.

Now we consider the following some hypothesis,

(B₀) $g : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ and $k : \mathbb{R}_+ \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$,

(B₁) For each $\omega \in \Omega$, the function $x \rightarrow g(t, \omega)$ in increasing almost every for each $t \in \mathbb{R}_+$,

(B₂) For each $\omega \in \Omega$, the function $x \rightarrow k(t, x(t, \omega); \omega)$ in increasing every for each $t \in \mathbb{R}_+$ and $x \in \mathbb{R}$,

(B₃) The nonlinear Volterra type random integral equation (1.1) has a lower random solution p and upper random solution q with $p \leq q$ on \mathbb{R}_+ ,

(B₄) The function $q : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ is defined by

$$q(t, \omega) = |k(t, p(t, \omega); \omega)| + |k(t, q(t, \omega); \omega)|$$

is Lebesgue integrable on \mathbb{R}_+ and for all $\omega \in \Omega$.

Remark 4.1: The hypothesis (B₃) is in general and used in several research papers on random differential equation and random integral equations (see [7 and 8]), in particular, if there exists a measurable functions $\beta, \gamma : \Omega \rightarrow C(\mathbb{R}_+, \mathbb{R})$ such that for every $\omega \in \Omega$,

$$\beta(t, \omega) \leq k(t, x(t, \omega); \omega) \leq \gamma(t, \omega)$$

for all $t \in \mathbb{R}_+, x \in \mathbb{R}$.

here in this case, the lower and upper solutions to the nonlinear Volterra type random integral equation (1.1) are given by,

$$p(t, \omega) \leq g(t, \omega) + \int_0^t k(s, \beta(s, \omega); \omega) ds$$

and

$$q(t, \omega) \geq g(t, \omega) + \int_0^t k(s, \gamma(s, \omega); \omega) ds$$

respectively, and finally the hypothesis (B₃) is remains valid as the function k is L^1_0 -caratheodory on $\mathbb{R}_+ \times \mathbb{R} \times \Omega$.

Theorem 4.2: Assume that the hypothesis (B₀) - (B₄) holds, then the nonlinear Volterra type random integral equation (1.1) has a minimal random solution $x_m(\omega)$ and a maximal random solution $x_M(\omega)$ defined on \mathbb{R}_+ , moreover

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$$x_m(\omega) = \lim_{n \rightarrow \infty} x_n(t, \omega)$$

$$\text{And } x_M(\omega) = \lim_{n \rightarrow \infty} y_n(t, \omega)$$

for all $t \in \mathbb{R}_+$, and for all $\omega \in \Omega$, where the random sequence $\{x_n(\omega)\}$ and $\{y_n(\omega)\}$ are given as,

$$x_n(t, \omega) = g(t, \omega) + \int_0^t k(s, x_{n-1}(s, \omega); \omega) ds$$

a.e for $t \in \mathbb{R}_+$, with $x_0 = p$ (4.1)

and

$$y_n(t, \omega) = g(t, \omega) + \int_0^t k(s, y_{n-1}(s, \omega); \omega) ds$$

a.e for $t \in \mathbb{R}_+$, with $y_0 = q$ (4.2)

$\forall n \geq 0$, for all $t \in \mathbb{R}_+$, and for all $\omega \in \Omega$.

Proof: Let the set $G = C(\mathbb{R}_+, \mathbb{R})$ be the separable Banach space, and define the random operator $C : \Omega \times [p, q] \rightarrow G$, by equation (3.2), then we will show that the random operator C satisfy all the conditions of the theorem 4.1 on $[p, q]$.

Now we define the operator $C(\omega)$ on $G = C(\mathbb{R}_+, \mathbb{R})$, by

$$C(\omega)x(t) = g(t, \omega) + \int_0^t k(s, x(s, \omega); \omega) ds \tag{4.3}$$

then the nonlinear Volterra type random integral equation (1.1) is equivalent to the operator equation

$$C(\omega)x(t) = x(t, \omega) \tag{4.4}$$

for all $t \in \mathbb{R}_+$, then it can be easily shown as in the proof of the theorem 3.1 that the operator $C(\omega)$ is an random operator on $\Omega \times [p, q]$.

Hypothesis (B_1) and (B_2) implies that the functions $x \rightarrow g(t, \omega)$ and $x \rightarrow k(t, x(t, \omega); \omega)$ in increasing on $[p, q]$ and as we know that the addition of two increasing functions are again increasing, hence we get the random operator $C(\omega)$ is an increasing on $[p, q]$. To see that,

Let $x, y : \Omega \rightarrow [p, q]$ be any arbitrary functions such that $x \leq y$ on Ω , then by (B_1)

$$\begin{aligned} C(\omega)x(t) &= g(t, \omega) + \int_0^t k(s, x(s, \omega); \omega) ds \\ &\leq g(t, \omega) + \int_0^t k(s, y(s, \omega); \omega) ds \\ &= C(\omega)y(t) \end{aligned}$$

for all $t \in \mathbb{R}_+$ and for all $\omega \in \Omega$.

This gives that for any $x \leq y$, we get $C(\omega)x(t) \leq C(\omega)y(t)$, for all $t \in \mathbb{R}_+$ and for all $\omega \in \Omega$, hence we get the random operator $C(\omega)$ is an increasing random operator on $[p, q]$.

Similarly by the hypothesis (B_3) ,

$$\begin{aligned} p(t, \omega) &\leq C(\omega)p(t) \\ &= g(t, \omega) + \int_0^t k(s, p(s, \omega); \omega) ds \\ &\leq g(t, \omega) + \int_0^t k(s, x(s, \omega); \omega) ds \\ &= C(\omega)x(t) \\ &\leq C(\omega)q(t) \\ &= g(t, \omega) + \int_0^t k(s, q(s, \omega); \omega) ds \end{aligned}$$

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$$\leq q(t, \omega)$$

Thus $p(t, \omega) \leq q(t, \omega)$ for all $t \in \mathbb{R}_+$, and for all $\omega \in \Omega$, thus $C(\omega) \in [p, q]$ for all $x \in [p, q]$.

Now it can easily shown in the proof of theorem 3.1 that the random operator $C(\omega)$ is completely continuous random operator on $[p, q]$, and thus the random operator $C(\omega)$ satisfies all these conditions theorem 4.1 and so that the random operator $C(\omega)x(t) = x(t, \omega)$ has a least random solution and greatest random solution in $[p, q]$ and consequently the has a minimal and maximal random solution defined on \mathbb{R}_+ .

This completes the proof.

V. CONCLUSION

In this paper, we introduced a nonlinear Volterra random integral equation in separable Banach space. It is clear that hybrid fixed point method forms an interesting powerful tool for proving existence as well as other qualitative aspects of the solutions of certain nonlinear Volterra random integral equation. The work along this line is the most active area of research and the field is growing very rapidly. These and other similar problems form the further scope for research work in the theory of Volterra random integral equations.

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