

Application of Homotopy Analysis Method for Solving Higher Order Differential Equations

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ABSTRACT: The present paper discusses higher order boundary value and initial value problems and their solution by using homotopy analysis method. Homotopy analysis method applied successfully on this type of problems. Numerical calculation has been completed using MATHEMATICA.

KEYWORDS: Homotopy analysis method, higher order ordinary differential equation, boundary value problem, initial value problem

I. INTRODUCTION

Higher order boundary value problems occur in the study of fluid dynamics, astrophysics, hydrodynamic, hydro magnetic stability, astronomy, beam and long wave theory, induction motors, engineering, and applied physics. The boundary value problems of higher order have been examined due to their mathematical importance and applications in diversified applied sciences.

The seventh-order boundary value problems generally arise in modelling induction motors with two rotor circuits. The induction motor behaviour is represented by a fifth-order differential equation model. This model contains two stator state variables, two rotor state variables, and one shaft speed. Normally, two more variables must be added to account for the effects of a second rotor circuit representing deep bars, a starting cage, or rotor distributed parameters. To avoid the computational burden of additional state variables when additional rotor circuits are required, model is often limited to the fifth-order and rotor impedance is algebraically altered as function of rotor speed under the assumption that the frequency of rotor currents depends on rotor speed. This approach is efficient for the steady state response with sinusoidal voltage, but it does not hold up during the transient conditions, when rotor frequency is not a single value. So, the behaviour of such models shows up in the seventh order.

Siddiqi and Iftikhar used the variation of parameter method for solving the seventh-order boundary value problems in [7]. Liu and Wu [6] give the general differential quadrature rule (GDQR) for the solution of eighth-order differential equation. Explicit weighting coefficients are formulated to implement the GDQR for eighth-order differential equations. Siddiqi and Akram [8] used nonic spline and non polynomial spline technique for the numerical solution of eighth-order linear special case boundary value problems.

In the present paper, the seventh, eighth and tenth order boundary value problems are solved using the homotopy analysis method (HAM). The following seventh, eighth and tenth-order boundary value problems are considered:

II. HOMOTOPY ANALYSIS METHOD

We consider the following differential equations,

$$N_i[S_i(x, t)] = 0, i = 1, 2, \dots, n$$

Where N_i are nonlinear operators that represents the whole equations, x and t are independent variables and $S_i(x, t)$ are unknown functions respectively.

By means of generalizing the traditional homotopy method, Liao constructed the so-called zero-order deformation equations

$$(1 - q)L[\Phi_i(x, t; q) - S_{i,0}(x, t)] = qh_i N_i[\Phi_i(x, t; q)] \quad (1)$$

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Where $q \in [0,1]$ is an embedding operators, h_i are nonzero auxiliary functions, L is an auxiliary linear operator, $S_{i,0}(x, t)$ are initial guesses of $S_i(x, t)$ and $\Phi_i(x, t, ; q)$ are unknown functions.

It is important to note that, one has great freedom to choose auxiliary objects such as h_i and L in HAM.

When $q = 0$ and $q = 1$ we get by (1),

$$\Phi_i(x, t, ; 0) = S_{i,0}(x, t) \text{ and } \Phi_i(x, t, ; 1) = S_i(x, t)$$

Thus q increase from 0 to 1, the solutions $\Phi_i(x, t, ; q)$ varies from initial guesses $S_{i,0}(x, t)$ to $S_i(x, t)$.

Expanding $\Phi_i(x, t, ; q)$ in Taylor series with respect to ,

$$\Phi_i(x, t, ; q) = S_{i,0}(x, t) + \sum_{m=1}^{\infty} S_{i,m}(x, t) \cdot q^m \tag{2}$$

Where

$$S_{i,m}(x, t) = \left[\frac{1}{m!} \cdot \frac{\partial^m \Phi_i(x, t, ; q)}{\partial q^m} \right]_{q=0}, \tag{3}$$

If the auxiliary linear operator, initial guesses, the auxiliary parameter h_i and auxiliary functions are properly chosen than the series equation (2) converges at $q = 1$.

$$\Phi_i(x, t, ; 1) = S_{i,0}(x, t) + \sum_{m=1}^{\infty} S_{i,m}(x, t) \tag{4}$$

This must be one of solutions of the original nonlinear equations.

According to (3), the governing equations can be deduced from the zero-order deformation equations (1).

Define the vectors

$$\vec{S}_{i,n} = \{S_{i,0}(x, t), S_{i,1}(x, t), S_{i,2}(x, t), \dots, S_{i,n}(x, t)\}$$

Differentiating (1) m times with respect to the embedding parameter q and the setting $q = 0$ and finally dividing them by $m!$.

We have the so-called m^{th} order deformation equations

$$L[S_{i,m}(x, t) - \chi_m S_{i,m-1}(x, t)] = h_i R_{i,m}(\vec{S}_{i,m-1}) \tag{5}$$

Where

$$R_{i,m}(S_{i,m-1}) = \left[\frac{1}{(m-1)!} \cdot \frac{\partial^{m-1} N_i[\Phi_i(x, t, ; q)]}{\partial q^{m-1}} \right]_{q=0} \tag{6}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases}$$

III. NUMERICAL EXAMPLES

Example-1

Consider the seventh order boundary value problem

$$u^{(7)}(x) = u(x) - e^x(35 + 12x + 2x^2), \quad 0 \leq x \leq 1 \tag{7}$$

$$u(0) = 0, \quad u(1) = 0, \quad u^{(1)}(0) = 1, \quad u^{(1)}(1) = -e, \quad u^{(2)}(0) = 0, \quad u^{(2)}(1) = -4e, \quad u^{(3)}(0) = -3, \tag{8}$$

Exact solution of this is

$$u(x) = x(1-x)e^x$$

Solution

Using 7th order deformation equation,

$$(1-q)L[\phi(x; q) - S_0(x)] - qhH[S^{(7)}(x) + S(x) + e^x(35 + 12x + 2x^2)] = 0 \tag{9}$$

, where $q \in [0,1]$

Now the initial approximation $S_0(x)$ is the solution of $u^{(7)}(x)$ with given boundary condition. So

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$$S_0(x) = x - \frac{x^3}{2} + \frac{1}{2}(-17 + 6e)x^4 + \frac{1}{2}(-27 + 10e)x^5 + \frac{1}{2}(-11 + 4e)x^6 \quad (10)$$

The linear operator L normally consists of the homogeneous part of non linear operator N where parameter h and function $H(x)$ are introduced in order to optimize the initial guess. Try to choose h in such a way that they get convergent series.

Hence m^{th} -order deformation can be given by

$$L[S_m(x) - \chi_m S_{m-1}(x)] = hH(x)R_m(\bar{S}_{m-1}) \quad (11)$$

Where

$$R_1(\bar{S}_0) = \frac{\partial^7 S_0(x; q)}{\partial q^7} + S_0(x; q) + e^x(35 + 12x + 2x^2) \quad (12)$$

$$R_m(\bar{S}_{m-1}) = \frac{\partial^7 S_{m-1}(x; q)}{\partial q^7} + S_{m-1}(x; q), m \geq 2 \quad (13)$$

Now the solution of m^{th} -order deformation equations (3.6) and (3.7) for $m \geq 1$ becomes

$$S_m(x) = \chi_m S_{m-1}(x) + hL^{-1}[R_m(\bar{S}_{m-1})] \quad (14)$$

Consequently, the first few terms of the homotopy series solution are as follows:

$$S_0(x) = x - \frac{x^3}{2} + \frac{1}{2}(-17 + 6e)x^4 + \frac{1}{2}(-27 + 10e)x^5 + \frac{1}{2}(-11 + 4e)x^6 \quad (15)$$

$$S_1(x) = \frac{h}{518918400} \begin{bmatrix} -32691859200 + 32691859200e^x - 24389164800x \\ -8302694400xe^x - 9081072000x^2 + 1037836800e^x x^2 \\ -2335132800x^3 + 795242426295x^4 - 292929440550e^x x^4 \\ -1230096003906x^5 + 452496853920e^x x^5 + 502831877247x^6 \\ -184994415376e^x x^6 + 518918400x^7 + 12870x^8 - 429x^{10} \\ -2652x^{11} + 936e^x x^{11} + 1735x^{12} - 650e^x x^{12} - 330x^{13} + 120e^x x^{13} \end{bmatrix} \quad (16)$$

The N^{th} order approximation of $u(x)$ can be expressed by

$$S_N(x) = S_0(x) + \sum_{m=1}^{N-1} S_m(x) \quad (17)$$

Example-2

Consider the seventh order boundary value problem

$$u^{(7)}(x) = u(x)u'(x) + e^{-2x}(2 + e^x(x-8) - 3x + x^2), \quad 0 \leq x \leq 1 \quad (18)$$

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$$\begin{aligned}
 u(0) = 0, \quad u(1) = 0, \quad u^{(1)}(0) = -2, \quad u^{(1)}(1) = -1/e, \quad u^{(2)}(0) = 3, \\
 u^{(2)}(1) = 2/e, \quad u^{(3)}(0) = -4,
 \end{aligned}
 \tag{19}$$

Exact solution of this is

$$u(x) = (1-x)e^{-x}$$

Solution

Using 7th order deformation equation,

$$(1-q)L[\phi(x;q) - S_0(x)] - qhH \left[\begin{array}{l} S^{(7)}(x) + S(x)S^{(1)}(x) \\ + e^{-2x}(2 + e^x(x-8) - 3x + x^2) \end{array} \right] = 0
 \tag{20}$$

, where $q \in [0,1]$

Now the initial approximation $S_0(x)$ is the solution of $u^{(7)}(x)$ with given boundary condition. So

$$S_0(x) = 1 - 2x + \frac{3x^2}{2} - \frac{2x^3}{3} + \frac{(36-12e)x^4}{6e} + \frac{(66-24e)x^5}{6e} + \frac{(30-11e)x^6}{6e}
 \tag{21}$$

The linear operator L normally consists of the homogeneous part of non linear operator N where parameter h and function $H(x)$ are introduced in order to optimize the initial guess. Try to choose h in such a way that they get convergent series.

Hence m^{th} -order deformation can be given by

$$L[S_m(x) - \chi_m S_{m-1}(x)] = hH(x)R_m(\bar{S}_{m-1})
 \tag{22}$$

Where

$$R_1(\bar{S}_0) = \frac{\partial^7 S_0(x;q)}{\partial q^7} + S_0(x;q) \frac{\partial S_0(x;q)}{\partial q} + e^{-2x}(2 + e^x(x-8) - 3x + x^2)
 \tag{23}$$

$$R_m(\bar{S}_{m-1}) = \frac{\partial^7 S_{m-1}(x;q)}{\partial q^7} + S_{m-1}(x;q) \frac{\partial S_{m-1}(x;q)}{\partial q}, \quad m \geq 2
 \tag{3.17}$$

Now the solution of m^{th} -order deformation equations (3.16) and (3.17) for $m \geq 1$ becomes

$$S_m(x) = \chi_m S_{m-1}(x) + hL^{-1}[R_m(\bar{S}_{m-1})]
 \tag{24}$$

Consequently, the first few terms of the homotopy series solution are as follows:

$$S_0(x) = 1 - 2x + \frac{3x^2}{2} - \frac{2x^3}{3} + \frac{(36-12e)x^4}{6e} + \frac{(66-24e)x^5}{6e} + \frac{(30-11e)x^6}{6e}
 \tag{25}$$

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$$S_1(x) = \frac{h}{52929676800} \left[\begin{aligned} &2274322050e^2 - 52929676800e^{2+x} + 50655354750e^{2+2x} + 1654052400e^2x \\ &+ 52929676800e^{2+x}x - 102964761900e^{2+2x}x + 413513100e^2x^2 \\ &+ 77740462800e^{2+2x}x^2 - 3473500400e^{2+2x}x^3 \\ &- 100276694820e^{2x}x^4 + 317578525200e^{1+2x}x^4 - \\ &9242637280e^{2+2x}x^4 + 1682992519e^{2x}x^5 - 58222791918e^{1+2x}x^5 \\ &+ 188997561564e^{2+2x}x^5 - 72364427100e^{2x}x^6 \\ &+ 264649706520e^{1+2x}x^6 - 87281523925e^{2+2x}x^6 + 21003840e^{2+2x}x^7 \\ &- 9189180e^{2+2x}x^8 + 3208920e^{2+2x}x^9 - 2100384e^{1+2x}x^{10} \\ &+ 160446e^{2+2x}x^{10} + 3659760e^{1+2x}x^{11} - 1113840e^{2+2x}x^{11} \\ &- 2864160e^{1+2x}x^{12} + 1003340e^{2+2x}x^{12} + 1306620e^{1+2x}x^{13} \\ &- 471240e^{2+2x}x^{13} - 440640e^{2x}x^{14} + 69360e^{1+2x}x^{14} + 83640e^{2+2x}x^{14} \\ &+ 969408e^{2x}x^{15} - 626688e^{1+2x}x^{15} + 99552e^{2+2x}x^{15} - 830790e^{2x}x^{16} \\ &+ 596700e^{1+2x}x^{16} - 107100e^{2+2x}x^{16} + 326700e^{2x}x^{17} - 238590e^{1+2x}x^{17} \\ &+ 43560e^{2+2x}x^{17} - 49500e^{2x}x^{18} + 36300e^{1+2x}x^{18} - 6655e^{2+2x}x^{18} \end{aligned} \right] \quad (26)$$

The N^{th} order approximation of $u(x)$ can be expressed by

$$S_N(x) = S_0(x) + \sum_{m=1}^{N-1} S_m(x) \quad (27)$$

IV. EXERCISE

Here we have some more unsolved example which can solve by using Homotopy Analysis Method.

Example-I

Consider the seventh order boundary value problem

$$u^{(7)}(x) = x \cdot u(x) - e^x(x^2 - 2x - 6), \quad 0 \leq x \leq 1$$

$$u(0) = 0, \quad u(1) = 0, \quad u^{(1)}(0) = 0, \quad u^{(1)}(1) = -e, \quad u^{(2)}(0) = -1, \quad u^{(2)}(1) = -2e, \quad u^{(3)}(0) = -2,$$

Exact solution of this is

$$u(x) = (1-x)e^x$$

Example-II

Consider the eighth order boundary value problem

$$u^{(8)}(x) = -x \cdot u(x) - e^x(48 + 15x + 2x^3), \quad 0 \leq x \leq 1$$

$$u(0) = 0, \quad u(1) = 0, \quad u^{(1)}(0) = 1, \quad u^{(1)}(1) = -e, \quad u^{(2)}(0) = 0, \quad u^{(2)}(1) = -4e, \quad u^{(3)}(0) = -3, \quad u^{(3)}(1) = -9e$$

Exact solution of this is

$$u(x) = x(1-x)e^x$$

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Example-III

Consider the eighth order boundary value problem

$$u^{(8)}(x) = u(x) - 8(2x \cos(x) + 7 \sin(x)), \quad -1 \leq x \leq 1$$

$$u(-1) = 0, \quad u(1) = 0, \quad u^{(1)}(-1) = u^{(1)}(1) = 2 \sin(1), \quad u^{(2)}(-1) = u^{(2)}(1) = -4 \cos(1) - 2 \sin(1),$$

$$u^{(3)}(-1) = u^{(3)}(1) = 6 \cos(1) - 6 \sin(1)$$

Exact solution of this is

$$u(x) = (x^2 - 1) \sin(x)$$

Example-IV

Consider the tenth order boundary value problem

$$u^{(10)}(x) = xu(x) - e^x(89 + 21x + x^2 - x^3), \quad -1 \leq x \leq 1$$

$$u(-1) = 1, \quad u(1) = 0, \quad u^{(1)}(-1) = 2/e, \quad u^{(1)}(1) = -2e, \quad u^{(2)}(-1) = 2/e,$$

$$u^{(2)}(1) = 6e, \quad u^{(3)}(-1) = 0, \quad u^{(3)}(1) = -12e, \quad u^{(4)}(-1) = -4/e, \quad u^{(4)}(1) = -20e$$

Exact solution of this is

$$u(x) = (1 - x^2)e^x \quad [1]$$

V. CONCLUSION

In this paper, the homotopy analysis method (HAM) has been applied to obtain the numerical solutions of seventh-order boundary value problems. The numerical results show that only a few number of approximations can be used for numerical purpose with a high degree of accuracy. It is also observed that our proposed method is well suited for the solution of higher order boundary value problems and reduces the computational work. HAM converges to exact solutions more rapidly as compared to the other method. Therefore, the present method is an accurate and reliable analytical technique for boundary value problems.

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