

# Thermoelastic Problem of Finite Length Hollow Cylinder

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**ABSTRACT:** The paper is concerned with the inverse unsteady-state problem of determining the temperature and displacement functions on the upper plane surface of a finite length hollow cylinder. Homogeneous boundary conditions of the third kind are maintained on curved surfaces of the cylinder while on the lower plane surface the heat flux is maintained at  $u(r, t)$  which is a known function of  $r$  and  $t$ . The Finite Marchi-Zgrablich and Laplace transform techniques are used to find the solutions of the inverse transient thermoelastic problems of a finite length hollow cylinder.

## I. INTRODUCTION

The inverse thermoelastic problem consists of the determination of temperature of the heating medium, the heat flux on the boundary surfaces of the solid when the conditions of the displacement and stresses are known at some points of the solid under consideration. The inverse problem is very important in view of its relevance to various industrial machines subjected to heating such as main shaft of the lathe and turbine and roll of a rolling mill.

Sierakowski and Sun [3] studied the direct problem of an exact solution to the elastic deformation of a finite length hollow cylinder. In [4] the inverse transient thermoelastic problem of determining temperature and displacement functions on the upper plane surface of a finite length hollow cylinder is investigated when the interior third kind boundary condition is known. The expressions of the temperature and displacement functions obtained at an arbitrary point of the cylinder in [4] are erroneous. The corresponding correct expressions are derived in the present paper. Also, the numerical results are obtained and presented graphically.

## II. THERMOELASTIC PROBLEM OF A FINITE LENGTH HOLLOW CYLINDER

Of concern in this section is an inverse transient thermoelastic problem of determining the temperature and displacement and functions on the upper plane surface of a finite length hollow cylinder occupying the space  $D : a \leq r \leq b, 0 \leq z \leq h$  when the interior third kind boundary condition is known.

Consider a hollow cylinder of length  $h$  occupying the space  $D$ . The differential equation governing the displacement function  $\phi(r, z, t)$  is

$$\nabla^2 \phi = \frac{(1+\nu)}{(1-\nu)} a_t T ; \quad \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}, \quad (2.1)$$

where  $\nu$  and  $a_t$  are Poisson's ratio and linear coefficient of thermal expansion of the material of the cylinder respectively, and  $T$  is the temperature of the cylinder satisfying the differential equation

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{k} \frac{\partial T}{\partial t}, \quad (2.2)$$

where  $k$  is the thermal diffusivity of the material of the cylinder, subject to the initial condition

$$[T(r, z, t)]_{t=0} = 0 \quad (2.3)$$

the interior condition

$$\left[ T(r, z, t) + \frac{\partial}{\partial z} T(r, z, t) \right]_{z=\xi} = f(r, t) \quad (2.4)$$

and the boundary conditions

$$\left[ T(r, z, t) + \frac{\partial}{\partial z} T(r, z, t) \right]_{z=0} = u(r, t) \quad (2.5)$$

$$\left[ T(r, z, t) + k_1 \frac{\partial}{\partial r} T(r, z, t) \right]_{r=a} = 0, \quad (2.6)$$

$$\left[ T(r, z, t) + k_2 \frac{\partial}{\partial r} T(r, z, t) \right]_{r=b} = 0, \quad (2.7)$$

The constants  $k_1$  and  $k_2$  are the radiation constants on the two curved surfaces.

In terms of the function  $\phi$  in (2.1) the radial and axial displacements  $U$  and  $W$  are given by

$$U = \frac{\partial \phi}{\partial r}; \quad W = \frac{\partial \phi}{\partial z} \quad (2.8)$$

In order to solve (2.2) subject to the conditions (2.3) – (2.7) we make use of the FMZ transform (1), the inverse FMZ transform (2) – (3) and the operator property with  $p=0$  discussed in the Appendix. Thus we find

$$\frac{\partial^2}{\partial z^2} \bar{T} - \mu_n^2 \bar{T} = \frac{1}{k} \frac{\partial}{\partial t} \bar{T}, \quad (2.9)$$

$$\bar{T}(\mu_n, z, 0) = 0 \quad (2.10)$$

$$\left[ \bar{T}(\mu_n, z, t) + \frac{\partial}{\partial z} \bar{T}(\mu_n, z, t) \right]_{z=0} = \bar{u}(\mu_n, t) \quad (2.11)$$

$$\left[ \bar{T}(\mu_n z, t) + \frac{\partial}{\partial z} \bar{T}(\mu_n, z, t) \right]_{z=\xi} = \bar{f}(\mu_n, t) \quad (2.12)$$

Introducing the Laplace transform (2.9) to (2.12) we can show that

$$\bar{T}^*(\mu_n, z, s) = \frac{[\sinh(qz) - q \cosh(qz)]}{(1 - q^2) \sinh(q\xi)} \bar{f}^*(\mu_n, s) - \frac{[\sinh[q(z - \xi)] - q \cosh[q(z - \xi)]]}{(1 - q^2) \sinh(q\xi)} \bar{u}(\mu_n, s) \quad (2.13)$$

The inverse Laplace transform of  $\bar{T}^*$  can be determined if we find the inverse Laplace transform of the function

$$G_0(s) = \frac{\sinh(qz) - q \cosh(qz)}{(1 - q^2) \sinh(q\xi)}; \quad q^2 = \mu_n^2 + \frac{s}{k} \quad (2.14)$$

The poles and residues of  $e^{st}G_0(s)$  are as follows:

$$(i) \quad q\xi = im\pi \quad \text{or} \quad s = -k(\mu_n^2 + \lambda_m^2) \\ \beta_{mn}^2 = (\mu_n^2 + \lambda_m^2); \quad \lambda_m = \frac{m\pi}{\xi}, \quad m = 1, 2, \dots \quad (2.15)$$

$$\begin{aligned} \text{Res}_{s = -k\beta_{mn}^2} [e^{st}G_0(s)] &= \lim_{s \rightarrow -k\beta_{mn}^2} \frac{e^{st} [\sinh(qz) - q \cosh(qz)]}{\xi(1 - q^2) \cosh(q\xi) (dq/ds)} \\ &= \frac{2k(-1)^m [\lambda_m \cos(\lambda_m z) - \sin(\lambda_m z)] \lambda_m e^{-k\beta_{mn}^2 t}}{\xi (1 + \lambda_m^2)} \end{aligned} \quad (2.16)$$

$$(ii) \quad q^2 = 1 \quad \text{or} \quad s = -k(\mu_n^2 - 1); \quad v_n^2 = \mu_n^2 - 1 \quad (2.17)$$

$$\text{Res}_{s = -kv_n^2} [e^{st}G_0(s)] = \lim_{s \rightarrow -kv_n^2} \frac{e^{st} [\sinh(qz) - \cosh(qz)]}{(-2q) \sinh(q\xi) (dq/ds)} = \frac{ke^{-z} e^{-kv_n^2 t}}{\sinh \xi} \quad (2.18)$$

Using the convolution theorem, formula and Eqns. (2.15) – (2.18) the inverse Laplace transform of  $\bar{T}^*$  is given by

$$\bar{T}(\mu_n, z, t) = \bar{T}_1(\mu_n, z, t) + \bar{T}_2(\mu_n, z, t) \quad (2.19)$$

$$\bar{T}_1(\mu_n, z, t) = \frac{ke^{-z}}{\sinh \xi} \int_0^t \bar{f}(\mu_n, t') e^{-kv_n^2(t-t')} dt'$$

$$+ \frac{2k}{\xi} \sum_{m=1}^{\infty} \frac{(-1)^m \lambda_m [\lambda_m \cos(\lambda_m z) - \sin(\lambda_m z)]}{(1 + \lambda_m^2)} \int_0^t \bar{f}(\mu_n, t') e^{-k\beta_{mn}^2(t-t')} dt' \quad (2.20)$$

$$\begin{aligned} \bar{T}_2(\mu_n, z, t) &= \frac{-ke^{-(z-\xi)}}{\sinh \xi} \int_0^t \bar{u}(\mu_n, t') e^{-k\nu_n^2(t-t')} dt' \\ &- \frac{2k}{\xi} \sum_{m=1}^{\infty} \frac{(-1)^m \lambda_m [\lambda_m \cos[\lambda_m(z-\xi) - \sin[\lambda_m(z-\xi)]]}{(1 + \lambda_m^2)} \\ &\quad \times \int_0^t \bar{u}(\mu_n, t') e^{-k\beta_{mn}^2(t-t')} dt'. \end{aligned} \quad (2.21)$$

Thus the expression for the temperature  $T(r, z, t)$  satisfying Eqns.(2.2) – (2.7) is given by (5) and(2.19) – (2.21).

The problem formulated in this section has been studied earlier by Pidadi and Khobragade[4]. Because of the erroneous inverse Laplace transform the expression in [4] corresponding to  $\bar{T}_1(\mu_n, z, t)$  does not contain the first term independent of the infinite series. Similarly, the expression corresponding to  $\bar{T}_2(\mu_n, z, t)$  also does not contain the first term independent of the infinite series. In addition, the factors  $(1 + \lambda_m^2)^{-1}$  and  $-(1 + \lambda_m^2)^{-1}$  are missing from the corresponding expressions of  $\bar{T}_1$  and  $\bar{T}_2$  respectively. The correctness of  $\bar{T}(\mu_n, z, t)$  given by (2.19) – (2.21) .

Speaking purely mathematically the expressions for  $\bar{T}_1$  and  $\bar{T}_2$  are analogous. The displacement components can be obtained using  $\bar{T}_1$  if  $u(r,t) \equiv 0$ . Similarly they can be obtained using  $\bar{T}_2$  if  $f(r, t) \equiv 0$ . The results for the general case can be obtained using the principle of superposition. In what follows therefore we restrict our attention to the case when  $u(r,t) \equiv 0$ . Having determined  $T_1$  given by

$$T_1(r, z, t) = \sum_{n=1}^{\infty} \frac{S_0(k_1, k_2, \mu_n r) \bar{T}_1(\mu_n, z, t)}{C_n}, \quad (2.22)$$

together with (2.20) we have to find the displacement function  $\phi_1(r, z, t)$  satisfying the Eqn.(2.1) which may be rewritten

$$\nabla^2 \phi_1 = [(1 + \nu)/(1 - \nu)] a_t T_1 \quad (2.23)$$

A particular solution of the above equation is given by

$$\phi_1(r, z, t) = \frac{(1 + \nu)}{(1 - \nu)} a_t \left[ \frac{ke^{-z}}{\sinh \xi} \sum_{n=1}^{\infty} \frac{S_0(k_1, k_2, \mu_n r)}{\nu_n^2 C_n} \int_0^t \bar{f}(\mu_n, t') e^{-k\nu_n^2(t-t')} dt' \right]$$

$$+ \frac{2k}{\xi} \sum_{n=1}^{\infty} \frac{S_0(k_1, k_2, \mu_n r)}{C_n} \sum_{m=1}^{\infty} \frac{(-1)^m \lambda_m [\lambda_m \cos(\lambda_m z) - \sin(\lambda_m z)]}{(1 + \lambda_m^2) \beta_{mn}^2} \int_0^t \bar{f}(\mu_n, t') e^{-k\beta_{mn}^2(t-t')} dt' \quad (2.24)$$

where  $v_n$  and  $\beta_{mn}$  are given by (2.15) and (2.17).

The expression for  $\phi(r, z, t)$  obtained in Pidadi and Khobragade [4] is of the form

$$\varphi(r, z, t) = \frac{(1+v)}{(1-v)} a_t \frac{r^2}{4} T(r, z, t) \quad (2.25)$$

where  $T(r, z, t)$  is the temperature distribution in [4]. Obviously, the displacement function  $\phi$  given by (2.25) does not satisfy the basic equation (2.1). Therefore, the expressions for displacement components obtained using (2.25) in [4] cannot be correct.

### III. SPECIAL CASE AND NUMERICAL RESULTS

Let the prescribed function  $f(r, t)$  in (2.4) be given by

$$f(r, t) = (1 - e^{-t})(h - \xi)\delta(r - 0.9) \quad (3.1)$$

then we have

$$\int_0^t \bar{f}(\mu_n, t') e^{-k\beta_{mn}^2(t-t')} dt' = \alpha_n B_{mn}(t) / k, \quad (3.2)$$

where  $\alpha_n$  and  $B_{mn}(t)$  are given by (3.5) and (3.6). Using the integral (3.2) in (2.20) gives

$$\begin{aligned} \bar{T}_1(\mu_n, z, t) &= \frac{\alpha_n e^{-z}}{\sinh \xi} G_n(t) \\ &+ \frac{2\alpha_n}{\xi} \sum_{m=1}^{\infty} \frac{(-1)^m \lambda_m}{(1 + \lambda_m^2)} [\lambda_m \cos(\lambda_m z) - \sin(\lambda_m z)] B_{mn}(t), \end{aligned} \quad (3.3)$$

where

$$G_n(t) = \frac{k(1 - e^{-t}) + (e^{-k v_n^2 t} - 1) / v_n^2}{(k v_n^2 - 1)}. \quad (3.4)$$

where  $\alpha_n$ ,  $B_{mn}$  and  $D_n$  are given by below equations,

$$\alpha_n = (h - \xi)(0.9) S_0(k_1, k_2, 0.9 \mu_n) \quad (3.5)$$

$$B_{mn}(t) = \frac{k(1 - e^{-t}) + (e^{-k\beta_{mn}^2 t} - 1) / \beta_{mn}^2}{(k\beta_{mn}^2 - 1)}. \quad (3.6)$$

$$D_n(t) = \frac{k(1 - e^{-t}) + (e^{-k\mu_n^2 t} - 1) / \mu_n^2}{(k\mu_n^2 - 1)}, \tag{3.7}$$

Thus the temperature  $T_1(r, z, t)$  is given by (2.22) and (3.3).

Next, using the integral (3.2) in the expression for displacement function  $\phi_1$  given by (2.24) we find

$$\begin{aligned} \phi_1(r, z, t) = & -\frac{(1 + \nu)}{(1 - \nu)} a_t \left[ \frac{e^{-z}}{\sinh \xi} \sum_{n=1}^{\infty} \frac{\alpha_n S_0(k_1, k_2, \mu_n r)}{\nu_n^2 C_n} G_n(t) \right. \\ & \left. + \frac{2}{\xi} \sum_{n=1}^{\infty} \frac{\alpha_n S_0(k_1, k_2, \mu_n r)}{C_n} \sum_{m=1}^{\infty} \frac{(-1)^m \lambda_m [\lambda_m \cos(\lambda_m z) - \sin(\lambda_m z)]}{(1 + \lambda_m^2) \beta_{mn}^2} \right] B_{mn}(t). \end{aligned} \tag{3.5}$$

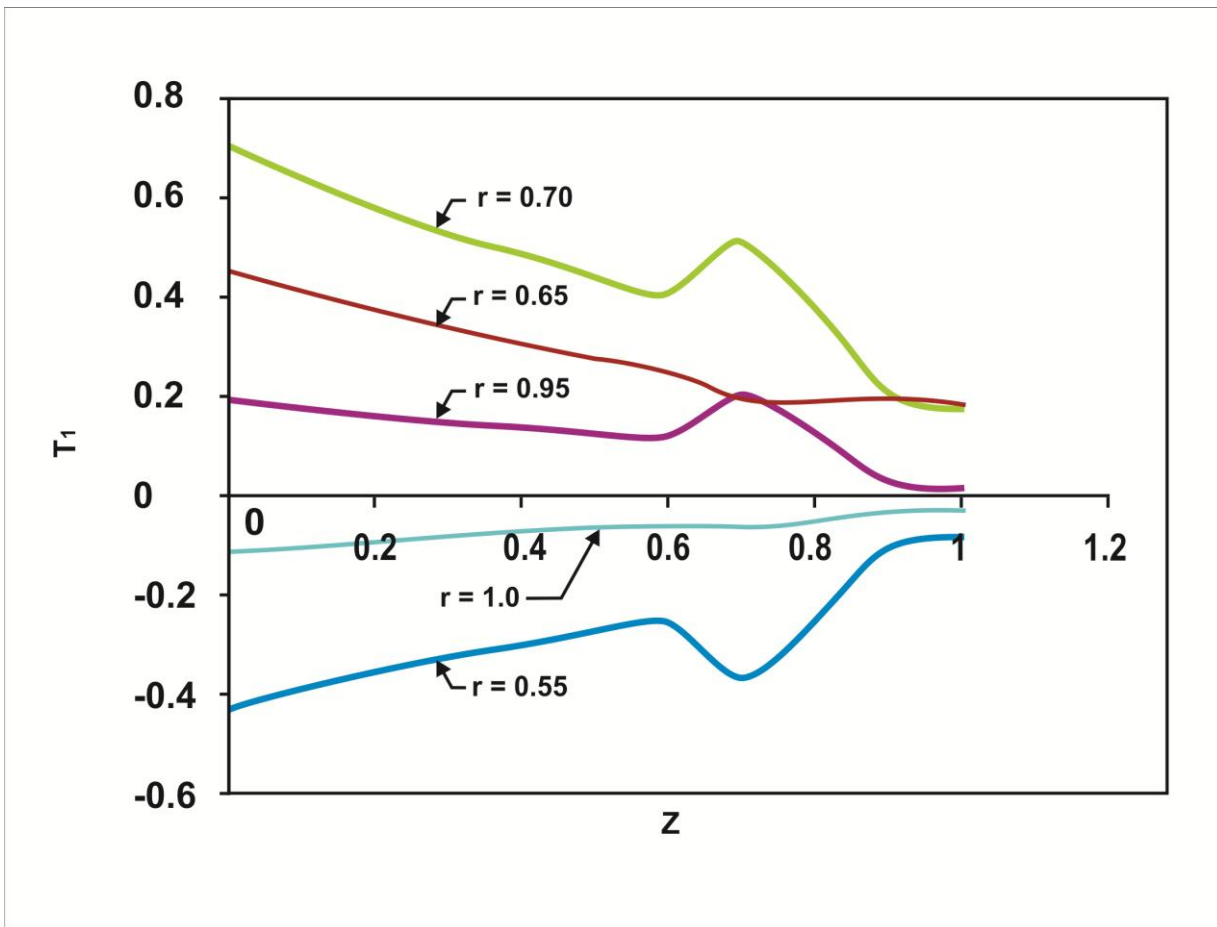


Fig. : The variation of  $T_1(r, z, t)$  given by equations (2.22), (3.3), (3.4), (3.5), (3.6) with  $z$  for  $r = 0.5, 0.65, 0.70, 0.95, 1.0$ . The values of other parameters are  $t = 1, h = 1, a = 0.5, b = 1.0, \xi = 0.75, k = 0.375$  and  $k_1 = k_2 = 0.25$ .

**APPENDIX**

**FINITE MARCHI-ZGRABLICH TRANSFORM**

FMZ transform may be rewritten

$$\bar{f}(\mu_n) = \int_a^b r f(r) S_0(k_1, k_2, \mu_n r) dr \quad (1)$$

together with the inversion formula

$$f(r) = \sum_{n=1}^{\infty} \frac{\bar{f}(\mu_n) S_0(k_1, k_2, \mu_n r)}{C_n} \quad (2)$$

where

$$C_n = \frac{b^2}{2} \left[ 1 + \frac{1}{k_2^2 \mu_n^2} \right] S_0^2(k_1, k_2, \mu_n b) - \frac{a^2}{2} \left[ 1 + \frac{1}{k_1^2 \mu_n^2} \right] S_0^2(k_1, k_2, \mu_n a) \quad (3)$$

The Finite Marchi-Zgrablich (FMZ) transform of T and inversion formula

$$\bar{T}(\mu_n, z, t) = \int_a^b r T(r, z, t) S_0(k_1, k_2, \mu_n r) dr \quad (4)$$

$$T(r, z, t) = \sum_{n=1}^{\infty} \frac{\bar{T}(\mu_n, z, t) S_0(k_1, k_2, \mu_n r)}{C_n} \quad (5)$$

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