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Contraction Conditions using Comparison Functions on b-2-Metric Spaces

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ABSTRACT: In this paper we prove fixed point theorem on b-2-metric spaces which is a generalization to a fixed point theorem of two mappings using comparison function on b-metric spaces.

KEYWORDS: fixed point, b-metric spaces comparison function, b-2-metric spaces.

I. INTRODUCTION

The contraction principle of Banach [1], proved in 1922, was followed by diverse works about fixed points theory regarding different classes of contractive conditions on some spaces such as: quasi-metric spaces [2,3], cone metric spaces [4,5], partially ordered metric spaces [6,7], Menger spaces [8], and fuzzy metric spaces [9,10].
The concept of 2-metric space was initiated by S. Gahler [11,12,13]. The study was further enhanced by Iseki [14], B.E. Rhoades [15], Miczko, Palezewski [16], and Khan [17]. They introduced several properties of 2-metric spaces and proved some fixed point and common fixed point theorems for contractive and expansion mappings.
The concept of b-metric space was introduced by Bakhtin [18] in 1989, who used it to prove a generalization of the Banach principle in a spaces endowed with such kind of metric. Since then, this notion has been used by many authors to obtain various fixed point theorems. Aydi *et al,* in [19] proved common fixed point results for single-valued and multi-valued mappings satisfying $\phi - contraction$ in b-metric spaces. Roshan *et al,* in [20] used the notation of almost generalized contractive mappings in ordered complete b-metric space and established some fixed and common fixed point results. In [21] Pacurar proved the existence and uniqueness of fixed points of $\phi - contractions$ on b-metric spaces. Hussain and shah in [22] introduced the notation of cone b-metric space, generalizing both notations of b-metric spaces and cone metric spaces. Fixed point theorems of contractive mappings in cone b-metric spaces without assumption of the normality of a corresponding cone are proved by Huang and Xu in [23]. The setting of partially ordered b-metric spaces was used by Husain et al, in [24]to study tripled coincidence points of mappings which satisfy nonlinear contractive conditions, extending those results of Berinde and Borcut [25] for metric spaces to b-metric spaces. Using the concept of a g-monotone mapping, Shah and Hussain in [26] proved common fixed point theorems involving g-non-decreasing mappings in b-metric spaces, generalizing several results of Agarwal et al [27] and Ciric et al [28]. Some results of Suzuki [29] are extended to the case of metric-type spaces and cone metric-type spaces. In [30] Shatanawi et al consider and establish on setting of b-metric spaces, regarding common fixed points of two mappings, using a contraction condition defined by means of a comparison function.
The aim of this paper is to establish results in b-2-metric spaces. In this paper we obtain common fixed point theorem for two mappings, using a contraction condition defined by means of comparison function, extending the main result of Shatanawi et al [30].

II. PRELIMINARIES

Definition 2.1. Let X be a non empty set. A real valued function d on $X \times X \times X$ is said to be 2-metric in X if

(i) To each of distinct points x, y, z in X, $d(x, y, z) > 0$

(ii) $d(x, y, z) = 0$, when at least two of x, y, z are equal

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(iii) $d(x,y,z) = d(y,z,x) = d(x,z,y)$

(iv) $d(x,y,z) \le d(x,y,a) + d(x,a,z) + d(a,y,z)$ for all $x,y,z,a \in X$

when d is a 2-metric on X, then the order pair (X,d) is called 2-metric space.

Definition 2.2. A sequence $\{x_n\}$ in 2-metric space (X,d) is said to be convergent to an element $x \in X$ if $\lim_{n\to\infty} d(x_n, x, a) = 0$ for all $a \in X$

It follows that if the sequence $\{x_n\}$ converges to x then $\lim_{n\to\infty} d(x_n, a, b) = d(x,a,b)$

for all $a, b \in X$

Definition 2.3. A sequence $\{x_n\}$ in 2-metric space (X,d) is said to be Cauchy sequence if $\lim_{n\to\infty} d(x_m, x_n, a) = 0$ as $m, n \to \infty$ for all $a \in X$

Proposition 2.4. If a sequence is convergent in a 2-metric space then it is a Cauchy sequence.

Definition 2.5. A 2-metric space (X,d) is said to be complete if every Cauchy sequence in X is convergent.

Proposition 2.6. If a sequence $\{x_n\}$ in a 2-metric space converges to x then every subsequence of $\{x_n\}$ also converges to the same limit x

Proposition 2.7. Limit of a sequence in 2-metric space, if exists, is unique.

Definition 2.8. Let X be nonempty set and $d : X \times X \to [0,\infty)$ .A function d is called b-metric space with constant (base) $s \ge 1$ if:

(1) $d(x, y) = 0 \Leftrightarrow x = y$

(2) $d(x, y) = d(y, x)$ *for* all $x, y \in X$

(3) $d(x, y) \le s(d(x, z) + d(z, y))$ *for* all $x, y, z \in X$

The pair (X,d) is called b-metric space.

It is obvious that a b-metric space with base $s = 1$ is a metric space. spaces (see, e.g., Singh and Prasad [32].

Definition 2.9. Let X be a non empty set. A real valued function d on $X \times X \times X$ is said to be b-2-metric with constant (base) $s \ge 1$ if

(1) To each of distinct points x, y, z in X, $d(x, y, z) > 0$

(2) $d(x, y, z) = 0$, when at least two of x, y, z are equal

(3) $d(x, y, z) = d(z, y, x) = d(x, z, y)$

(4) $d(x, y, z) \le s(d(x, y, a) + d(x, a, z) + d(a, y, z))$ for all $x, y, z, a \in X$

where d is a b-2-metric on X, then the order pair (X,d) is called b-2-metric space [33].

III. MAIN RESULTS

Now we are ready to prove our main results

Theorem 3.1. Let (X,d) be complete b-2-metric space with constant s and $T, S : X \to X$ two mappings on X. Suppose that there is a constant $L < \frac{1}{2+s}$ and a comparison function φ such that the inequality

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$$sd(Tx,Sy,a) \le \varphi(\max\{ sd(x,Tx,a), sd(y,Sy,a), L[d(x,Sy,a)+d(Tx,y,a)]\}) \quad (3.1)$$

holds for each $x, y, a \in X$. Suppose one of the mappings S or T is continuous. Then S and T have a unique common fixed point.

Proof: Let $x_0 \in X$ be arbitrary. We define a sequence $\{x_n\}$ as follows:

$$x_{2n+1} = Tx_{2n}, \quad x_{2n+2} = Sx_{2n+1}, \quad n \in N \quad (3.2)$$

Suppose that there is some $n \in N$ such that $x_n = x_{n+1}$. If $n = 2k$ then $x_{2k} = x_{2k+1}$ and from the contraction condition (3.1) with $x = x_{2k}$, $y = x_{2k+1}$ and for all $a \in X$ we have

$sd(x_{2k+1}, x_{2k+2}, a) = sd(Tx_{2k}, Sx_{2k+1}, a)$

$\le \varphi(\max\{ sd(x_{2k}, Tx_{2k}, a), sd(x_{2k+1}, Sx_{2k+1}, a)), L[d(x_{2k}, Sx_{2k+1}, a) + d(Tx_{2k}, x_{2k+1}, a)]\})$

$= \varphi(\max\{ sd(x_{2k}, x_{2k+1}, a), sd(x_{2k+1}, x_{2k+2}, a)), Ld(x_{2k}, x_{2k+2}, a\})$

Hence, as we suppose that $x_{2k} = x_{2k+1}$ and as a comparison function φ is non-decreasing,

$sd(x_{2k+1}, x_{2k+2}, a) \le \varphi(\max\{ sd(x_{2k+1}, x_{2k+2}, a), L[sd(x_{2k}, x_{2k+2}, x_{2k+1}) + sd(x_{2k}, x_{2k+1}, a) + sd(x_{2k+1}, x_{2k+2}, a)]\}$

$= \varphi(\max\{ sd(x_{2k+1}, x_{2k+2}, a), Lsd(x_{2k+1}, x_{2k+2}, a)\})$

$= \varphi(sd(x_{2k+1}, x_{2k+2}, a))$

If we assume that $d(x_{2k+1}, x_{2k+2}, a) > 0$,then we have, as $\varphi(t) < t$ for $t > 0$,

$sd(x_{2k+1}, x_{2k+2}, a) \le \varphi(sd(x_{2k+1}, x_{2k+2}, a)) < sd(x_{2k+1}, x_{2k+2}, a)$

a contradiction. Therefore $d(x_{2k+1}, x_{2k+2}, a) = 0$ .Hence $x_{2k+1} = x_{2k+2}$. Thus we have $x_{2k} = x_{2k+1} = x_{2k+2}$.

By (3.2), it means $x_{2k} = Tx_{2k} = Sx_{2k}$. That is x_{2k} is a common fixed point of T and S.

If $n = 2k+1$ then $x_{2k+1} = x_{2k+2}$ and from contraction condition (3.1) with $x = x_{2k+2}$ and $y = x_{2k+1}$ we have

$sd(x_{2k+2}, x_{2k+3}, a) = sd(Sx_{2k+1}, Tx_{2k+2}, a)$

$\le \varphi(\max\{ sd(x_{2k+2}, Tx_{2k+2}, a), sd(x_{2k+1}, Sx_{2k+1}, a), L[d(x_{2k+1}, Tx_{2k+2}, a) + d(x_{2k+2}, Sx_{2k+1}, a)]\})$

$= \varphi(\max\{ sd(x_{2k+2}, x_{2k+3}, a), sd(x_{2k+1}, x_{2k+2}, a), Ld(x_{2k+1}, x_{2k+3}, a)\})$

.

Hence, as we supposed that $x_{2k+1} = x_{2k+2}$ and as a comparison function φ is non-decreasing,

$sd(x_{2k+2}, x_{2k+3}, a) \le \varphi(\max\{ sd(x_{2k+2}, x_{2k+3}, a), L[d(x_{2k+1}, x_{2k+3}, a)]\})$

$\le \varphi(\max\{ sd(x_{2k+2}, x_{2k+3}, a), L[d(x_{2k+2}, x_{2k+3}, a) + d(x_{2k+1}, x_{2k+2}, a) + d(x_{2k+1}, x_{2k+3}, x_{2k+2})]\})$

$= \varphi(\max\{ sd(x_{2k+2}, x_{2k+3}, a), L[d(x_{2k+2}, x_{2k+3}, a)]\})$

$= \varphi(sd(x_{2k+2}, x_{2k+3}, a))$

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If we assume that $d(x_{2k+2}, x_{2k+3}, a) > 0$, then we have, as $\varphi(t) < t$ for $t > 0$,

$sd(x_{2k+2}, x_{2k+3}, a) \leq \varphi(sd(x_{2k+2}, x_{2k+3}, a)) < sd(x_{2k+2}, x_{2k+3}, a)$,

a contradiction. Therefore, $sd(x_{2k+2}, x_{2k+3}, a) = 0$. Hence $x_{2k+2} = x_{2k+3}$. Thus we have $x_{2k+1} = x_{2k+2} = x_{2k+3}$. By (3.2), it means $x_{2k+1} = Tx_{2k+1} = Sx_{2k+1}$, that is, x_{2k+1} is a common fixed point of T and S.

From now on, we suppose that $x_n \neq x_{n+1}$ each $n \in N$.

Now we shall prove that

$$sd(x_n, x_{n+1}, a) \leq \varphi(sd(x_{n-1}, x_n, a)) \quad \text{for each } n \in N \text{ and } a \in X \tag{3.3}$$

There are two cases which we have to consider

Case (i). $n = 2k, \mathrm{k} \in \mathrm{N}$.

From the contraction condition (3.1) with $x = x_{2k}$ and $y = x_{2k-1}$ we get

$$sd(x_{2k+1}, x_{2k}, a) = sd(Tx_{2k}, Sx_{2k-1}, a)$$
$$\leq \varphi(\max\{ sd(x_{2k}, Tx_{2k}, a), sd(x_{2k-1}, Sx_{2k-1}, a), L[d(x_{2k}, Sx_{2k-1}, a) + d(x_{2k-1}, Tx_{2k}, a)]\})$$
$$= \varphi(\max\{ sd(x_{2k}, x_{2k+1}, a), sd(x_{2k-1}, x_{2k}, a), L[d(x_{2k-1}, x_{2k+1}, a)]\})$$

Since $L < \frac{1}{3}$, we get

$$sd(x_{2k+1}, x_{2k}, a) \leq \varphi(\max\{ sd(x_{2k}, x_{2k+1}, a), sd(x_{2k-1}, x_{2k}, a), \frac{s}{3}[d(x_{2k-1}, x_{2k+1}, a)]\})$$

$$\leq \varphi(\max\{ sd(x_{2k}, x_{2k+1}, a), sd(x_{2k-1}, x_{2k}, a), \frac{s}{3}[d(x_{2k}, x_{2k+1}, a) + d(x_{2k-1}, x_{2k}, a) + d(x_{2k-1}, x_{2k+1}, x_{2k})]\})$$

$$= \varphi(\max\{ sd(x_{2k}, x_{2k+1}, a), sd(x_{2k-1}, x_{2k}, a)\})$$

Now, if we suppose that $\max\{ sd(x_{2k}, x_{2k+1}, a), sd(x_{2k-1}, x_{2k}, a)\} = sd(x_{2k}, x_{2k+1}, a)$, then by property (a) of φ in Definition 4 we get

$$sd(x_{2k}, x_{2k+1}, a) \leq \varphi(sd(x_{2k}, x_{2k+1}, a)) < sd(x_{2k}, x_{2k+1}, a)$$

a contradiction. Therefore, from the above inequality we have

$$sd(x_{2k}, x_{2k+1}, a) \leq \varphi(sd(x_{2k-1}, x_{2k}, a)) \tag{3.4}$$

Thus we proved that (3.3) holds for $n = 2k$.

Case(ii) $n = 2k+1, \ \mathrm{n} \in \mathrm{N}$.

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Using the same argument as in the case (i), it can be proved that (3.3) holds for $n = 2k+1$, that is

$$sd(x_{2k+1}, x_{2k+2}, a) \le \varphi(sd(x_{2k}, x_{2k+1}, a)) \quad (3.5)$$

From (3.4) and (3.5) we conclude that the inequality (3.3) holds for all $n \in N$.

From (3.3), by the induction it is easy to prove that

$$sd(x_n, x_{n+1}, a) \le \varphi^n(sd(x_0, x_1, a)), \forall n \in N \quad (3.6)$$

Since $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t > 0$, from (3.6) it follows that

$$\lim_{n\to\infty} d(x_n, x_{n+1}, a) = 0 \quad (3.7)$$

Now we shall prove that $\{x_n\}$ is a Cauchy sequence. Let $\varepsilon > 0$. Since $L < \frac{1}{2+s}$ implies $s - 3L > 0$ and $1 - L(2+s) > 0$ from (3.7) we conclude that there exists $n_0 \in N$ such that

$$d(x_n, x_{n-1}, a) < \frac{1-L-Ls}{3s}\varepsilon \quad (3.8)$$

for all $n \ge n_0$

Let $m, n \in N$ with $m > n$. By induction on m we shall prove that

$$d(x_n, x_m, a) < \varepsilon, \forall m > n \ge n_0 \quad (3.9)$$

Let $n \ge n_0$ and $m = n+1$ then from (3.3) and (3.8) we get

$$d(x_n, x_m, a) = d(x_n, x_{n+1}, a) \le d(x_n, x_{n-1}, a) < \frac{1-L-Ls}{3s}\varepsilon < \varepsilon .$$

Thus (3.9) holds for $m = n+1$.

Assume now that (3.9) holds for some $m \ge n+1$. We have to prove that (3.9) holds for $m+1$. We have to consider four cases.

Case1. n is odd, $m+1$ is even.

From contraction condition (3.1) we get

$$sd(x_n, x_{m+1}, a) = sd(Tx_{n-1}, Sx_m, a)$$
$$\le \varphi(\max\{ sd(x_{n-1}, x_n, a), sd(x_m, x_{m+1}, a), L[d(x_{n-1}, x_{m+1}, a) + d(x_n, Tx_m, a)]\})$$

Hence we get, as $d(x_m, x_{m+1}, a) < d(x_{n-1}, x_n, a)$ and $\varphi(t) < t$ for all $t > 0$,

$$sd(x_n, x_{m+1}, a) < (\max\{ sd(x_{n-1}, x_n, a), L[d(x_{n-1}, x_{m+1}, a) + d(x_n, x_m, a)]\}) \quad (3.10)$$

If from (3.10) we have $sd(x_n, x_{m+1}, a) < sd(x_{n-1}, x_n, a)$, then by (3.8),

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$$sd(x_n, x_{m+1}, a) < sd(x_{n-1}, x_n, a) < \frac{1-L-Ls}{3s}\varepsilon < \varepsilon$$

.

If (3.10) implies $sd(x_n, x_{m+1}, a) < L[d(x_{n-1}, x_{m+1}, a) + d(x_n, x_m, a)]$, then by triangle inequality,

$$sd(x_n, x_{m+1}, a) < L[d(x_{n-1}, x_{m+1}, x_n) + d(x_{n-1}, x_n, a) + d(x_n, x_{m+1}, a) + d(x_n, x_m, a)]$$

and*

As $L < \frac{1}{2+s}$ implies $\frac{L}{1-L} < 3L < 1 \le s$

$$d(x_n, x_{m+1}, a) < \frac{L}{1-L}[d(x_{n-1}, x_n, a) + \frac{1}{s}d(x_n, x_m, a)]$$
$$< 3L[d(x_{n-1}, x_n, a) + \frac{1}{s}d(x_n, x_m, a)]$$

Using (3.8) and the induction hypothesis (3.9) we have,

$$d(x_n, x_{m+1}, a) < 3L\frac{1-L-Ls}{3s}\varepsilon + \frac{3L}{s}\varepsilon < \frac{1-3L-L(s-1)}{s}\varepsilon + \frac{3L}{s}\varepsilon$$
$$\le \frac{1-3L}{s}\varepsilon + \frac{3L}{s}\varepsilon = \frac{1}{s}\varepsilon \le \varepsilon$$

Thus we proved that in this case (3.9) holds for $m+1$. Therefore, by induction, we conclude that in Case I the inequality (3.9) holds for all $m < n$

Case II. n is even $m+1$ is odd. The proof of (3.9) in this case is similar to given in Case III. n is even $m+1$ is even.

Using the triangle inequality and the contraction condition (3.1) we obtain

$$d(x_n, x_{m+1}, a) \le sd(x_n, x_{m+1}, x_{n+1}) + sd(x_n, x_{n+1}, a) + sd(x_{n+1}, x_{m+1}, a)$$
$$= sd(x_n, x_{m+1}, x_{n+1}) + sd(x_n, x_{n+1}, a) + sd(Tx_n, Sx_m, a)$$

therefore

$$d(x_n, x_{m+1}, a)$$
$$\le sd(x_n, x_{m+1}, x_{n+1}) + sd(x_n, x_{n+1}, a) + \varphi(\max\{ sd(x_n, Tx_n, a), sd(x_m, Sx_m, a), L[d(x_n, Sx_m, a) + d(x_m, Tx_n, a)]\})$$

$$= sd(x_n, x_{m+1}, x_{n+1}) + sd(x_n, x_{n+1}, a) + \varphi(\max\{ sd(x_n, x_{n+1}, a), sd(x_m, x_{m+1}, a), L[d(x_n, x_{m+1}, a) + d(x_m, x_{n+1}, a)]\})$$
$$= sd(x_n, x_{m+1}, x_{n+1}) + sd(x_n, x_{n+1}, a) + \varphi(\max\{ sd(x_n, x_{n+1}, a), L[d(x_n, x_{m+1}, a) + d(x_m, x_{n+1}, a)]\})$$

Hence we get, as $d(x_m, x_{m+1}, a) < d(x_n, x_{n+1}, a)$ and $\varphi(t) < t$ for all $t > 0$

$$d(x_n, x_{m+1}, a) < sd(x_n, x_{m+1}, x_{n+1}) + sd(x_n, x_{n+1}, a) + (\max\{ sd(x_n, x_{n+1}, a), L[d(x_n, x_{m+1}, a) + d(x_m, x_{n+1}, a)]\}) \quad (3.11)$$

If the inequality (3.11) implies

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$$d(x_n, x_{m+1}, a) < sd(x_n, x_{m+1}, x_{n+1}) + sd(x_n, x_{n+1}, a) + sd(x_n, x_{n+1}, a)$$

then from (3.8) we get

$$d(x_n, x_{m+1}, a) < sd(x_n, x_{m+1}, x_{n+1}) + sd(x_n, x_{n+1}, a) + sd(x_n, x_{n+1}, a)$$

$$d(x_n, x_{m+1}, a) < 3s\frac{1-L-Ls}{3s}\varepsilon$$

Hence

$$d(x_n, x_{m+1}, a) < \varepsilon$$

If the inequality (3.11) implies

$$d(x_n, x_{m+1}, a) < sd(x_n, x_{m+1}, x_{n+1}) + sd(x_n, x_{n+1}, a) + L[d(x_n, x_{m+1}, a) + d(x_m, x_{n+1}, a)]$$

Hence we get

$$(1-L)d(x_n, x_{m+1}, a) < sd(x_n, x_{m+1}, x_{n+1}) + sd(x_n, x_{n+1}, a) + Ld(x_m, x_{n+1}, a)$$

Then by triangle inequality we have

$$(1-L)d(x_n, x_{m+1}, a) < sd(x_n, x_{m+1}, x_{n+1}) + sd(x_n, x_{n+1}, a) + L[d(x_m, x_{n+1}, x_n) + d(x_m, x_n, a) + d(x_n, x_{n+1}, a)]$$

Now, by using (3.3) and (3.8) and induction hypothesis we obtain

$$(1-L)d(x_n, x_{m+1}, a) < sd(x_{n-1}, x_n, x_{m+1}) + sd(x_{n-1}, x_n, a) + L[d(x_{n-1}, x_n, x_m) + d(x_m, x_n, a) + d(x_{n-1}, x_n, a)]$$

$$(1-L)d(x_n, x_{m+1}, a) < 2s\frac{1-L-Ls}{3s}\varepsilon + 2L\frac{1-L-Ls}{3s}\varepsilon + L\varepsilon = \frac{2(s+L)(1-L-Ls)}{3s}\varepsilon + L\varepsilon$$

Hence we get

$$(1-L)d(x_n, x_{m+1}, a) < \frac{2(s+L)(1-L-Ls)+3sL}{3s}\varepsilon = \frac{2s+2L-2L^2-2s^2L-2sL^2+sL}{3s}\varepsilon$$

Since $L < \frac{1}{2+s}$ using $2L+Ls<1$ we have

$$(1-L)d(x_n, x_{m+1}, a) < \frac{2s-2L^2-2s^2L-2sL^2+1}{3s}\varepsilon \le \frac{3s-2L^2-2s^2L-2sL^2}{3s}\varepsilon$$

and

$$(1-L)d(x_n, x_{m+1}, a) < \frac{3s-2L^2-2s^2L-2sL^2}{3s}\varepsilon < \frac{3s-2sL^2}{3s}\varepsilon < \frac{3s-2sL}{3s}\varepsilon < \frac{3s-3sL}{3s}$$

$$(1-L)d(x_n, x_{m+1}, a) < \frac{3s(1-L)}{3s}\varepsilon$$

Hence

$$d(x_n, x_{m+1}, a) < \varepsilon$$

Thus we proved that (3.9) and holds for $m+1$. Therefore by induction we conclude that in Case III the inequality (3.9) holds for all $m > n$.

Case IV. n is odd, $m+1$ is odd. The proof of (3.9) in this case is similar to one given in Case III.

Therefore, we proved that in all of four cases the inequality (3.9) holds.

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From (3.9) it follows that $\{x_n\}$ is a Cauchy sequence. Since (X,d) is a complete b-2-metric space, then $\{x_n\}$ converges to some $u \in X$ as $n \to +\infty$.

Now we shall prove that if one of the mappings T or S is continuous, then $Tu = Su = u$. Without loss of generality, we can suppose that S is continuous. Clearly, as $x_n \to u$ then by (3.2) we have $Sx_{2n+1} = x_{2n+2} \to u$ as $n \to +\infty$. Since $x_{2n+1} \to u$ and S is continuous then, $Sx_{2n+1} \to Su$. Thus, by the uniqueness of the limit in b-2-metric space, we have $Su = u$. Now from contraction condition (3.1)

$$sd(Tu,u,a) = sd(Tu,Su,a) \le \varphi(\max\{sd(u,Tu,a), sd(u,Su,a), L[d(u,Su,a)+d(Tu,u,a)]\}) = \varphi(sd(u,Tu,a))$$

If we suppose that, $d(u,Tu,a) > 0$ then we have

$$sd(u,Tu,a) \le \varphi(sd(u,Tu,a) < sd(u,Tu,a)$$

is a contradiction. Therefore, $d(u,Tu,a) = 0$. Hence $Tu = u$ Thus we proved that u is a common fixed point of T and S.

Suppose now that u and v are different common fixed points of T and S, that is $d(u,v,a) > 0$. Then

$$\begin{aligned} sd(u,v,a) &= sd(Tu,Sv,a) \\ &\le \varphi(\max\{sd(u,Tu,a), sd(v,Sv,a), L[d(u,Sv,a)+d(v,Tu,a)]\}) \\ &= \varphi(2Ld(u,v,a)) \end{aligned}$$

Since $2L < 1 \le s$, then we get $sd(u,v,a) \le \varphi(sd(u,v,a)) < sd(u,v,a)$, a contradiction. Thus we proved that T and S, have a unique common fixed point in X.

If $S = T$ in Theorem3.1., then we have the following result.

Corollary3.1. Let (X,d) be a complete b-2-metric space with constant s and $T: X \to X$ a mapping on X. Suppose that there is a constant $L < \frac{1}{3}$ and a comparison function φ such that the inequality

$$sd(Tx,Ty,a) \le \varphi(\max\{sd(x,Tx,a), sd(y,Ty,a), L[d(x,Ty,a)+d(Tx,y,a)]\}) \qquad (3.12)$$

holds for each $x, y, a \in X$. Suppose that the mapping T is continuous. Then T has a unique fixed point.

Omitting the continuity assumption of mapping T or S in Theorem3.1., modifying the contraction condition (3.1) and imposing on a comparison function φ a corresponding condition, then we can prove the following

Theorem3.2. Let (X,d) be complete b-2-metric space with constant s and $T,S: X \to X$ two mappings on X. Suppose that there is a constant $L < \frac{1}{2+s}$ and a comparison function φ such that the inequality

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$$sd(Tx,Ty,a) \le \varphi(\max\{ sd(x,Tx,a), d(y,Ty,a), L[d(x,Ty,a)+d(Tx,y,a)]\}) \quad (3.13)$$

holds for each $x, y, a \in X$. If in addition function φ satisfies the following condition:

$$\lim_{n\to\infty} \sup \varphi(\beta) < \alpha \,,\ \alpha > 0 \quad (3.14)$$

Then S and T have a unique common fixed point.

Proof Since the contraction condition (3.13) implies the contraction condition (3.1) in Theorem3.1., then from the proof of Theorem3.1. it follows that the sequence $\{x_n\}$ defined as in (3.3), converges to some $u \in X$, that is

$$Tx_{2n} = x_{2n+1} \to u \ \text{ and } \ Sx_{2n+1} = x_{2n+2} \to u \ \text{ as } \ n \to \infty \quad (3.15)$$

Now we we prove that $Su = u$. From the contraction condition (3.13) and by monotonicity of φ we obtain

$$\begin{aligned} sd(x_{2n+1},Su,a) &= sd(Tx_{2n},Su,a) \\ &\le \varphi(\max\{sd(x_{2n},x_{2n+1},a), d(u,Su,a), L(d(x_{2n+1},u,a)+d(x_{2n},Su,a))\}) \quad (3.16) \\ &\le \varphi(\max\{sd(x_{2n},x_{2n+1},a), sd(u,Su,x_{2n+1}) + sd(u,x_{2n+1},a) + sd(x_{2n+1},Su,a), \\ &\quad L(d(x_{2n+1},u,a) + sd(x_{2n},Su,x_{2n+1}) + sd(x_{2n},x_{2n+1},a) + sd(x_{2n+1},Su,a))\}). \end{aligned}$$

Since φ is non decreasing and $L < \frac{1}{3}$, from (3.16) we get

$$sd(x_{2n+1},Su,a) \le \varphi(sd(x_{2n+1},u,a) + sd(x_{2n},Su,x_{2n+1}) + sd(x_{2n},x_{2n+1},a) + sd(x_{2n+1},Su,a)). \quad (3.17)$$

If we set

$$t_n = sd(x_{2n+1},u,a) + sd(x_{2n},Su,x_{2n+1}) + sd(x_{2n},x_{2n+1},a) + sd(x_{2n+1},Su,a)$$

Then in virtue of (3.15),

$$\lim_{n\to\infty} \sup t_n = \lim_{n\to\infty} \sup sd(x_{2n+1},Su,a) = r \quad (3.18)$$

where $r \ge 0$. Let $\{t_{n_k}\}$ be a subsequence of $\{t_n\}$ such that $t_{n_k} \to r$ as $k \to \infty$, For simplicity, denote $\{t_{n_k}\}$ again by $\{t_n\}$. Then from (3.18)

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} sd(x_{2n+1},Su,a) = r \quad (3.19)$$

Suppose that $r > 0$. Then from (3.19), (3.17), and the assumption (3.14) of φ, we have

$$r = \lim_{n\to\infty} t_n = \lim_{n\to\infty} sd(x_{2n+1},Su,a) \le \lim_{t_n \to r} \varphi(t_n) < r$$

a contradiction. Therefore,

$$\lim_{n\to} t_n = \lim_{n\to\infty} sd(x_{2n+1},Su,a) = 0$$

Hence we have $x_{2n+1} \to Su \ \text{ as } \ n \to \infty$. Since by (3.15), $x_{2n+1} \to u$ and as the limit in a b-2-metric space is unique, it follows that $Su = u$. Now by (3.13)

$$\begin{aligned} sd(Tu,u,a) &= sd(Tu,Su,a) \\ &\le \varphi(\max\{sd(u,Tu,a), d(u,Su,a), L(d(Tu,u,a)+d(u,Su,a))\}) \\ &= \varphi(sd(u,Tu,a)) \end{aligned}$$

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If we suppose that $d(u,Tu,a)>0$ then we have $sd(u,Tu,a)\le\varphi(sd(u,Tu,a)\le sd(u,Tu,a)$ a contradiction. Therefore, $d(u,Tu,a)=0$, that is $Tu=u$. Thus we proved that $Tu=Su=u$.
If $S=T$ in Theorem3.2., then we get the following result.

Corollary3.2. Let (X,d) be a complete b-2-metric space with constant s and $T:X\to X$ a mapping on X. Suppose that there is a constant $L<\frac{1}{1+s}$ and a comparison function φ such that the inequality

$$sd(Tx,Ty,a)\le\varphi(\max\{sd(x,Tx,a),sd(y,Ty,a),L[d(x,Ty,a)+d(Tx,y,a)]\}) \quad (3.12)$$

holds for each $x,y,a\in X$. If in addition a comparison function ϕ satisfies the inequality (3.14) ,then T has a unique fixed point.

IV. CONCLUSION

Generalization of theorems from b-metric spaces to b-2-metric spaces can be direct by using definitions, as in [33], but in this paper, generalization is not direct because of restrictions s (base in b-2-metric spaces) and L (comparison function constant).

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