

Global Analysis of A SEIR Epidemic Model With New Modulated Saturated Incidence Rate under Treatment

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ABSTRACT: On account of the effect of limited resources on the control of an epidemic disease, the epidemic model of SEIR type with saturated incidence rate under treatment is investigated. Treatment plays an important role to control or decrease the spread of diseases such as flue, tuberculosis, and measles. In practical view, the treatment function $T(I)$ is introduced and justified in the SEIR epidemic model. It is assumed that the treatment rate is proportional to the number of patients as long as this number is below a certain limit and it becomes constant when that number of patients exceeds this limit. Existence and stability of the disease free and endemic equilibrium are investigated for both the cases. Numerical simulations are given to illustrate the analytical results.

KEYWORDS: SEIR Epidemic Model, Basic Reproduction Number, Stability Analysis, Disease-free Equilibrium.

I. INTRODUCTION

Mathematical models have become important tools in analyzing the spread and control of infectious diseases. Many researchers constructed the epidemic models in different ways and those are analyzed in a different manner. In the course of disease control, the treatment is an important method to decrease the spread of diseases. Recently, many researchers have explored the role of treatment function $T(I)$, which represents the number of infectious individuals of healing, within epidemic disease models. In classical epidemic models, the treatment rate is assumed to be proportional to the number of infective individuals. Wendi and Wang [61] considered the treatment function has the form

$$T(I) = \begin{cases} rI, & \text{if } 0 \leq I \leq I_0 \\ rI_0, & \text{if } I > I_0 \end{cases}$$

where r is a constant value.

II. THE MATHEMATICAL MODEL

The proposed model is

$$\left. \begin{aligned} \frac{dS}{dt} &= A - dS - \frac{\lambda SI}{1 + \alpha I^2} + \beta R \\ \frac{dE}{dt} &= \frac{\lambda SI}{1 + \alpha I^2} - (d + \epsilon)E \\ \frac{dI}{dt} &= \epsilon E - (d + m)I - T(I) \\ \frac{dR}{dt} &= mI - (d + \beta)R + T(I) \end{aligned} \right\} (1)$$

where

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- $S(t)$ - The number of susceptible individuals
- $E(t)$ - The number of exposed infected individuals but not infection
- $I(t)$ - The number of infected individuals
- $R(t)$ - The number of recovered individuals with new modulated saturated incidence rate.

A is the recruitment rate of the population, d is the natural death rate, λ is the proportionality constant, β is the natural recovery rate of the infective individuals, m is the rate at which recovered individuals lose immunity and to the susceptible class, α is the parameter measures of psychological effect. It assumes that the birth and death rate are not equal. Wendi and Wang [10] considered the treatment function has the form

$$T(I) = \begin{cases} rI, & \text{if } 0 \leq I \leq I_0 \\ K, & \text{if } I > I_0 \end{cases}$$

where $K=rI_0$. This means that the treatment rate is proportional to the infective when the number of infective is less or equal to some fixed value I_0 and the treatment is constant when the number of infective crosses the fixed value I_0 .

Case (i): SEIR model with $0 \leq I \leq I_0$. In this case the system reduces to

$$\left. \begin{aligned} \frac{dS}{dt} &= A - dS - \frac{\lambda SI}{1 + \alpha I^2} + \beta R \\ \frac{dE}{dt} &= \frac{\lambda SI}{1 + \alpha I^2} - (d + \epsilon)E \\ \frac{dR}{dt} &= mI - (d + \beta)R + rI \end{aligned} \right\} \quad (2)$$

Consider a non-monotonic saturated incidence rate of the form $\frac{\lambda SI}{1 + \alpha I^2}$. This represents the inhibition effect of the behavioral change of the susceptible individuals where there is an increase in the number of infective individuals.

III. EQUILIBRIUM POINTS

Biological meaning of the components $(S(t), E(t), I(t), R(t))$. We focus on the model in the first octant R^3 . We first consider the existence of equilibria of system (2). For any values of parameters, model (2) always has a disease – free equilibrium $P_0(S_0, E_0, I_0, R_0) = (A/d, 0, 0, 0)$ is achieved for the endemic equilibrium $P^*(S^*, E^*, I^*, R^*)$

$$A - dS - \frac{\lambda SI}{1 + \alpha I^2} + \beta R = 0 \quad (A)$$

$$\frac{\lambda SI}{1 + \alpha I^2} - (d + \epsilon)E = 0 \quad (B)$$

$$\epsilon E - (d + m)I - rI = 0 \quad (C)$$

$$mI - (d + \beta)R + rI = 0 \quad (D)$$

From (C) and (D), we get

$$E^* = \frac{(d + m + r)I}{\epsilon}$$

$$R^* = \frac{(m + r)I}{d + \beta}$$

Using this in (B), we get

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$$\frac{\lambda SI}{1 + \alpha I^2} = (d + \epsilon)E$$

$$\Rightarrow S^* = \frac{(d + \epsilon)(d + m + r)(1 + \alpha I^2)}{\epsilon \lambda}$$

Substitute S , E and R value in equation (A), we get

$$\alpha d(d + \epsilon)(d + m + r)I^2 + \left[\lambda(d + \epsilon)(d + m + r) - \frac{\lambda \epsilon \beta(m + r)}{d + \beta} \right] I + d(d + \epsilon)(d + m + r) - A\lambda \epsilon = 0 \tag{3}$$

It is given as a root of the quadratic equation $aI^2 + bI + c = 0$ where

$$a = \alpha d(d + \epsilon)(d + m + r), \quad b = \lambda(d + \epsilon)(d + m + r) - \frac{\lambda \epsilon \beta(m + r)}{d + \beta}$$

$$c = d(d + \epsilon)(d + m + r) - A\lambda \epsilon$$

Clearly the above equation will have a positive root if $\Delta > 0$ and $R_0 > 1$, where R_0 is the basic reproduction number

$$R_0 = \frac{A\lambda \epsilon}{d(d + \epsilon)(d + m + r)}$$

From equation (3), we can see that

i) If $R_0 \leq 1$, then there is no positive equilibrium

$$I = -\left(\lambda(d + \epsilon)(d + m + r) - \frac{\lambda \epsilon \beta(m + r)}{d + \beta} \right) \pm \frac{\sqrt{\left(\lambda(d + \epsilon)(d + m + r) - \frac{\lambda \epsilon \beta(m + r)}{d + \beta} \right)^2 - 4[\alpha d(d + \epsilon)(d + m + r)][d(d + \epsilon)(d + m + r) - A\lambda \epsilon]}}{2[\alpha d(d + \epsilon)(d + m + r)]}$$

ii) If $R_0 > 1$, there is a unique positive equilibrium $P^* (S^*, E^*, I^*, R^*)$ called the endemic equilibrium.

Now $\Delta = b^2 - 4ac$

$$\Delta = \left(\lambda(d + \epsilon)(d + m + r) - \frac{\lambda \epsilon \beta(m + r)}{d + \beta} \right)^2 - 4[\alpha d(d + \epsilon)(d + m + r)][d(d + \epsilon)(d + m + r) - A\lambda \epsilon]$$

$$\Delta = \left(\lambda(d + \epsilon)(d + m + r) - \frac{\lambda \epsilon \beta(m + r)}{d + \beta} \right)^2 - 4[\alpha d(d + \epsilon)(d + m + r)][1 - R_0]$$

where $R_0 = \frac{A\lambda \epsilon}{d(d + \epsilon)(d + m + r)}$

Therefore

$$S^* = \frac{(d + \epsilon)(d + m + r)(1 + \alpha I^2)}{\epsilon \lambda} \tag{4}$$

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$$E^* = \frac{(d+m+r)I}{\epsilon} \tag{5}$$

$$R^* = \frac{(m+r)I}{d+\beta} \tag{6}$$

$$I^* = \frac{-(\lambda(d+\epsilon)(d+m+r) - \frac{\lambda\epsilon\beta(m+r)}{d+\beta}) \pm \sqrt{\Delta}}{2[\alpha d(d+\epsilon)(d+m+r)]} \tag{7}$$

In the next section, we shall study the property of these equilibria and perform a global qualitative analysis of the model (2).

IV. MATHEMATICAL ANALYSIS

Lemma:

The system (2) has a disease free-equilibrium points if $N = \frac{A}{d}$, which is attracting in the first octant.

Proof.

Consider $N(t) = S(t) + E(t) + I(t) + R(t)$

$$\frac{dN}{dt} = A - dN(t) \tag{8}$$

It is clear that $N = \frac{A}{d}$ is a solution of equation (8) and for any $N(t_0) \geq 0$.

The general solution of equation (8) is

$$\frac{dN}{A - dN(t)} = dt$$

Integrating this we get,

$$N(t) = \frac{A - e^{-t+t_0}(A - dN(t_0))}{d}. \text{ Thus, } \lim_{t \rightarrow \infty} N = \frac{A}{d} \text{ which implies the conclusion.}$$

It is clear that the limit set at system (2) is on the plane $S + E + I + R = \frac{A}{d}$.

Then we focus on the reduced system

$$\left. \begin{aligned} \frac{dI}{dt} &= \frac{\lambda I \epsilon}{[\lambda I + (d+\epsilon)](1+\alpha I^2)} \left(\frac{A}{d} - I - R \right) - (d+m+r) @ P(I, R) \\ \frac{dR}{dt} &= (m+r)I - (d+\beta)R @ Q(I, R) \end{aligned} \right\} \tag{9}$$

We have the following result regarding the non-existence of periodic orbits in system (9) which implies the non-existence of periodic orbits of system (2) by lemma.

Theorem:

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System (9) does not have non trivial periodic orbits.

Proof.

Consider the system (9) for $I > 0, R > 0$. Take a Dulac function.

$$D(I, R) = \frac{1 + \alpha I^2}{\lambda I}$$

We have $\frac{\partial(DP)}{\partial I} + \frac{\partial(DQ)}{\partial R} < 0$

Here the conclusion follows. In order to study the properties of the disease free equilibrium P_0 and the endemic equilibrium P^* , we rescale (9) by

$$x = \frac{\lambda I}{d + \beta}, y = \frac{\lambda R}{d + \beta}, z = (d + \beta)t, w = \frac{m + r}{d + \beta}$$

$$\frac{dx}{dz} = \frac{x}{1 + vx^2} (K - x - y) - ux$$

$$\frac{dy}{dz} = \frac{\lambda}{(d + \beta)^2} [(m + r)I - (d + \beta)R] \tag{10}$$

$$= wx - y$$

where $v = \frac{\alpha(d + \beta)^2}{\lambda^2}, K = \frac{A\lambda \epsilon}{d(d + \beta)}, u = \frac{d + m + r}{d + \beta}, w = \frac{m + r}{d + \beta}$

Note that, the trivial equilibrium $(0,0)$ of system (10) is the disease free equilibrium P_0 of model (2) and the unique positive equilibrium (x^*, y^*) of system (10) is the endemic equilibrium P^* of model (2) iff $u - K < 0$.

From equilibrium (10), we get

$$\frac{x}{1 + vx^2} (K - x - y) - ux = 0 \tag{11}$$

$$wx - y = 0 \tag{12}$$

From equation (11) & (12), we get $y^* = wx^* & K - x^* - y^* - u - uvx^{*2} = 0$

Using $y^* = wx^*$, then we get

$$K - x^* - wx^* - u - uvx^{*2} = 0$$

i.e., $uvx^{*2} + (1 + w)x^* + (u - K) = 0$

$$x^* = \frac{-(1 + w) \pm \sqrt{(1 + w)^2 - 4uv(u - K)}}{2uv} \tag{13}$$

$$y^* = wx^* \tag{14}$$

We first determine the stability and topological type of $(0, 0)$. The Jacobian matrix of system (10) at $(0, 0)$ is

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$$M_0 = \begin{bmatrix} K - u & 0 \\ w & -1 \end{bmatrix}$$

If $K - u = 0$, then there exist a small neighbourhood N_0 of $(0, 0)$ such that the dynamics of system (10) is equivalent to

$$\frac{dx}{dz} = -x^2 - xy + 0((x, y)^3)$$

$$\frac{dy}{dz} = wx - y$$

We know that $(0,0)$ is a saddle node. Hence, we obtain the following result.

Theorem:

The disease - free equilibrium $(0,0)$ of system (10) is

- i) a stable hyperbolic node if $u - K > 0$
- ii) a saddle node if $u - K = 0$
- iii) a hyperbolic saddle if $u - K < 0$

when $u - K < 0$, we discuss the stability and topological type of the endemic equilibrium (x^*, y^*) .

The Jacobian matrix of (10) at (x^*, y^*) is

$$M_1 = \begin{bmatrix} \frac{x^* (vx^{*2} + 2vwx^{*2} - 2Kvx^* - 1)}{(1 + vx^{*2})^2} & \frac{-x^*}{1 + vx^{*2}} \\ w & -1 \end{bmatrix}$$

We have

$$\det(M_1) = \frac{x^* [1 + w + 2Kvx^* - (1 + w)vx^{*2}]}{(1 + vx^{*2})^2}$$

The sign of $\det(M_1)$ is determined by $S_1 = (1+w)+2Kvx^* - (1+w)vx^{*2}$

Note that $uvx^{*2} + (1+w)x^* + u - K = 0$, we have

$$x^* = \frac{-(1+w) + \Delta_1}{2uv}, \text{ where } \Delta_1 = \sqrt{(1+w)^2 - 4uv(u-K)}, \text{ into } S_1 \text{ and using a straightforward calculation, we have}$$

$$uS_1 = -\Delta_1[(1+w)\Delta_1 - (1+w)^2 - 2uKv]$$

$$S_1 = \frac{\Delta_1(1+w)}{u} \left[-\Delta_1 + \frac{2uKv}{1+w} + (1+w) \right] = \frac{(1+w)\Delta_1}{u} \left[\frac{2uKv}{1+w} + (1+w) - \Delta_1 \right]$$

$$\begin{aligned} \text{Since, } \left((1+w) + \frac{2uKv}{1+w} \right)^2 - \Delta_1^2 &= \left(1+w + \frac{2uKv}{1+w} \right)^2 - [(1+w)^2 - 4u^2v + 4uKv] \\ &= (1+w)^2 + 2(1+w) \frac{2uKv}{1+w} + \frac{4u^2K^2v^2}{(1+w)^2} - (1+w)^2 + 4u^2v - 4uKv \end{aligned}$$

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$$= \frac{4u^2 K^2 v^2}{(1+w)^2} + 4u^2 v > 0$$

It follows that $S_1 > 0$. Hence $\det(M_1) > 0$ and (x^*, y^*) is a node or a focus or a centre. Furthermore, we have the following result on the stability of (x^*, y^*) .

Theorem:

Suppose $u - K < 0$, then there is a unique endemic equilibrium (x^*, y^*) of model (10), which is stable node.

Proof.

We know that the stability of (x^*, y^*) is determined by $Trace(M_1)$.

$$Trace(M_1) = \frac{-v^2 x^{*4} + (1+2w)x^{*3} - 2(K+1)vx^{*2} - x^* - 1}{(1+vx^{*2})^2}$$

The sign of $Trace(M_1)$ is determined by

$$P_1 = -v^2 x^{*4} + (1+2w)vx^{*3} - 2(I+K)vx^{*2} - x^* - 1$$

We claim that $P_2 \neq 0$

To see this, note that $uvx^{*2} + (1+w)x^* + u - K = 0$

Then we have, $u^3 v P_2 = (B_1 K + B_2)x^* - (B_3 K + B_4)$ where $B_1 = uv(2 + 3u + 2w + 4uw)$,

$B_2 = (1+w)[(1+w)^2 + u(1+w)(1+2w) - 2u^3 v]$, $B_3 = -(1+w)^2 - u(1+w)(1+2w) + 2u^3 v$

and $B_4 = u[(1+w)^2 + u(1+w)(1+2w) - v(1+2u)K^2]$

When $u - K < 0$, we can see that $B_1 K + B_2 > 0$.

Let $\xi = uvx^{*2} + (1+w)x^* + u - K$

Similarly, we have $(B_1 K + B_2)^2 \xi = u^3 v P_2 - P_3$ where P is a polynomial of x^* and

$$P_3 = u^3 v (1 + K^2 v + 2w + w^2) [(K + 2ku - 2u^2)^2 v + (1 + K - u + w)(1 + u + w + 2uw)]$$

Assume that $P_2 = 0$. Since $\xi = 0$, it follows that $P_3 = 0$. However, when $u - K < 0$, we have $P_3 > 0$.

Therefore, $P_2 \neq 0$ for any positive value of the parameters v, w and K , that is $Trace(M_1) \neq 0$. Thus $u - K < 0$ implies that (x^*, y^*) does not change stability.

Take $u=1, K=2, v=1, w=1$.

$$\text{Then } x^* = -1 + \sqrt{2}, y^* = -1 + \sqrt{2}, Trace(M_1) = -1.465 < 0$$

By the continuity of $Trace(M_1)$ on the parameters, we know that $Trace(M_1) < 0$ for $u - K < 0$. This completes the proof.

Theorem:

Let R_0 be defined by (A).

i) If $R_0 < 1$, then model (2) has a unique disease-free equilibrium.

$P_0 = (A/d, 0, 0, 0)$ which is global attractor in the first octant.

ii) If $R_0 = 1$, then model (2) has a unique disease-free equilibrium.

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$P_0 = (A/d, 0, 0, 0)$ which attracts all orbits in the interior of the first octant.

iii) If $R_0 > 1$, then model (2) has two equilibria, a disease-free equilibrium.

$P_0 = (A/d, 0, 0, 0)$ and an endemic equilibrium $P^* = (S^*, E^*, I^*, R^*)$. The endemic equilibrium P^* is a global attractor in the interior of the first octant.

Case (ii): SIR model with $I > I_0$ In this case the model is

$$\begin{aligned} \frac{dS}{dt} &= A - dS - \frac{\lambda SI}{1 + \alpha I^2} + \beta R \\ \frac{dE}{dt} &= \frac{\lambda SI}{1 + \alpha I^2} - (d + \epsilon)E \\ \frac{dI}{dt} &= \epsilon E - (d + m)I - K_1 \\ \frac{dR}{dt} &= mI - (d + \beta)R + K_1 \end{aligned} \tag{15}$$

Since $S+E+I+R=A/d$ is invariant manifold of the system (15), the model reduces to

$$\left. \begin{aligned} \frac{dI}{dt} &= \frac{A\lambda I(A/d - I - R)}{[\lambda I + (d + \epsilon)](1 + \alpha I^2)} - (d + m)I - K_1 \\ \frac{dR}{dt} &= mI - (d + \beta)R + K_1 \end{aligned} \right\} \tag{16}$$

Rescale the system by $x = \frac{\lambda I}{d + \beta}$, $y = \frac{\lambda R}{d + \beta}$, $T = (d + \beta)t$ we get

$$\frac{dx}{dT} = \frac{x}{1 + v_1 x^2} (K - x - y) - u_1 x - c \tag{17}$$

$$\frac{dy}{dT} = w_1 x - y + c \tag{18}$$

where $K = \frac{A\lambda \epsilon}{d(d + \beta)}$, $v_1 = \frac{\alpha(d + \beta)^2}{\lambda^2}$, $u_1 = \frac{d + m}{d + \beta}$, $w_1 = \frac{m}{d + \beta}$, $c = \frac{\lambda K_1}{(d + \beta)^2}$

For the equilibrium,

$$u_1 v x^3 + (1 + w_1 + c v_1) x^2 + (u_1 + c - K) x + c = 0 \tag{19}$$

If $u_1 + c > K$, equation (19) has no positive solution. But if $u_1 + c < K$, then it has either two positive roots or no positive roots. By the theory of equations,

$$P_0 x^3 + 3P_1 x^2 + 3P_2 x + P_3 = 0 \tag{20}$$

has all roots are real if the conditions $G^2 + 4H^3 < 0$ and $H < 0$.

Where $H = P_0 P_2 - P_1^2$ and $G = P_0^2 P_3 - 3P_0 P_1 P_2 + 2P_1^3$ are satisfied.

Comparing equation (19) and (20) we have

$$P_0 = u_1 v, P_1 = (1 + w_1 + c v_1) / 3, P_2 = (u_1 + c - K) / 3 \text{ and } P_3 = c.$$

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$$H = P_0 P_2 - P_1^2 = u_1 v [(u_1 c - K)/3] - [(1+w_1+cv_1)/3]^2 < 0,$$

For $u_1+c < K$ when $u_1+c < K$ and $H < 0$ and

$$\begin{aligned} G^2 + 4H^3 &= (P_0^2 P_3 - 3P_0 P_1 P_2 + 2P_1^3)^2 + 4(P_0 P_2 - P_1^2)^3 \\ &= P_0^2 (P_0^2 P_3^2 - 6P_0 P_1 P_2 P_3 + 4P_3 P_1^3 + 4P_0 P_2^3 - 3P_1^2 P_2^2) \end{aligned}$$

To fulfill the condition $G^2 + 4H^3 < 0$ we must have

$$(P_0^2 P_3^2 + 3P_3 P_1^3 + 4P_0 P_2^3) < (3P_1^2 P_2^2 + 6P_0 P_1 P_2 P_3) \tag{21}$$

To investigate the local stability of the positive equilibrium (\bar{x}, \bar{y}) of the system (17) & (18), we consider the Jacobian matrix.

$$M_2(\bar{x}, \bar{y}) = \begin{bmatrix} \frac{(1+v_1\bar{x}^2)(K-\bar{x}-\bar{y}-\bar{x})-2v_1\bar{x}(K\bar{x}-\bar{x}^2-\bar{x}\bar{y})}{1+v_1\bar{x}^2} - u_1 & \frac{-\bar{x}}{1+v_1\bar{x}^2} \\ w_1 & -1 \end{bmatrix}$$

$$\begin{aligned} \text{Now } \det(M_2) &= \frac{1}{(1+v_1\bar{x}^2)^2} [-(1+v_1\bar{x}^2)(K-2\bar{x}-w_1\bar{x}-c) \\ &+ 2v_1\bar{x}(K\bar{x}-\bar{x}^2-w_1\bar{x}^2-c\bar{x}) + u_1(1+v_1\bar{x}^2)^2 + w_1\bar{x}(1+v_1\bar{x}^2)] \end{aligned}$$

Solving this we get the sign of $\det(M_2)$ is governed by the factor

$$P_5 = -v(1+w_1+cv)\bar{x}^3 + v(2K+u_1-2c)\bar{x}^2 + (2+2w_1-vc)\bar{x} + (c+u_1-K) \tag{22}$$

We recall equation (19) and use it in (22). After some calculations we get,

$$\begin{aligned} u_1 P_5 &= \bar{x}^2 [2Ku_1 v + u_1^2 v - 2u_1 v c + (1+w_1+cv)^2] + \bar{x} [(2u_1+2u_1 w_1 - u_1 v c) + (1+w_1+cv)(u_1+c-K)] \\ &+ [(cu_1+u_1^2 - Ku_1) + c(1+w_1+cv)] \end{aligned}$$

$$\therefore u_1 P_5 = \xi_1 \bar{x}^2 + \xi_2 \bar{x} + \xi_3 \tag{23}$$

Where $\xi_i = \{u_1^2 v + (1+w_1+cv)^2 + 2u_1 v(K-c)\} > 0$ for $K > c$ (i.e.) $\xi_i > 0$ if $K < c$.

Hence with the above condition, sufficient condition for P_5 to be positive if $\xi_2^2 - 4\xi_1 \xi_3 \leq 0$ (24)

$$\text{Trace}(M_2) = \frac{\{(1+v\bar{x}^{-2})(K-2\bar{x}-w_1\bar{x}-c) - 2v\bar{x}^2(K-\bar{x}-w_1\bar{x}-c)\} - (u_1+1)(1+v\bar{x}^2)^2}{(1+v\bar{x}^2)^2}$$

Solving we get the sign of $\text{Trace}(M_2)$ is determined by the sign of

$$P_6 = -(u_1+1)v^2\bar{x}^4 + vw_1\bar{x}^3 + (vc-Kv-2vu_1-2v)\bar{x}^2 - (2+w_1)\bar{x} + (K-c-u_1-l) \tag{25}$$

After some algebraic calculation using (19) and (25) we get

$$u_1^2 P_6 = \eta_1 \bar{x}^2 + \eta_2 \bar{x} + \eta_3$$

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$$\eta_1 = -[(I + w_1 + cv)(2u_1w_1 + u_1cv + u_1 + w_1 + cv + I) + u_1v(K - c)(I + 2u_1) + u_1^2 + u_1] < 0 \text{ for } K > u_1 + c,$$

$$\eta_2 = u_1(cvu_1 + cv - 2u_1 - w_1u_1) - (u_1 + c - K)(2u_1w_1 + u_1cv + u_1 + w_1 + cv + I)$$

$$\eta_3 = u_1(K - c - u_1 - I) - c(2u_1w_1 + u_1cv + u_1 + w_1 + cv + I)$$

Therefore the sufficient condition for which $P_6 < 0$ is

$$\eta_2^2 - 4\eta_1\eta_3 \leq 0 \tag{26}$$

We conclude the above finding in the following theorem.

Theorem:

When $K > u_1 + c$, the system (17) – (18) has two positive equilibrium points (\bar{x}_1, \bar{y}_1) and (\bar{x}_2, \bar{y}_2) , where \bar{x}_1 and \bar{x}_2 are the solution of the equation (19) under the parametric conditions given by (21) and when the conditions (24) and (26) are fulfilled at some equilibrium point that point must be asymptotically stable.

V. NUMERICAL SIMULATION AND DISCUSSIONS

Case (i): When the treatment rate is ∞ to the infective so that $0 \leq I \leq I_0$.

We choose the parameters in the model as follows.

$$A = 1.0, \lambda = 0.5, d = 0.2, m = 0.5, r = 0.2, \alpha = 0.1, \varepsilon = 0.1$$

Here the basic reproduction number $R_0 = 1.666 > 1$. The equilibrium position goes lower and when the new parameter α increases. Simultaneously when we increase the treatment function r we see that $\alpha = 4$ and $r = 1.5$ susceptible population increases significantly.

When $\alpha = 1, A = 15, \beta = 0.1, d = 2, m = 10, r = 0.1, \lambda = 0.5, \varepsilon = 0.1$ we have $R_0 = 0.0150 < 1$.

In this case the disease dies out.

Consider the values of the parameters

$$A = 5, d = 0.1, \alpha = 0.1, \beta = 0.4, \lambda = 1, m = 0.01, r = 0.2, u = 0.55, w = 0.05$$

By rescaling the system we see that $u - K < 1$. Hence there exists the unique positive equilibrium $x^* = 3.9$ and $y^* = 0.195$.

Case (ii): When $I > I_0$, the parameters are $A = 2.8, d = 0.0453, \alpha = 2, v = 0.13, K = 0.4, \gamma = 0.05, K_1 = 0.7,$

$$\varepsilon = 0.2, r = 0.01$$

In this case $S + E + I + R = 61.8102$ is invariant manifold. The system reduces to

$$\frac{dI}{dt} = \frac{0.08I(61.8102 - I - R)}{(0.4I + 0.2453)(1 + 2I^2)} - 0.0853I - 0.7$$

The rescaling system reduces to

$$\frac{dx}{dz} = \frac{x(28.3544 - x - y)}{1 + 0.0614x^2} - 0.5436x - 9.1205$$

$$\frac{dy}{dz} = 0.7416x - y + 9.1205$$

Here $u - K < 0$ and hence there exists unique positive equilibrium point (x^*, y^*) where $x^* = 3.9$ and $y^* = 0.195$. In order to see the equilibrium point we have $u_1 + c < K - cv_1$ and $u_1v_1 < cv_2$. Thus the above equation has two positive roots. It can be easily calculate that increasing values of α slightly lowers down (x_1, y_1) whereas pushes up

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the pair (x_2, y_2) .

V.CONCLUSION

The SEIR epidemic model is proposed to simulate the limited resources for the treatment of patients, which can occur as a consequence of lack of medications or limited beds in hospitals. As the treatment rate plays a major role to control the disease, this model is used to find the capacity of the treatment. If the treatment rate I_0 is smaller than I , then the hospitals will have to offer constant treatment to all the infective individuals. This model determines the basic reproduction number R_0 . When $R_0 \leq 1$, there exists no positive equilibrium and the disease-free equilibrium is globally asymptotically stable, that is the disease dies out. When $R_0 > 1$, the disease-free equilibrium becomes unstable and the disease persists.

REFERENCES

- [1] Anderson R.M, May R.N : Population Biology of Infectious Disease, I. Nature, 316-367, 1979.
- [2] Capasso V and SerioG : A Generalization of the Kermack-Mckindrick Deterministic Epidemic Model, Math.Bio.Sci.42, 43-61, 1978.
- [3] Diekmann O. and Heesterbeek J. A. P.,Mathematical Epidemiology of Infectious Diseases, Model buildinganalysis and interpretation, John Wiley & Sons, 2000.
- [4] GajendraUjjainkar, Gupta V.K, Singh B, KhandeludR : An Epidemic Model with Modified Non-monotonicIncidence Rate under Treatment, Vol. 6,1159-1171, 2012.
- [5] Hethcote H. W.,The mathematics of infectious diseases. SIAM review, 42, 599-653, 2000.
- [6] Jasmine D. and Henry Amirtharaj E.C., Modeling and Simulation of Modified SIR Epidemic Model with Immigration and Non-monotonic Incidence under Treatment, Indian Journal of Applied Research, 3(7), 43-44, 2013.
- [7] Jasmine D. and Henry Amirtharaj E.C., Global Analysis of SIR Epidemic Model with Immigration and Non-monotonic Incidence under Treatment, International Journal of Applied Mathematics and Statistical Research , Vol.2(5), 83-92, 2013.
- [8] Kar T.K and AshimBatabyal : Modeling and Analysis of an Epidemic Model with Non-monotonic Incidence Rate under Treatment, Journal of Mathematical Research Vol.2, No.1, 103-115, 2010.
- [9] Ruan S and Wang W: Dynamical Behavior of an Epidemic Model with Non-linear Incidence Rate. J.DifferentialEquations,1S, 135-163, 2003.
- [10] Van Den P. Driessche and Watmough J. Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission, Math. Bioscience, 18029-48, 2002.
- [11] Wendi& Wang: Backward Bifurcation of an Epidemic Model with Treatment, J. Math, Bio. Sci.201,58-71, 2006.
- [12] Xiao D and RuanS : Global Analysis of an Epidemic Model with Non-monotone Incidence Rate. Math..Bio Science, 208, 419-429, 2007. International Journal of Applied Mathematics and Statistical Research ,Vol.2(5), 83-92, 2013.