

Comparative Study of Gompertz and Logistic Laws of Growth Functions

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ABSTRACT: This paper deals with the comparative study of the Gompertz [1] and Logistic law of growth functions [2]. It is considered here that the fish species obey the two laws of growth functions separately. We then discuss their growth rate, steady states and dynamical behaviours, in case of both unexploited and exploited fishery. The general and particular solutions satisfying the initial conditions are determined for both the growth equations in case of exploited as well as unexploited system. All the results are illustrated graphically.

KEYWORDS: Gompertz law of growth, Logistic law of growth, environmental carrying capacity, steady states, exploited and unexploited fishery, harvesting effort.

I. INTRODUCTION

The Logistic law of growth function is $F(x) = rx \left(1 - \frac{x}{k}\right)$, where r is a positive constant, called the intrinsic growth rate of the population and k is also a positive constant, called the environmental carrying capacity. This growth function was first proposed by P.F.Verhulst [2] in the year 1838 and first used in the model of human population. Logistic growth function is first used in the commercial marine fishery by the biologist M.B.Schaefer [3]. Schaefer considered a fish species obeying the logistic law of growth and adopted the CPUE (catch-per-unit-effort) hypothesis [1] to represent the catch-rate function.

In spite of various limitations of the Logistic growth function, this function is still being used in the commercial marine fishery for its simplicity. Researchers like Pradhan and Chaudhuri [4], Ganguly and Chaudhuri [5], Ray and Pradhan [6] and many other researchers discussed several mathematical fishery models by using Logistic growth function as the growth function of the fish species.

Many alternative terms for the growth function $F(x)$ have been suggested by May [7]. Gompertz law of growth function is one of the important growth functions. Gompertz law of growth function is in the form $F(x) = rx \ln\left(\frac{k}{x}\right)$, where r is a positive constant, called the intrinsic growth rate of the population and k is a positive constant, called the environmental carrying capacity. Pradhan and Chaudhuri [8] discussed a bioeconomic modeling of a single species fishery with Gompertz law of growth.

In this paper we have discussed the growth rate, the steady states and the dynamical behaviours of a fish species obeying both the growth laws separately in cases of exploited as well as unexploited fishery. It is also proved that the non trivial steady state for the Logistic growth function exists only when the harvesting effort is less than the BTP, whereas no restriction is required for existence of the non trivial steady state of the Gompertz law of growth function. In the exploited fishery it is proved that in the Gompertz law of growth the population reaches earlier to the steady state compared to the Logistic law of growth. It is also seen that the growth rate of the exploited fishery in the Gompertz law of growth ceases later than the Logistic growth law. The general and particular solutions of both the growth equations are found for the exploited and unexploited systems and the solutions are then compared for both the laws analytically and graphically. Comparing all the results of the two cases we have proved that the Gompertz law of growth is more

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realistic and flexible than the Logistic law of growth. Graphical representation of the results helps us to compare these two growth laws.

II. COMPARISON OF GROWTH FUNCTIONS

Let $x(t)$ be the population density of a fish species at time t . The Gompertz growth function [1] is $G(x) = rx \ln\left(\frac{k}{x}\right)$, where r is the intrinsic growth rate of the population and k is a constant, called the carrying capacity of the environment, such that $k = \lim_{t \rightarrow \infty} x(t)$.

$$\begin{aligned} \text{Now, } rx \ln\left(\frac{k}{x}\right) &= -rx \ln\left(\frac{x}{k}\right), 0 < \frac{x}{k} < 1 \\ &= -rx \left\{ \left(\frac{x}{k} - 1\right) - \frac{1}{2} \left(\frac{x-k}{k}\right)^2 + \frac{1}{3} \left(\frac{x-k}{k}\right)^3 + \dots \dots \dots \right\} \end{aligned}$$

Since $\left|\frac{x-k}{k}\right| < 1$, so the above infinite series is a convergent series.

When $\frac{x-k}{k} \ll 1$, neglecting higher order terms of $\frac{x-k}{k}$, the above expression becomes

$$rx \ln\left(\frac{k}{x}\right) \approx rx \left(1 - \frac{x}{k}\right) = L(x) \text{ (say), which is the Logistic growth function [2].}$$

So the Logistic growth function is an approximation of the Gompertz growth function.

Let us consider the Gompertz growth function $G(x) = rx \ln\left(\frac{k}{x}\right)$.

$$\text{Therefore, } \frac{dG(x)}{dx} = r \left\{ \ln\left(\frac{k}{x}\right) - 1 \right\}.$$

$$\frac{dG(x)}{dx} = 0 \Rightarrow x = \frac{k}{e} \text{ and } G\left(\frac{k}{e}\right) = \frac{rk}{e}.$$

$$\text{Now, } \frac{d^2G(x)}{dx^2} = -\frac{r}{k} < 0.$$

Therefore, in the Gompertz law of growth, $\frac{k}{e}$ is an extreme value of the growth function $G(x), \forall x \in [0, k]$ and the growth function attains its maximum value at $x = \frac{k}{e}$.

Again $\frac{dG(x)}{dx} > 0$ for all x such that $0 < x < \frac{k}{e}$ and $\frac{dG(x)}{dx} < 0$ for all x such that $\frac{k}{e} < x < k$.

So in the interval $\left(0, \frac{k}{e}\right)$ the Gompertz growth function is an increasing function of population density, increasing from 0 to $\frac{rk}{e}$ and in the interval $\left(\frac{k}{e}, k\right)$, the growth function is a decreasing function of population density, decreasing from $\frac{rk}{e}$ to 0. The maximum growth is $\frac{rk}{e}$ when the population size is $\frac{k}{e}$.

Now, we consider the Logistic growth function $L(x) = rx \left(1 - \frac{x}{k}\right)$.

$$\text{Therefore, } \frac{dL(x)}{dx} = r - \frac{2rx}{k}.$$

$$\frac{dL(x)}{dx} = 0 \Rightarrow x = \frac{k}{2} \text{ and } L\left(\frac{k}{2}\right) = \frac{rk}{4}.$$

$$\text{Now, } \frac{d^2L(x)}{dx^2} = -\frac{2r}{k} < 0.$$

Therefore, in the Logistic law of growth, $\frac{k}{2}$ is an extreme value of the growth function $L(x), \forall x \in [0, k]$ and the growth function attains its maximum value at $x = \frac{k}{2}$ i.e. at the half level of environmental carrying capacity.

Again $\frac{dL(x)}{dx} > 0$ for all x such that $0 < x < \frac{k}{2}$ and $\frac{dL(x)}{dx} < 0$ for all x such that $\frac{k}{2} < x < k$.

So in the interval $\left(0, \frac{k}{2}\right)$ the Logistic growth function is an increasing function of population density, increasing from 0 to $\frac{rk}{4}$ and in the interval $\left(\frac{k}{2}, k\right)$, the growth function is a decreasing function of population density, decreasing from $\frac{rk}{4}$ to 0. The maximum growth for this function is $\frac{rk}{4}$ when the population size is $\frac{k}{2}$.

From the above results we see that the growth of the population initially increases to a certain level and then declines to zero for both the growth functions. Biological implication of the above result is that, the ideal living conditions prevail

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in the initial stage and there should be a very rapid growth initially. Thereafter, as the population grows, the limitation of resources forces the growth rate decline and the population gradually approaches the saturation level.

Since $\frac{rk}{e} > \frac{rk}{4}$, so $\max G(x) > \max L(x) \forall x \in [0, k]$ i.e. the maximum growth of the population obeying the Gompertz law is greater than the maximum growth of the population obeying the Logistic law of growth.

Again, since $\frac{k}{e} < \frac{k}{2}$, so the growth of the population obeying Gompertz law of growth increases in short interval and decreases in a long interval compared to the Logistic law of growth.

Therefore, the growth of the population obeying Gompertz growth law is faster than the growth of the population obeying Logistic growth law.

Figure-1 represents the growth curves corresponding to the two growth laws for the intrinsic growth rate $r = 5$ and carrying capacity $k = 100$.

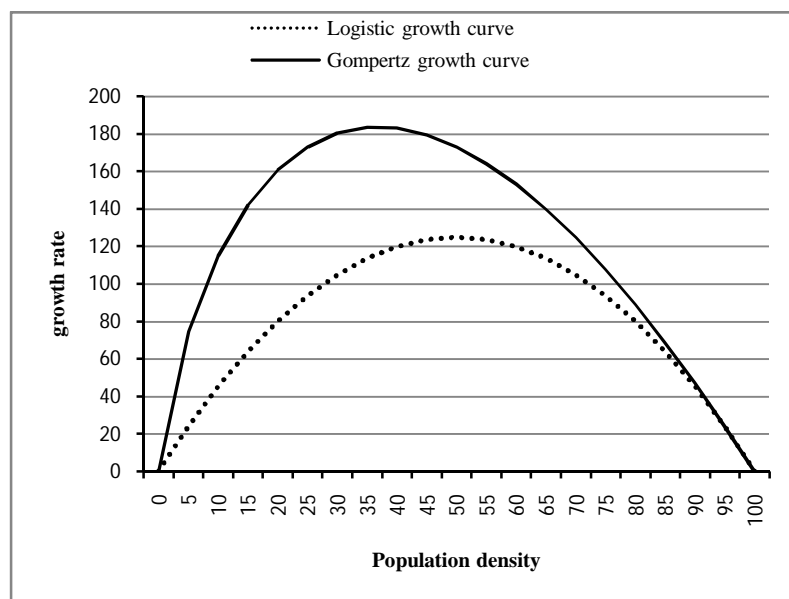


Fig.1: Growth curves in two laws for $r = 5$ and $k = 100$

III. SOLUTIONS OF GROWTH EQUATIONS FOR UNEXPLOITED SPECIES

Let at any time t , $x(t)$ be the population density of a fish species which obeys the Gompertz law of growth. Therefore, the growth equation becomes

$$\frac{dx(t)}{dt} = rx \ln\left(\frac{k}{x}\right), x > 0 \forall t \text{ and } x_0 = x(0) \tag{1}$$

$$\text{Solving (1), we have } x(t) = e^{\ln k - (\ln k - \ln x_0)e^{-rt}} \tag{2}$$

Now, $x(t) \rightarrow k$ as $t \rightarrow \infty$ by (2)

If the fish species obeys the logistic law of growth, then the growth equation becomes

$$\frac{dx(t)}{dt} = rx \left(1 - \frac{x}{k}\right), x > 0 \forall t \text{ and } x_0 = x(0) \tag{3}$$

$$\text{Solving (3), we have } x(t) = \frac{k}{1 + \left(\frac{k-x_0}{x_0}\right)e^{-rt}} \tag{4}$$

Therefore, $x(t) \rightarrow k$ as $t \rightarrow \infty$ by (4)

From the above results it is clear that, the limiting value of the population density for both the growth laws is equal to the environmental carrying capacity in the unexploited fishery.

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Figure-2 and Figure-3 represent the solution curves of logistic growth equation and Gompertz growth equation respectively.

Here we take the intrinsic growth rate $r = 5$ and the environmental carrying capacity $k = 100$ in appropriate units.

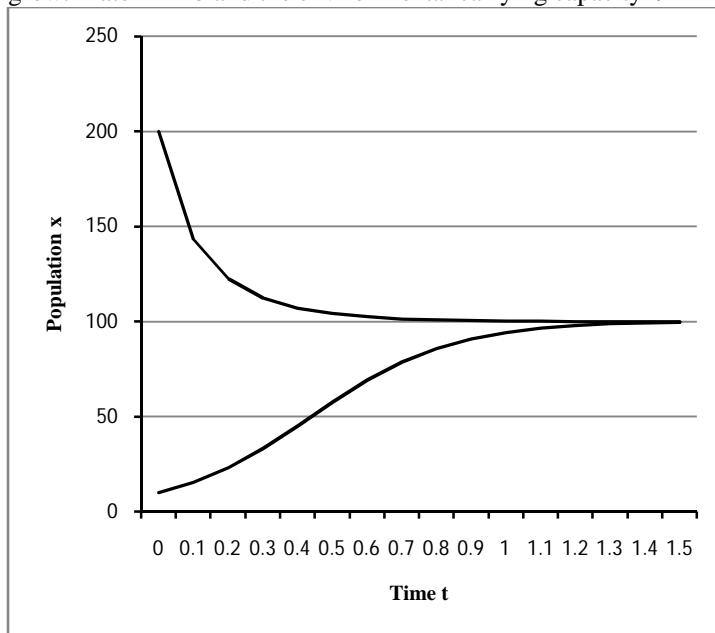


Fig.2: Population density curve in Logistic law of growth for unexploited fishery

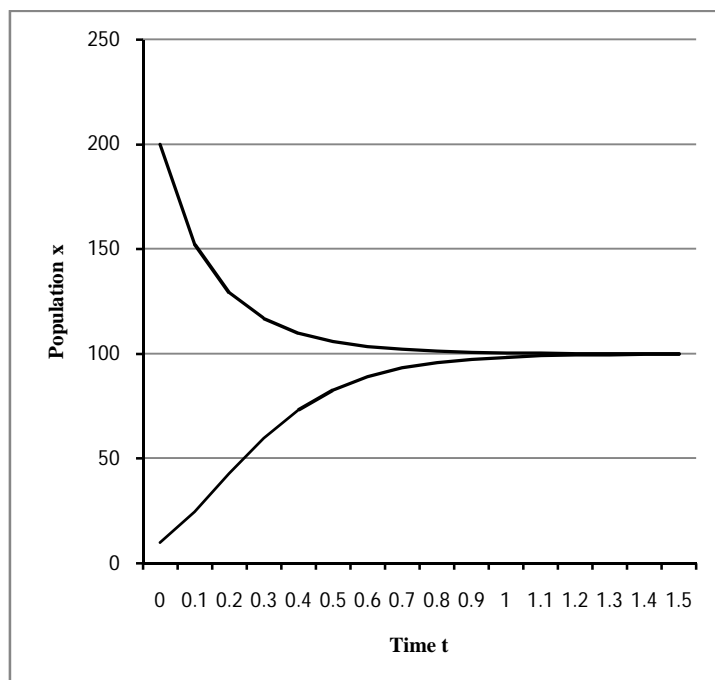


Fig.3: Population density curve in Gompertz law of growth for unexploited fishery

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When the initial population density $x_0 = 10$, below the carrying capacity $k = 100$, we see that the population density gradually increases and tends to the carrying capacity k in both the curves. When we take initial population density $x_0 = 200$, above the carrying capacity, then the population density gradually decreases and tends to the carrying capacity k in both the curves.

Biologically it is true that when the population density is below the carrying capacity then for sufficient resources the species grows up to the maximum limit of the carrying capacity and when the population density is above the carrying capacity then due to limitation of resources and crowding effects, the population density declines to the level of carrying capacity.

But comparing the two solution curves, we see that, when the initial population density is below the carrying capacity then for the Gompertz law of growth the population density increases very fast compared to the Logistic growth law. Whereas the population density for the Gompertz law of growth decreases slowly compared to the logistic law of growth when the initial population density is above the carrying capacity.

For both the growth equations $x = k$ is the non-trivial equilibrium point.

Since $\frac{dx}{dt} > 0$ for all x such that $0 < x < k$ and $\frac{dx}{dt} < 0$ for all x such that $x > k$, so $x = k$ is globally asymptotically stable equilibrium point for both the growth equations.

IV. COMPARISON OF TWO LAWS FOR EXPLOITED FISH SPECIES

Now it is assumed that the species is subjected to harvesting at a constant rate qEx where E denotes the harvesting effort and q is a constant, called the catchability coefficient. Then the growth equations (1) and (3) becomes

$$\frac{dx(t)}{dt} = rx \ln\left(\frac{k}{x}\right) - qEx, \quad x > 0 \quad \forall t \text{ and } x_0 = x(0) \tag{5}$$

$$\text{and } \frac{dx(t)}{dt} = rx \left(1 - \frac{x}{k}\right) - qEx, \quad x > 0 \quad \forall t \text{ and } x_0 = x(0) \tag{6}$$

respectively.

For a constant effort $E = 30$, catchability coefficient $q = 0.1$, intrinsic growth rate $r = 5$ and the carrying capacity $k = 100$ in appropriate units the growth curves of the equations (5) and (6) are given in Figure-4.

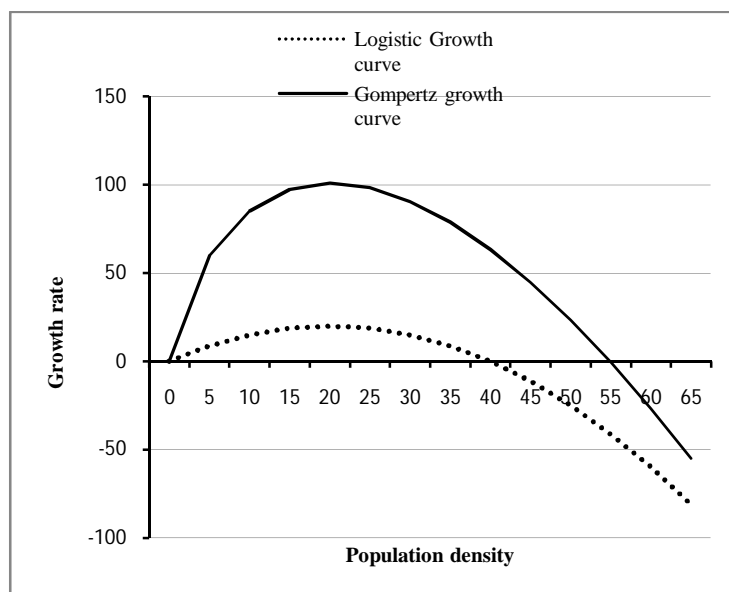


Fig.4: Growth curves in two laws for $r = 5, q = 0.1, E = 30$ and $k = 100$

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From the above figure we see that the growth rate of the exploited fish species is very slow and ceases early if the species obeys logistic law of growth instead of Gompertz law of growth. In a mathematical model if the species obey the Logistic law of growth and due to harvest the growth rate becomes zero at any time, then the actual growth rate is not zero but very low. Since the logistic law of growth function is an approximation of the Gompertz law of growth function, so such a situation occurs. For this reason we can say that the Gompertz law of growth function is more realistic than the logistic law of growth function.

V. SOLUTIONS OF GROWTH EQUATIONS FOR EXPLOITED SPECIES

Equation (5) can be written as

$$\frac{dx}{dt} = rx(A - \ln x) \quad \text{where } A = \ln k - \frac{qE}{r}$$

$$\Rightarrow \frac{dx}{x(\ln x - A)} = -r dt$$

Solving the above equation we have, $\ln x - A = Ce^{-rt}$.

Using the initial condition $x_0 = x(0)$, we have $C = \ln x_0 - A$

Therefore, $\ln x - A = (\ln x_0 - A)e^{-rt} \Rightarrow x(t) = e^{A + (\ln x_0 - A)e^{-rt}}$

$$\text{i.e. } x(t) = e^{\left\{(\ln k - \frac{qE}{r}) - (\ln k - \frac{qE}{r} - \ln x_0)e^{-rt}\right\}} \quad (7)$$

Now, $x(t) \rightarrow e^{\ln k - \frac{qE}{r}} = ke^{-\frac{qE}{r}} < k$ as $t \rightarrow \infty$, by (7)

Therefore, in the exploited system if the fish species obeys the Gompertz law of growth, then the population density tends to a finite limit which is less than the environmental carrying capacity.

Again equation (6) can be written as

$$\frac{dx}{dt} = -\frac{rx}{k} (x - B) \quad \text{where } B = k \left(1 - \frac{qE}{r}\right)$$

$$\Rightarrow \frac{dx}{x(x-B)} = -\frac{r}{k} dt$$

Solving the above equation we have, $\ln \left(\frac{x-B}{x}\right) = -\frac{rt}{kB} + D$

Using the initial condition $x_0 = x(0)$, we have $D = \ln \left(\frac{x_0-B}{x_0}\right)$

Therefore, $\ln \left(\frac{x-B}{x}\right) = -\frac{rt}{kB} + \ln \left(\frac{x_0-B}{x_0}\right)$

$$\Rightarrow \frac{x-B}{x} = \frac{x_0-B}{x_0} e^{-\frac{rt}{kB}} \Rightarrow x(t) = \frac{B}{1 - \frac{x_0-B}{x_0} e^{-\frac{rt}{kB}}} \quad (8)$$

Now, $x(t) \rightarrow B = k \left(1 - \frac{qE}{r}\right) < k$ as $t \rightarrow \infty$, by (8)

Therefore, in the exploited system if the fish species obeys the logistic law of growth, then the population density tends to a finite limit which is less than the environmental carrying capacity.

Since $k \left(1 - \frac{qE}{r}\right) < ke^{-\frac{qE}{r}} < k$, so in the exploited system, the limiting value of the population density obeying the Gompertz law of growth is greater than the limiting value of the population density obeying the logistic law of growth but less than the environmental carrying capacity k which is the limiting value of the population density for both the growth laws in the unexploited fishery.

Figure-5 and figure-6 represent the solution curves corresponding to the Gompertz law of growth and the logistic law of growth respectively. Here we take a constant effort $E = 30$, catchability coefficient $q = 0.1$, intrinsic growth rate $r = 5$ and the carrying capacity $k = 100$ in appropriate units.

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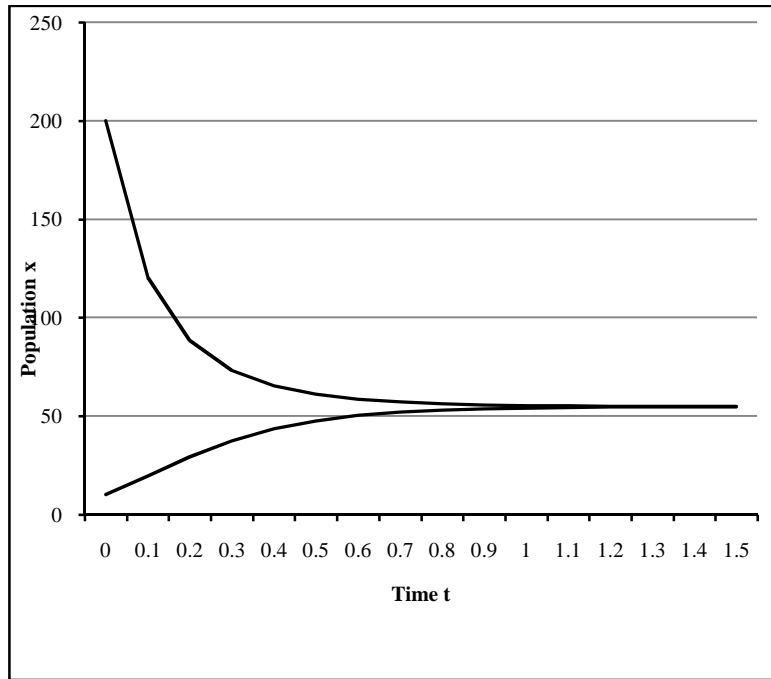


Fig.5: Population density curve in Gompertz law of growth for exploited fishery

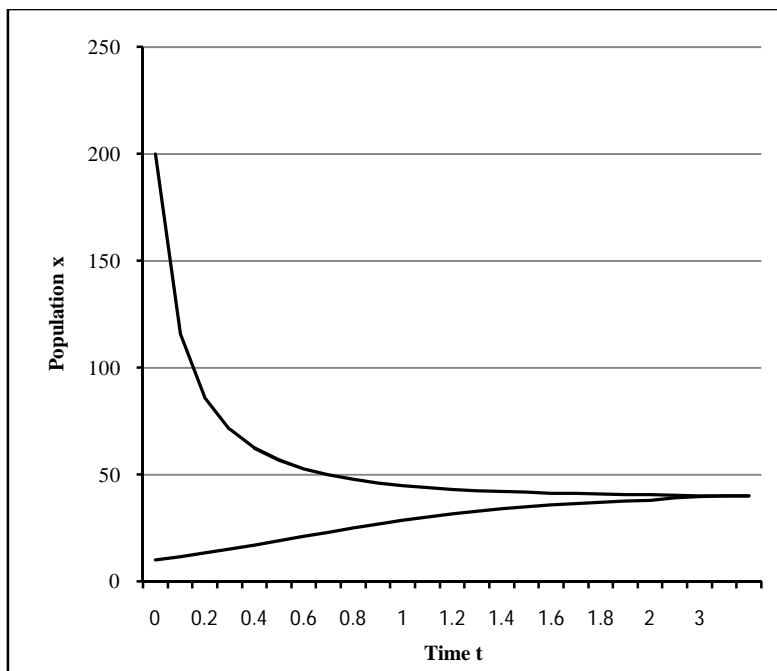


Fig.6: Population density curve in logistic law of growth for exploited fishery

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From these two figures we see that in the Gompertz law of growth for the exploited fishery the steady state level of the species is greater than the steady state level of the species obeying logistic law of growth. In the unexploited fishery these steady states are same and equal to the environmental carrying capacity. From these two figures it is also clear that in the Gompertz law of growth for the exploited fishery the species reaches its steady state level very fast compared to the Logistic law of growth.

VI. STEADY STATES OF THE EXPLOITED FISHERY

The steady state of the exploited system will be obtained by setting $\frac{dx}{dt} = 0$.

The non-trivial steady state of the Gompertz law of growth is $x_1 = ke^{-\frac{qE}{r}} > 0$ by (5).

and the non-trivial steady state of the logistic law of growth is $x_2 = k\left(1 - \frac{qE}{r}\right)$ by (6) provided $E < \frac{r}{q}$.

Thus we see that the non-trivial steady state of the Gompertz law of growth equation always exists but the non-trivial steady state of the logistic law of growth equation exists only when the harvesting effort is less than the *biotechnical productivity* (BTP) $[1] \frac{r}{q}$.

Now, $x_1 = ke^{-\frac{qE}{r}} = k\left(1 - \frac{qE}{r} + \frac{q^2E^2}{2r^2} - \dots\right) \approx k\left(1 - \frac{qE}{r}\right)$ when $\frac{qE}{r} \ll 1$ and this is the non-trivial steady state of the logistic growth equation.

Therefore, the non-trivial steady state of the logistic law of growth equation is approximately equal to the non-trivial steady state of the Gompertz law of growth equation if the harvesting effort is very lower than the BTP of the population.

Since $x(t) \rightarrow ke^{-\frac{qE}{r}}$ as $t \rightarrow \infty$, by (7) and $x(t) \rightarrow k\left(1 - \frac{qE}{r}\right)$ as $t \rightarrow \infty$, by (8), so both the steady states are globally asymptotically stable.

After discussing the above results we have the following comparative study for both the growth functions:

Gompertz law of growth	Logistic law of growth
i) It is a general law of growth function.	i) It is an approximation of Gompertz law of growth.
ii) Maximum growth is $\frac{k}{e}$ for unexploited fishery.	ii) Maximum growth is $\frac{k}{2}$ for unexploited fishery.
iii) Growth function increases in the interval $\left(0, \frac{k}{e}\right)$.	iii) Growth function increases in the interval $\left(0, \frac{k}{2}\right)$.
iv) Growth function decreases in the interval $\left(\frac{k}{e}, k\right)$.	iv) Growth function decreases in the interval $\left(\frac{k}{2}, k\right)$.
v) Faster growth compared to Logistic growth.	v) Slower growth compared to Gompertz growth.
vi) $x = k$ is the non-trivial stable steady state for the unexploited fishery.	vi) $x = k$ is the non-trivial stable steady state for the unexploited fishery.
vii) Growth rate of the exploited fishery ceases later than the logistic growth law.	vii) Growth rate of the exploited fishery ceases earlier than the Gompertz growth law.
viii) $x = ke^{-\frac{qE}{r}}$ is the non-trivial steady state for the exploited fishery.	viii) $x = k\left(1 - \frac{qE}{r}\right)$ is the non-trivial steady state for the exploited fishery.
ix) There is no restriction for existence of the non-trivial steady state for the exploited fishery.	ix) Non-trivial steady state exists only when the harvesting effort is less than the BTP.
x) In the exploited fishery the population reaches earlier to the steady state compared to the Logistic law of growth	x) In the exploited fishery the population reaches slowly to the steady state compared to the Gompertz law of growth

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VII. CONCLUSION

The peculiar feature of the Gompertz law of growth is the faster early growth and slower approach to the steady state compared to the Logistic law of growth. Gompertz law of growth is more realistic and flexible than the logistic law of growth. However, logistic law of growth function is commonly used by the researchers for its simplicity.

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