

Properties of Ideals in an M-Semigroup

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ABSTRACT: In this paper, we discuss the various properties of ideal of an M-semigroup. It is clear that every ideal of an M-semigroup $M \cong R \times S$, is a fine band R of its left ideals each if which is a semi-deflation of I. Also, we find few conditions for an M-semigroup to be simple.

KEYWORDS: Semideflation, semi inflation, minimal left and right ideals, fine band.

I. INTRODUCTION

Tamura (14) proved that an M – Groupoid S is the direct product of a right singular semigroup and a groupoid with a two sides identity. By an M-semigroup we mean a groupoid M in which

P₁: There is atleast one $e \in M$ such that $ex=x$ for all $x \in M$.

P₂: $x(yz)=(xy)z$ for all $x,y,z \in M$ (Assosiative law)

P₃: For each $x \in M$, there is unique left identity e such that $xe=x$.

we discuss some properties of ideal of an M-semigroup.

II. PRELIMINARIES

2.1A semigroup S is called a right group if it is right simple and left cancellative.

2.2 A subset S' of a semigroup S is said to be a sub semi group of S if S' is a semigroup with the same binary operation of S.

2.3 Let X and Y be any two nonempty sets. Then the system $(S=X \times Y; *)$ where

$$(x,y) * (x',y') = (x,y')$$

For all x,x' in X and y,y' in Y is a band. It is called a rectangular band on $X \times Y$.

2.4 If A is band of type ζ , S is a band A of semigroups S_i ($i \in A$), and each semigroup S_i ($i \in A$) is a semigroup of type \mathfrak{S} , then S is called as a ζ - band A of \mathfrak{S} - semigroup.

2.5 The concept that is being defined now is due to Clifford and Preston (4). Let B be a semigroup. With each i of B, associate a set G_i consisting i (i in B) which are mutually disjoint. Let $G = \cup G_i$ (i in B) and let the product in B be extended to a product in G by defining $xy=ij$ if x is in G_i and y in G_j (i,j in B). Then G is a semigroup which is called an Inflation of B.

The following result is also due to the above authors:

Let B be a subsemigroup of a semigroup G such that B contains G^2 . Define $f: G \rightarrow B$ by

(i) f is a surjective mapping,

(ii) $f = f^2$ and

(iii) $xf.yf = xy$ for all x,y in G.

Then G is an inflation of B.

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2.6 The definition of inflation as given below is due to Tamura(14).

Let B be a given semigroup. Let S be any semigroup. Then S is an inflation of B if and only if,

- (i) S contains B as a semigroup,
- (ii) B contains a homomorphic image of S,

Say $(s)f$; such that

- (a) f/B is the identity map and
- (b) for all x,y in S, $xy= xf. yf$.

2.7 By an idempotent element of a semigroup S, is meant an element x in S, such that $x = x^2$; A semigroup S is said to be an idempotent semigroup if every element of S is idempotent.

2.8 A band is an idempotent semigroup.

2.9 By an M - groupoid S we mean a groupoid which satisfies the following conditions:

- (i) There is atleast one $e \in S$.
- (ii) If y or z is a left identity of S then $x(yz) = (xy)z$ for all $x \in S$.
- (iii) For any $x \in S$, there is a unique left identity e such that $xe = x$.

2.10 By an M – semigroup M we mean a semigroup that satisfies the following two conditions:

- P_1 : There is at least one $e \in M$ such that $ex =x$ for all $x \in M$.
- P_2 : For every $x \in M$ there is a unique left identity e such that $xe = x$.

2.11 An M – semigroup M is isomorphic to the direct product of a right zero semigroup and a semigroup with a two sided identity; and conversely.

That is $M \cong R \times S$ where R is a right zero semi – group and S is a semigroup with a two sided identity. The right zero semigroup R is the set of all left identities of M and $S \cong Me$, for a left identity e of M .

2.12 By a sub – M – semigroup M' of a semigroup S we mean a subsemigroup M' of S which is also an M – semigroup.

2.13 By an induced sub – M – semigroup M' of an M – semigroup M, we mean a sub – M – semigroup M' of M whose set of left identities is same as that of M.

2.14 By a normal sub – M – semigroup K of an M – semigroup we mean that K is a sub – M – semigroup of M satisfies:

- (i) $ka \subseteq ak$; for all a belong to M , and
- (ii) Every left identity of K is a left identity of M.

2.15 $M=\{e,f,a,b\}$ with the following operation is an M-semigroup

	e	f	a	b
e	e	f	a	b
f	e	f	a	b
a	a	b	a	b
b	a	b	a	b

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2.16 An M-Semigroup M is a disjoint union of its sub M-semigroups which are left ideals of M containing exactly one identity and each such subM-semigroup is a semi deflation of M.

III. IDEAL OF AN M-SEMIGROUP

In this part, we find that every ideal of an M-semigroup $M \cong R \times S$, is a fine band R of its left ideals each if which is a semi-deflation of I. Also, we find few conditions for an M-semigroup to be simple. In an M-semigroup $M \cong R \times S$, for a proper ideal I of M, $R \cap I = \emptyset$. Example 2.15 is an example of an M-semigroup which contains an ideal $I = \{a, b\}$.

3.1 Lemma:

Every ideal I of an M-semigroup $M \cong R \times S$ is a disjoint union of subsemigroup $I \cap Me, e \in R$. That is,

$$I = \bigcup_{e \in R} (I \cap Me).$$

Proof:

For any left identity e of M and $x \in I, xe \in I$. Since xe is an element of $Me, I \cap Me \neq \emptyset$; for all $e \in R$.

For any two distinct left identities e, f of M, Me and Mf are disjoint because, if $x \in Me \cap Mf$ then $x = le = mf$, for some l, m of M.

$mf = le = lee = mfe = me$.

Also, $(mf)f = mf; (mf)e = mfe = me = mf$.

By the uniqueness of e, $e=f$ which is a contradiction.

Therefore, $I \cap Me$ and $I \cap Mf$ are disjoint.

For any $xe, ye \in I \cap Me$,

$$xe.ye = xye \in I \cap Me$$

i.e., $I \cap Me$ is a semigroup.

Consider $\bigcup_{e \in R} (I \cap Me)$. For every $x \in I$, there exists a unique $e \in R$ such that $xe = x$. Therefore, $x \in I$ is an

element of a unique Me .

That is,
$$I = \bigcup_{e \in R} (I \cap Me)$$

3.2 Lemma:

Each $I \cap Me, e \in R$ is a left ideal of I.

Proof:

For any $xe \in I \cap Me$, for a fixed e of R and $y \in I$,

$$y(xe) = yxe \in (I \cap Me) \text{ proves the lemma.}$$

3.3 Lemma:

Each $I \cap Me, e \in R$ is a semi-deflation of I.

Proof:

Define $f : I \rightarrow I \cap Me$ for a fixed $e \in R$, such that $(x)f = xe$, for all $x \in I$.

It is well defined because for any $x \in I$, xe is an element of $I \cap Me$.

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$$\begin{aligned} (xy)f &= xye \\ &= xe.ye \\ &= (x)f.(y)f. \end{aligned}$$

For any $le \in I \cap Me$, $(le)f = lee = le$. Therefore, f is a homomorphism fixing the elements of $I \cap Me$. Hence, f is a semi-inflation map and each $I \cap Me$, $e \in R$ is a semi-deflation of I .

3.4 Lemma:

For any ideal I of an M -semigroup M , $I \cap Me \cong I \cap Mf$; $e, f \in R$.

Proof:

For any element $xe \in I \cap Me$

$xe.f \in I$ since I is an ideal.

Also, $xe.f = xf \in Mf$

Therefore, $xf = xe.f \in I \cap Mf$. That is, for every, $xe \in I \cap Me$ there exists an $xf \in I \cap Mf$. Similarly we can prove that for every $xf \in I \cap Mf$ there exists an $xe \in I \cap Me$.

Therefore, the map $\phi: I \cap Me \rightarrow I \cap Mf$ is onto.

Also, $(xe)\phi = (ye)\phi$

$$\begin{aligned} \Rightarrow xf &= yf \text{ since } (xe)\phi = xef = xf \text{ and} \\ & \quad (ye)\phi = yef = yf \end{aligned}$$

$$\Rightarrow xfe = yfe$$

$$\Rightarrow xe = ye.$$

Therefore, ϕ is one-one and hence $I \cap Me \cong I \cap Mf$.

3.5 Lemma:

I is a fine band R of subsemigroups $I \cap Me$, $e \in R$.

Proof:

Since $I = \bigcup_{e \in R} (I \cap Me)$, any element of I is of the form xe , for any $e \in R$

Define, $f: I \rightarrow R$ such that $(xe)f = e$; for all $xe \in I$, $e \in R$.

Clearly f is a homomorphism onto a band R .

$$(e)f^{-1} = I \cap Me, e \in R.$$

For any $xe \in I \cap Me$ and $yf \in I \cap Mf$

$xe.yf = xyf \in I \cap Mf$. That is,

$$\begin{aligned} (I \cap Me)(I \cap Mf) &\subseteq (I \cap Mf) \\ &= (I \cap Mf)^2, \text{ since } I \cap Mf \text{ is total.} \end{aligned}$$

That is, I is a fine band R of subsemigroup $I \cap Me$, $e \in R$.

From lemmas 3.1, 3.2, 3.3, 3.4 and 3.5 we have,

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3.6 Theorem:

Every ideal I of an M -semigroup M is a fine band R of isomorphic left ideals of I , which are semi-deflations of I .

Since M is a trivial ideal of itself we have

3.7 Corollary:

An M -semigroup $M \cong R \times S$ is a fine band R of isomorphic left ideals of M which are semi-deflation of M [2.16].

Now, we shall find few characterizations for an M -semigroup M to be simple.

3.8 Lemma:

Let $M \cong R \times S$ be an M -semigroup containing both minimal right ideals. If $Me, e \in R$ is a minimal left ideal of M , then M is simple.

Proof:

For any proper ideal I of M , $Ie \subseteq Me$, and Ie is a left ideal of M contained in Me , which is a contradiction. Therefore, M cannot have any ideal. That is, M is simple.

We know that 'if a semigroup S is simple and containing both minimal left and minimal right ideals then every minimal left (right) ideal of S is of the form $Se(fS)$ where e, f are idempotents' [8]. Therefore, by 3.8 we have,

3.9 THEOREM:

An M -semigroup containing both minimal left and minimal right ideals is simple if and only if $Me, e \in R$ is its minimal left ideal. Also we have

3.10 THEOREM:

A simple M -semigroup containing both minimal left and minimal right ideals is a disjoint union of its minimal left ideals.

3.11 Lemma:

If an M -semigroup $M \cong R \times S$ containing both minimal left and minimal right ideal is simple then the set of idempotents E of M is R .

Proof:

In a simple M -semigroup M , containing both minimal left ideals and minimal right ideals, $Ma, a \in E$ is a minimal left ideal [13].

But $M = \bigcup_{e \in R} Me$ where each Me is a left ideal.

Since $e \in E, Me$ is a minimal left ideal. If $a \in E \setminus R$ then Ma is also a minimal left ideal. If $a = ae$ for an $e \in R$, then Ma is a minimal left ideal.

But $Ma \subseteq Me$, a contradiction to the fact that Me is a minimal left ideal. Therefore $a \in R$ and hence $E = R$. Hence the lemma.

3.12 Definition:

A semigroup S is called as a right group if it is right simple and left cancellative.

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3.13 THEOREM:

Let $M \cong R \times S$ be an M-semigroup containing both minimal left and minimal right ideals. M is simple if and only if M is a right group.

Proof:

Let M be simple. Clearly $M = \bigcup_{e \in R} Me$, where each Me is a minimal left ideal of M [13].

By 3.11, the set of idempotents of M is R. Therefore, the minimal right ideal of M is $eM = M$. That is, M is simple and containing a minimal left ideal Me and a minimal right ideal $eM = M$. Therefore M is the union of isomorphic groups [13]. That is M is a right group. Converse is clear.

3.14 Lemma:

If $Me, e \in R$ is a minimal left ideal of an M-semigroup $M \cong R \times S$, then M is a right group.

Proof:

Since $Me, e \in R$ contains an idempotent e, M contains a minimal right ideal also [13]. Therefore, from 3.8 and 3.13 M is a right group.

From 3.14 and [4] we have

3.15 THEOREM:

If $Me, e \in R$ is a minimal left ideal of an M-semigroup $M \cong R \times S$, then M is a right group and conversely.

In an M-semigroup $M \cong R \times S$, which is not a simple semigroup, let M_1 be a maximal ideal of M. We know $M_1 \cap R = \emptyset$. If possible let M_2 be another maximal ideal of M. Since M_1 is maximal and $M_2 \neq M_1$; $M_1 \cup M_2 = M$.

That is, M_2 contains R which is not possible unless $M_2 = M$. Therefore M_1 is a unique maximal ideal of M.

Hence we have

3.16 THEOREM:

A nonsimple M-semigroup contains a unique maximal ideal of M.

REFERENCES

1. 13. Ananth K. Atre, "On a semigroup which is an inflation of its set of factorizable elements" *vig. Bharati*(1975)69-72.
2. Clifford, A.H. "Semi groups containing minimal ideals". *Am.J. Maths.*70(1948) 521.
3. Clifford, A.H. "Bands of semi groups". *Proc. Am. Math. Soc.* 5(1954) 499-504.
4. Clifford, A.H. and Preston, G.B "The algebraic theory of semi groups". Vol.I. *Mathematical surveys*, No.7, 1961, American Mathematical society, Rhode Island.
5. Harishito and Hiroshi " On the structure of semigroups containing minimal left ideals and minimal right ideals". *Proc. Jap. Aca*> 31(1995) 264-266.
6. Lakshmanan.L " Certain studies in the structure of an algebraic semigroup" Thesis submitted to Bangalore University(1983).
7. Lakshmanan. L " On an M-semigroup and its set of idempotent elements" *Indian journal of pure and applied Mathematics* 21(5)423-433, May 1990.
8. Lakshmanan.L "Some properties of sub semigroups of an M-semigroup" *IJIRSET* vol.5 Issue 6, June 2016, 9473-9476
9. Petrich Mario " Prime ideals of the Cartesian product of two semigroups". *Czh. Math.J.*12(87) (1962) 150-152.
10. Plemmons Robert " Maximal ideals in the direct product of two semi groups" *Czh. Math.J.*17(92) (1967) 257-260
11. Preston, G.B. " Inverse semigroups with minimal right" *J.Lond.Math.Soc.*29(1954) 404-411.
12. Schwarz Steafan "On the structure of simple semigroups without zero" *Czh.Math.J.*1(76) (1951)41-53. MR 14,12.1
13. Schwarz Steafan "On the structure of simple semigroups without zero" *Czh.Math.J.*1(76) (1951)41-53. MR 14,12.
14. Tamura, T. "Semigroups satisfying the identity $xy = f(x,y)$ ". *Pac. J. Maths.*31(1969)513-521.
15. 10. Tully Edward T. Junior "Semigroups in which each ideal is a retract" *Jl. Of Aust.Math. Soc.*9 (1969) 239-245. MR 39#1582.
16. Venuraju, K. And Hanumanthachari, J. "A note on maximal and minimal prime ideals belonging to an ideal in semigroups" *Indian Journal of Pure and Applied Mathematics* 11 (12) 1566-71.