

Topological Structures on BCH-Algebras

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ABSTRACT. There has been some recent interest in applying topological notions to non-mainstream algebras [16, 17]. In [1], the authors introduce topological notions into MV-algebras. In their paper D.S. Lee and D. N. Ryu [2], initiates the study of topological BCK-algebras. Motivated by this, in this paper we define the notion of topological BCH-algebra, give some examples and prove some important theorems.

KEYWORDS: positive implicative BCH-algebra, TBCH-algebra, LS-topology, LSTBCH-algebra.

I. INTRODUCTION

Y. Imai and K. Iseki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([3, 4, 5, 6]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In ([12, 13]) Q. P. Hu and X. Li introduced a wider class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. K. H. Dar introduced the notions of left and right mappings on BCK-algebras in [9]. Further in [10], the authors discussed notions of endomorphisms on BCH-algebras. Left and Right mappings over BCI-algebras have been discussed in [11]. In [8] the authors discussed some more properties of left and right mappings on BCH-algebras.

In [14], the authors studied topological BL-algebras and proved some theorems that determine the relationship between them. There has been some recent interest in applying topological notions to non-mainstream algebras [16, 17]. In [15], Ahn and Kwon addresses the issue of attacking to BCC-algebras in as natural a manner as possible. Motivated by this, in this paper, we study the connection between topology and BCH-algebras.

II. PRELIMINARIES

In this section, we recall some basic definitions and results that are required for our work.

Definition 2.1: A BCH-algebra $(X, *, 0)$ is a non empty set X with a constant 0 and a binary operation satisfying the following conditions:

- (1) $x * x = 0$
- (2) $(x * y) * z = (x * z) * y$
- (3) $x * y = 0$ and $y * x = 0 \Rightarrow x = y \forall x, y, z \in X$.

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Example 2.2: Let $(X = \{0, 1, 2, 3\}, *, 0)$ be a set with the following cayley table,

*	0	1	2	3
0	0	0	2	2
1	1	0	2	2
2	2	2	0	0
3	3	3	1	0

Then $(X, *, 0)$ is a BCH-algebra.

Definition 2.3: Let X be a BCH-algebra. If X satisfies the condition,

$$(x*y)*z = (x*y)*(y*z), \text{ for all } x, y, z \text{ in } X$$

Then X is called positive implicative BCH-algebra.

Definition 2.4: [10] Let $(X, *)$ be a BCH-algebra, and $a \in X$. Define a left map $L_a : X \rightarrow X$ by, $L_a(x) = a*x$, for all $x \in X$.

Definition 2.5: [10] Let $(X, *)$ be a BCH-algebra, and $a \in X$. Define a right map $R_a : X \rightarrow X$ by, $R_a(x) = x*a$, for all $x \in X$.

Definition 2.6: [10] Let $(X, *)$ be a BCH-algebra.

The set of all left maps on X is defined as, $L(X)$.

III. TOPOLOGICAL STRUCTURES ON BCH-ALGEBRAS

In this section we discuss the topological properties in the set of all left maps on a BCH-algebra. And we define, what is a TBCH-algebra, LS-topology and LSTBCH-algebra. Finally, we discuss some algebraic properties on TBCH-algebra.

Definition 3.1: Let X be a BCH-algebra and U, V be any non empty subsets of X . We define a subset $U*V$ of X as follows, $U*V = \{x*y / x \in U, y \in V\}$.

Definition 3.2: Let $(X = \{0, a, b, c\}, *, 0)$ be a BCH-algebra the binary operation $*$ will be defined as follows,

*	0	a	b	c
0	0	0	0	0
a	a	0	b	b
b	b	b	0	0
c	c	b	a	0

Let $U = \{0, a\}$ and $V = \{a, b\}$ be two subsets of X .

$U*V = \{0*a, 0*b, a*a, a*b\} = \{0, 0, 0, b\} = \{0, b\}$ subset of X .

Definition 3.3: Let $(X, *)$ be a BCH-algebra. Let τ be the collection of subsets of X , τ is said to be a topology on X if

- (1) $X, \emptyset \in \tau$.

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- (2) arbitrary union of members of τ is in τ .
- (3) finite intersection of members of τ is in τ .

Definition 3.4: Let $(X, *)$ be a BCH-algebra and ι a topology on X . Then $X = (X, *, \tau)$ is called a topological BCH-algebra, if the operation ‘*’ is continuous, or equivalently, for any $x, y \in X$ and for any open set W of $x*y$ there exist two open sets U and V respectively such that $U*V$ is a subset of W .

Example 3.5: Let $(X = \{0, 1, 2, 3\}, *, 0)$ be a BCH-algebra the binary operation * will be defined as,

*	0	1	2	3
0	0	0	0	0
1	1	0	0	2
2	2	2	0	0
3	3	3	3	0

Define a topology $\iota = \{\emptyset, X, \{2\}, \{3\}, \{2, 3\}, \{0, 1\}, \{0, 1, 2\}, \{0, 1, 3\}\}$

One can easily prove, $(X, *, \tau)$ is a TBCH-algebra.

Example 3.6: Let $(X = \{0, 1, 2, 3\}, *, 0)$ be a BCH-algebra the binary operation * will be defined as follows,

*	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	2	0	2
3	3	3	3	0

The topology $\tau = \{\emptyset, X, \{2\}, \{2, 3\}\}$

Let $2, 0 \in X, 2*0 = 2 = \{2\} = W$

$U = \{2\}$ and $V = X$

Clearly, $\{2\} * X = X \not\subseteq \{2\}$.

Therefore, $(X, \tau, *)$ is not a topological BCH-algebra. But $(X, \tau, *)$ is both a topological space and BCH-algebra.

Remark 3.7: The members of τ are called an open set in X . The complement of $A \in \tau$, that is $X \setminus A$ is called a closed set in X , denoted by \bar{A} . The open set of an element $x \in X$, is a member of τ containing x .

Definition 3.8: Let X and Y be a TBCH-algebra. A map $f : X \rightarrow Y$ is continuous, if for every $x \in X$ and any open set W of $f(x)$, there exist an open set V of x such that $f(V)$ subset of W .

Example 3.9: Let $(X = \{0, 1, 2, 3\}, \tau, 0)$ and $(Y = \{0, 1, 2\}, *, 0)$ are the BCH-algebras the binary operation * will be defined on X and Y are as follows respectively. The topology $\tau_1 = \{\emptyset, X, \{1, 0\}, \{2, 3\}\}$ is on X and the topology $\tau_2 = \{\emptyset, X, \{1, 0\}, \{2\}\}$ is on Y .

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Vol. 6, Issue 12, December 2017

*	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

*	0	1	2
0	0	0	2
1	1	0	2
2	2	2	0

The map $f: X \rightarrow Y$ is defined by, $f(x) = 1$, for all $x \in X$ is continuous.

Definition 3.10: Let X be a TBCH-algebra, and $x \in X$. Define a left map $L_a : X \rightarrow X$ by, $L_a(x) = x*y$, for all $y \in X$.

Example 3.11: Let $(X = \{0, 1, 2, 3\}, *, 0)$ be a TBCH-algebra the binary operation $*$ will be defined as follows,

*	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

$\tau = \{\emptyset, X, \{1, 0\}, \{2, 3\}\}$ is a topology on X .

For $x \in X$, the map $L_2 : X \rightarrow Y$ is a left map defined by, $L_2(x) = 2*x$.

Theorem 3.12: Let X be a TBCH-algebra. Every left map on X is continuous.

Proof: Let $x \in X$. Define a left map $L_x: X \rightarrow X$ by, $L_x(y) = x*y$, for all $y \in X$.

Since X is TBCH, for every x, y in X and any open set W of $x*y$, there exist a two open sets U and V of x and y respectively, such that $U*V$ is a subset of W .

Clearly, $x*V \subseteq U*V$. $L_x(V)$ subset of W . Since x is arbitrary, L_x is continuous.

Definition 3.13: Let X be a BCH-algebra. The binary operation \odot will be defined on $L(X)$ is $(L_a \odot L_b)(x) = L_a(x) * L_b(x)$, for all $x \in X$.

Example 3.14: Let $(X = \{0, 1, 2, 3\}, *, 0)$ be a BCH-algebra, with the following cayley table,

*	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	2	0	0
3	3	3	3	0

$L(X) = \{L_0, L_1, L_2, L_3\}$. Let $L_1, L_2 \in L(X)$, $L_1 \odot L_2 = L_0$.

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Theorem 3.15: Let X be a positive implicative BCH-algebra, then $(L(X), \odot, L_0)$ is a BCH-algebra.

Proof: Let $L_a, L_b \in L(X)$. Consider, $(L_a \odot L_b)(x) = L_a(x) * L_b(x) = (a * x) * (b * x)$. Since X is positive implicative, $(a * x) * (b * x) = (a * b) * x$. Then $L_a \odot L_b = L_{a * b}, \forall a, b$ in X .

- (1) $L_a \odot L_a = L_{a * a} = L_0$.
- (2) $(L_a \odot L_b) \odot L_c = L_{((a * b) * c)} = L_{(a * c) * b} = (L_a \odot L_b) \odot L_c$.
- (3) If $L_a \odot L_b = L_0$ and $L_b \odot L_a = 0$
 $\Rightarrow L_{a * b} = L_0$ and $L_{b * a} = L_0$.
 $\Rightarrow a = b. \Rightarrow L_a = L_b$.

Here, $(L(X), \odot)$ satisfies the BCH conditions. Then $(L(X), \odot, L_0)$ is a BCH-algebra.

Remark 3.16: If X is both a positive implicative BCH-algebra and a topological BCH-algebra then X is called a positive implicative TBCH-algebra.

Definition 3.17: Let X is a positive implicative TBCH-algebra and A be any subset of X . Define, $L_A = \{L_a \mid a \in A\}$.

Definition 3.18: Let X be a TBCH-algebra. Define a map $\Phi: X \rightarrow L(X)$ by, $\Phi(x) = L_x$, for all $x \in X$.

Remark 3.19: Let X is a positive implicative TBCH-algebra, and A, B be any subset of X . If $A \subseteq B$ then $\Phi(A) = \Phi(B)$. Let X is a positive implicative TBCH-algebra and G_1, G_2 be any subset of X . If $G = G_1 \cup G_2$ then $\Phi(G) = \Phi(G_1) \cup \Phi(G_2) = \Phi(G_1 \cup G_2)$.

Theorem 3.20: The map Φ is a BCH-isomorphism.

Proof: Clearly, Φ is one-one and onto.

Consider, $\Phi(x * y) = L_{x * y}$
 $\Rightarrow L_{x * y}(t) = (x * y) * t = (x * t) * (y * t)$
 $\Rightarrow \Phi(x) \odot \Phi(y),$ for all x, y in X .

Hence, Φ is a BCH-isomorphism.

Theorem 3.21: Let $(X, *, \tau)$ be a positive implicative TBCH-algebra, then $\tau' = \{\Phi(G) \mid G \in \tau\}$ is a topology on $L(X)$.

Proof: It is trivial that, $L(X)$ and $\emptyset \in \tau'$.

Let $\{\Phi(G_i) \mid i \in I\} \in \tau'$, implies that $\{G_i \mid i \in I\} \in \tau$. Then $\cup_{i \in I} G_i \in \tau$. Therefore, $\cup_{i \in I} \Phi(G_i) \in \tau'$. Similarly, we can prove a finite intersection of elements of τ' is in τ' .

Remark 3.22: Let $(X, *, \tau)$ be a positive implicative TBCH-algebra, and the collection of subsets of $L(X), \tau' = \{L_G \subseteq L(X) \mid G \in \tau\}$ is called a LS-topology on the set $L(X)$.

Example 3.23: Let $(X = \{0, 1, 2, 3\}, *, \tau)$ be a positive implicative TBCH-algebra the binary operation $*$ will be defined as follows,

*	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	2	0	2
3	3	3	3	0

$\tau = \{\emptyset, X, \{1, 0\}, \{0, 1, 3\}, \{0, 1, 2\}, \{2\}, \{3\}, \{2, 3\}\}$ is a topology on X .

Now, $L(X) = \{L_0, L_1, L_2, L_3\}$ and the LS-topology

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Vol. 6, Issue 12, December 2017

$$\tau' = \{\emptyset, L(X), \Phi(\{1, 0\}), \Phi(\{0, 1, 3\}), \Phi(\{0, 1, 2\}), \Phi(\{2\}), \Phi(\{3\}), \Phi(\{2, 3\})\}.$$

Theorem 3.23: Let X be a positive implicative TBCH-algebra. Then $(L(X), \odot, \tau')$ is a TBCH-algebra.

Proof: Let $x, y \in X$, then $\Phi(x), \Phi(y) \in L(X)$.

Since X is TBCH, for any set W of $x*y$, there exist a two open sets U and V of x and y respectively such that $U * V \subseteq W$.

From definition 3.18, and above theorem, every open set $\Phi(W)$ of $L_x \odot L_y$, there exist a two open set $\Phi(U)$ and $\Phi(V)$ of L_x and L_y respectively, such that $\Phi(U * V) \subseteq \Phi(W)$. Therefore, \odot is continuous. Hence, $(L(X), \odot, \tau')$ is a TBCH-algebra.

Example 3.25: Consider a positive implicative TBCH-algebra $(X = \{0, 1, 2, 3\}, *, \tau)$ with the following cayley table,

*	0	1	2
0	0	0	0
1	1	0	1
2	2	2	0

and the topology $\tau = \{\emptyset, X, \{1, 0\}, \{2\}\}$.

Define the LS-topology $\tau' = \{\emptyset, L(X), \Phi(\{1, 0\}), \Phi(\{2\})\}$. Here, \odot is continuous.

Hence, $(L(X), \odot, \tau')$ is a TBCH-algebra.

Theorem 3.26: Let $(X, *, \tau)$ be a positive implicative TBCH-algebra. If $\{0\}$ is open in X , then τ' is a discrete topology in LSTBCH-algebra.

Proof: For every $x \in X$, $L_x \in L(X)$. Then there exist some open sets $\Phi(U)$ and $\Phi(V)$ of $L(X)$ such that $\Phi(U * V) \subseteq \Phi(L_0)$, since $x*x = 0$ and $\{0\}$ is open in X .

Let $W = U \cap V$. Then $\Phi(W * W) = \Phi(\{0\})$. Implies that $\Phi(W) = \Phi(\{L_x\})$.

Therefore, τ' is a discrete topology in LSTBCH-algebra.

IV. CONCLUSION

In this paper, we worked on left maps on a BCH-algebra. The set of all right maps never forms a BCH-algebra under the operation \odot . Because in a BCH-algebra the operation \odot is never be a binary operation on the set of all right maps. If X is an associative BCH-algebra then the left maps and right maps on X are equal. In that case $R(X)$ forms a TBCH-algebra under \odot .

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