

An Overview on Numerical Solution of System of Linear Equations through Direct Methods

Alok Kumar Maurya¹, Atendra Singh Yadav², Ashish Kumar³, Kishan Kumar Kushwaha⁴

M.Sc. Student, Department of Mathematics & Statistics, Sam Higginbottom Institute of Agriculture, Technology & Sciences Allahabad, U.P., India^{1,4}

Research Scholar, Department of Mathematics & Statistics, Sam Higginbottom University of Agriculture, Technology & Sciences Allahabad, U.P., India^{2,3}

ABSTRACT: In this paper we present a survey of three direct methods for the solution of system of linear equations. Various methods are introduced to solve systems of linear equations but in contrast to all methods not a single method are best for all situations (to get appropriate solution). These methods depend on speed and accuracy as these are an important factor in solving large systems of equations because the volume of computations involved for solution are bulky. Direct solution of simultaneous linear equations may be regarded as a slow process for large systems of equations and requires special treatment to avoid numerical instability.

KEYWORDS: Matrix, Gauss Elimination, Gauss Matrix Elimination, LU Decomposition Method linear equations.

I. INTRODUCTION

A collection of linear equations together are known as a system of linear equations. These are engaged with the same set of variables. Different methods are being used for the solution of system of linear equations. Not a single method which used to be best method to get appropriate result. Speed used as a significant factor for the solution of system of linear equations as the volume of computation engaged are large. These methods need fewer multiplications for huge system of equations as well as have fewer rounds off errors. The solution of a linear system of equation can be accomplished by a numerical method which directs method. Amongst the direct method, they will describe the elimination method by gauss as also its modification and the *LU* decomposition method. Numerical analysis, the subject which studies algorithms for computing expressions defined with real numbers. Numerical analysis made it possible. There are two different phases to address in numerical analysis. The second means that the behavior of the algorithm is continuous with respect to the parameters of the algorithm. In trying to improve the accuracy or efficiency of a stable algorithm, one often led to consider algorithms that turn out to be unstable and therefore of minimal (if any) value. These various aspects of numerical analysis are often intertwined, as ultimately we want an algorithm that we can analyze rigorously to ensure it is effective when using computer arithmetic.

II. METHODOLOGY

The coefficients a , b , c , d , and f are given constants, and the task is to find the unknowns values x , y . Let n be a positive integer. The general form for a system of n linear equation in the n unknown components x_1, x_2, \dots, x_n is

International Journal of Innovative Research in Science, Engineering and Technology

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Vol. 6, Issue 5, May 2017

$$\left. \begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \dots + a_{nn}x_n &= b_n \end{aligned} \right\}$$

The coefficients are given symbolically by a_{ij} . With i the number of the equation and j is the number of the associated unknown component. On some occasions, to avoid possible confusion, they also use the symbol a_{ij} . The right-hand sides b_1, \dots, b_n are given numbers; and the problem is to calculate the unknowns x_1, x_2, \dots, x_n . The linear system is said to be of order n .

Gauss Elimination Method

This is the elementary elimination method and it reduces the system of equation to an equivalent upper-triangular system, which can be solved by back substitution.

Let the system of n linear equation in n unknowns be given by

$$\left. \begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= b_n \end{aligned} \right\} \quad (1.1)$$

There are two steps in the solution of the system given in equation (1.1), viz., the elimination of unknowns and back substitution.

Step 1: The unknowns are eliminated to obtain an upper-triangular system. To eliminate x_1 from the second equation,

multiply the first equation by $\left(-\frac{a_{21}}{a_{11}}\right)$ and obtain

$$-a_{21}x_1 - a_{12}\frac{a_{21}}{a_{11}}x_2 - \dots - a_{1n}\frac{a_{21}}{a_{11}}x_n = -b_1\frac{a_{21}}{a_{11}}$$

Adding the above equation to the second equation of (1.1),

$$\text{obtain } \left(a_{22} - a_{12}\frac{a_{21}}{a_{11}}\right)x_2 + \left(a_{23} - a_{13}\frac{a_{21}}{a_{11}}\right)x_3 + \dots + \left(a_{2n} - a_{1n}\frac{a_{21}}{a_{11}}\right)x_n = b_2 - b_1\frac{a_{21}}{a_{11}}$$

Above can be written as

$$a'_{22}x_2 + a'_{23}x_3 + \dots + a'_{2n}x_n = b'_2 \quad \dots(1.2)$$

Thus the primes indicate that the original element has changed its value. Similarly, they can multiply the first equation by $-a_{31}/a_{11}$ and add it to the third equation of the system(1.1). This eliminates the unknown x_1 from the equation of equation (1.1) and obtain.

$$a'_{32}x_2 + a'_{33}x_3 + \dots + a'_{3n}x_n = b'_3 \quad \dots(1.3)$$

They can eliminate x_1 from the remaining equations and after eliminating x_1 from the last equation of equation(1.1), obtain the system

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$$\left. \begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n &= b_1 \\ a'_{22}x_2 + a'_{23}x_3 + \dots + a'_{2n}x_n &= b'_2 \\ a'_{32}x_2 + a'_{33}x_3 + \dots + a'_{3n}x_n &= b'_3 \\ &\vdots \\ a'_{n2}x_2 + a'_{n3}x_3 + \dots + a'_{nn}x_n &= b'_n \end{aligned} \right\} \dots (1.4)$$

They next eliminate x_2 from the last $(n - 2)$ equations of equation(1.3). Before this, it is important to notice that in the process of obtaining the above system, multiply the first row by $(-a_{21}/a_{11})$, i.e. we have divided it by a_{11} which is therefore assumed to be nonzero. For this reason, the first equation in the system (1.3) is called the pivot equation, and a_{11} is called the pivot or pivotal element. The method obviously fails if $a_{11} = 0$. They shall discuss this important point after completing the description of the elimination method. Now, to eliminate x_2 from the third equation. Repeating this process with the remaining equations, they obtain the system the third equation of equation(1.3). Multiply the second equation by $(-a'_{32}/a'_{22})$ and add it to m

$$\left. \begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n &= b_1 \\ a'_{22}x_2 + a'_{23}x_3 + \dots + a'_{2n}x_n &= b'_2 \\ a''_{33}x_3 + \dots + a''_{3n}x_n &= b''_3 \\ &\vdots \\ a''_{n3}x_3 + \dots + a''_{nn}x_n &= b''_n \end{aligned} \right\} \dots (1.5)$$

In Equation(1.4), the double primes indicate that the elements have changed twice. It is easily seen that this procedure can be continued to eliminate x_3 from fourth equation onwards, x_4 from the fifth equation onwards, etc., finally obtain the upper-triangular form;

$$\left. \begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{1n}x_n &= b_1 \\ a'_{22}x_2 + a'_{23}x_3 + \dots + a'_{2n}x_n &= b'_2 \\ a''_{33}x_3 + \dots + a''_{3n}x_n &= b''_3 \\ &\vdots \\ a^{(n-1)}_{nn}x_n &= b^{(n-1)}_n \end{aligned} \right\} \dots (1.6)$$

Were $a^{(n-1)}_{nn}$ indicates that the element a_{nn} has changed $(n - 1)$ times. They thus have completed the first step of elimination of unknowns and reduction to the upper-triangular form.

Step 2: They now have to obtain the required solution from the system(1.5), from the last equation of this system, they obtain

$$x_n = \frac{b^{(n-1)}_n}{a^{(n-1)}_{nn}} \dots(1.7)$$

This is then substituted in the $(n - 1)^{th}$ equation to obtain x_{n-1} and the process is repeated to compute the other unknowns. They have therefore first computed x_n , then $x_{n-1}, x_{n-2}, \dots, x_2, x_1$, in that order. Due to this reason, the process is called back substitution.

Example: Use gauss elimination to solve the system

$$10x + y + z = 12 \quad \dots(1.8)$$

$$2x + 10y + z = 13 \quad \dots(1.9)$$

$$x + 4y + 3z = 5 \quad \dots(1.10)$$

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Solution: First eliminate x from the (1.9) and (1.10), we get

$$8y - 5z = 3 \quad \dots (A)$$

Similarly, eliminate x from equations (1.8) and (1.10), we get

$$9y + 29z = 38 \quad \dots (B)$$

Eliminate y from equation (A) and (B), we get

$$z = 1$$

The upper-triangular form is therefore given by

$$10x + y + z = 12$$

$$8y - 5z = 3$$

$$z = 1$$

It follows that the required solution is $x = 1$, $y = 1$ and $z = 1$.

Gauss-Jordan method

By this method we eliminate unknown not only from the equations below but also from the equations above. In this way system is reduced in diagonal matrix. This method can be used to solve systems of linear equations involving two or more variables. Gauss-Jordan method is modification of the gauss elimination method, the essential difference being that when an unknown is eliminated, it is eliminated from all equation. The method does not require back substitution to obtain the solution and is the best illustrated by the following example.

Example: Use to Gauss-Jordan method solve the system

$$10x + y + z = 12$$

$$2x + 10y + z = 13$$

$$x + 4y + 3z = 5$$

Solution: First eliminate x from the system of linear equation taking first two we get

$$49y + 4z = 53 \quad \dots (C)$$

Eliminate x from the equation first and second we get

$$9y + 29z = 38 \quad \dots (D)$$

Eliminate y from equation (C) and (D), we get $z = 1$

Put values of z in equations above equation, we get

It follows that the required solution is $x = 1$, $y = 1$ and $z = 1$.

LU Decomposition Method

Introduced by Mathematician Alan Tango this LU method is used to solve the system of linear equations. The first step is to convert the given system to matrix form. And then after using this method the matrix is decomposed into the product of lower and upper triangular matrix. It can be viewed as a matrix form of gauss elimination.

In LU decomposition of matrix, they described a scheme for computing the matrices L and U such that

$$A = LU \quad \dots (E)$$

Where L is lower triangular and U an upper triangular matrix and these are given in equation

Let the system of equations be

$$\left. \begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned} \right\} \quad \dots (F)$$

This can be written in the matrix form

$$AX = B \quad \dots (2.0)$$

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$$LUX = B \quad \dots (2.1)$$

If they set

$$UX = Y \quad \dots (2.2)$$

Then equation (2.1) becomes

$$LY = B \quad \dots (2.3)$$

This is equivalent to the system

$$\left. \begin{aligned} y_1 &= b_1 \\ l_{21}y_1 + y_2 &= b_2 \\ l_{31}y_1 + l_{32}y_2 + y_3 &= b_3 \end{aligned} \right\} \dots (2.4)$$

Where, $(y_1, y_2, y_3)^T = Y$ and $(b_1, b_2, b_3)^T = B$

The system (2.4) can be solved for y_1, y_2 and y_3 by the forward substitution. When Y is known, the system (2.3) becomes

$$\left. \begin{aligned} u_{11}x_1 + u_{12}x_2 + u_{13}x_3 &= y_1 \\ u_{22}x_2 + u_{23}x_3 &= y_2 \\ u_{33}x_3 &= y_3 \end{aligned} \right\} \dots (2.5)$$

Which can be solved for x_1, x_2, x_3 by back substitution process. As noted earlier, this method has the advantage of being applicable to solve systems with the same coefficient matrix but different right-hand side vector.

Example: Use to LU Decomposition Method solve the system

$$\begin{aligned} 10x + y + z &= 12 \\ 2x + 10y + z &= 13 \\ x + 4y + 3z &= 5 \end{aligned}$$

Solution: Above equations in the matrix $(AX=B)$ form

$$\begin{bmatrix} 10 & 1 & 1 \\ 2 & 10 & 1 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 13 \\ 5 \end{bmatrix} \quad \dots(1.34)$$

By LU decomposition method first we take $LU=A$

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{12} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 & 1 & 1 \\ 2 & 10 & 1 \\ 1 & 3 & 3 \end{bmatrix} \quad \dots(1.35)$$

$$\begin{bmatrix} l_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix} = \begin{bmatrix} 10 & 1 & 1 \\ 2 & 10 & 1 \\ 1 & 3 & 3 \end{bmatrix} \quad \dots (1.36)$$

Solve equations (1.36) and find the values of $l_{11}, l_{12}, l_{22}, l_{31}, l_{32}, l_{33}, u_{12}, u_{13}, u_{23}$ in equation (1.35) we get

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$$\begin{bmatrix} 1 & 0 & 0 \\ 1/5 & 1 & 0 \\ 1/10 & 1/32 & 0 \end{bmatrix} \begin{bmatrix} 10 & 1 & 1 \\ 0 & 99/5 & 4/5 \\ 0 & 0 & 63/22 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 13 \\ 5 \end{bmatrix} \quad \dots(1.37) \quad \text{Putting}$$

$$\begin{bmatrix} 10 & 1 & 1 \\ 0 & 99/5 & 4/5 \\ 0 & 0 & 63/22 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \quad \dots (1.38)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1/5 & 1 & 0 \\ 1/10 & 1/32 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 12 \\ 13 \\ 5 \end{bmatrix} \quad \dots (1.39)$$

From equation (1.39) we get $u = 12$, $v = 53/5$ and $w = 608/5$

$$\begin{bmatrix} 10 & 1 & 1 \\ 0 & 99/5 & 4/5 \\ 0 & 0 & 63/22 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 53/5 \\ 608/5 \end{bmatrix} \quad \dots (1.40)$$

From equation (1.40) we get $x = -3.16$, $y = 1.18$, $z = 42.46$

III. CONCLUSIONS

Direct methods are not appropriate for solving large number of equations in a system, particularly when the coefficient matrix is sparse, i.e. when most of the elements in a matrix are zero as contrast to LU decomposition method.

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