

# Development of Special Functions in Mathematical Physics

Bhagvat K. Kumthekar<sup>1</sup>, Vijay B. Patare<sup>2</sup>

Department of Physics, Nutan Mahavidyalaya, Sailu Dist Parbhani, India<sup>1</sup>

Department of Mathematics, Nutan Mahavidyalaya, Sailu Dist Parbhani, India<sup>2</sup>

**ABSTRACT:** In the present paper, I would like to explain special functions and their developments in mathematical physics. Such as Bessel equation, Legendre equation, Laguerre equation and Hermite equation. These functions are developed only for special cases of equations. So that the names were given special functions.

## I. INTRODUCTION

There are some special functions in mathematics which are developed due to requirement of special cases in Physics. In science and technology there are large number problems which are in terms of differential equations or partial differential equations. Partial differential equations have two or more independent variables. When a function  $f$  is depending upon number of variables  $(x, y, z, t)$  then differentiation with respect to one of the variable is known as partial differentiation e.g.  $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}, \frac{\partial f}{\partial t}$ . If our function is depend upon only one variable and we differentiate with respect to that variable it becomes perfect differentiation  $\frac{df}{dx}$ .

A powerful tool for solving a partial differential equation may be to split it into (ordinary) differential equations by the method of the separation of variables.

A relation between independent variable, say  $x$ , dependent variable, say  $y$ , and derivatives of  $y$  with respect to  $x$  is known as a differential equation.

$$f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots, \frac{d^ny}{dx^n}\right) = 0 \quad (1)$$

Order of the given differential equation is the order of the highest derivative in the equation. Degree of the given differential equation is the degree (power) of the highest derivative in the equation [1]. Differential equations may be classified into two categories:

- (i) Linear differential equations, and
- (ii) Non-linear differential equations.

$$a_n \frac{d^ny}{dx^n} + a_{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \dots + a_1 \frac{dy}{dx} + a_0 y = f(x) \quad (2)$$

where  $a$ 's may be functions of  $x$  or constant, but not the function of  $y$ . Here, each term contains either  $y$  or derivative of  $y$  with respect to  $x$ , and there is only one degree of  $y$  and its derivatives [2]. When  $f(x)$  is zero, in above equation (2) is known as homogeneous linear differential equation. Differential equations, other than that of the form of

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equation (2) are nonlinear ones. Thus, in a non-linear differential equation, at least one term contains either product of  $y$  and its derivative, product of two or more derivatives, or has  $y$  or its derivative of degree more than one.

If  $y_1, y_2, \dots, y_r$  are  $r$  solutions of a homogeneous linear differential equation then a linear combination of the solutions also is a solution of the equation [3]. In the present communication I would like to discuss about Bessel equation, Legendre equation, Laguerre equation and Hermite equation.

## 1. Legendre Equation:

Legendre differential equation is,

$$(1 - x^2) \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + n(n + 1)y = 0 \quad (3)$$

Where  $n$  is positive integer. Solution of this equation is known as Legendre Function [4]. In order to find out singular points and possibility of the series solution for this equation, we have

$$P(x) = -\frac{2x}{1-x^2} \quad \text{and} \quad Q(x) = \frac{n(n-1)}{1-x^2}$$

Such type of equation comes in Schrodinger's equation for Hydrogen atom in spherical polar coordinate [5].

## Applications of Legendre functions

Legendre's equation occurs in many areas of applied mathematics, physics and chemistry in physical situation with a spherical geometry such as flow of an ideal fluid past a sphere, the determination of the electric field due to a charged sphere and the determination of the temperature distribution in a sphere given its influence. Legendre polynomials have an extensive usage area, particularly in physics and engineering. For example, Legendre and Associate Legendre polynomials are widely used in the determination of wave functions of electrons in the orbits of an atom, [5] and in the determination of potential functions in the spherically symmetric geometry etc. Also in nuclear reactor physics, Legendre polynomials have an extraordinary importance [6].

## 2. Laguerre equation:

Laguerre's differential equation is

$$x \frac{d^2y}{dx^2} + (1 - x) \frac{dy}{dx} + ny = 0 \dots \quad (4)$$

Where  $n$  is positive integer. Solution of this equation is known as Laguerre Function [4]. In order to find out singular points and possibility of the series solution for this equation, we have

$$P(x) = \frac{1-x}{x} \quad \text{and} \quad Q(x) = \frac{n}{x}$$

Such type of equation comes in Schrodinger's equation for Hydrogen atom in spherical polar coordinate.

## Applications of Laguerre's functions

A spectral method of inverting one- and two-dimensional semi-infinite convolutions using the Laguerre polynomials. Applying the Laguerre polynomials makes it possible to eliminate the discretization procedure, which may take the solution of the perturbed equation outside the region of well-posedness [7]. Tempered fractional diffusion equations (TFDEs) involving tempered fractional derivatives on the whole space [8].

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### 3. Hermite equation:

Hermite's differential equation is

$$\frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + 2ny = 0 \quad (5)$$

Where n is positive integer. Solution of this equation is known as Legendre Function [4]. In order to find out singular points and possibility of the series solution for this equation, we have

$$P(x) = -2x \quad \text{and} \quad Q(x) = 2n$$

### Applications of Hermite's functions

Harmonic oscillator problem in quantum mechanics can be solved by using Hermite polynomials. The problem is usual: to find all values of energies  $E$  for which such a non-trivial solution exists. The physical meaning of eigenvalues  $E_k$  is the energy levels, and eigenfunctions  $y$  are the "wave functions" which describe the state of the system [9]. The Landau levels, appearing in the quantum analysis of an electron (or any charged particle) moving in a classical magnetic field. Morse potential used to treat the dynamical behavior of diatomic molecules and its approximation in terms of harmonic oscillator potential. [10]

### 4. Bessel functions:

Bessel's differential equation is

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2)y = 0$$

where n is an integer or a half integer. Solution of this equation is known as Bessel function [4]. In order to find out singular points and possibility of the series solution for this equation, we have

$$P(x) = \frac{1}{x} \quad \text{and} \quad Q(x) = 1 - \frac{n^2}{x^2}$$

### Applications of Bessel functions

Bessel's equation arises when finding separable solutions to Laplace's equation and the Helmholtz equation in cylindrical or spherical coordinates. Bessel functions are therefore especially important for many problems of wave propagation and static potentials. In solving problems in cylindrical coordinate systems, one obtains Bessel functions of integer order ( $\alpha = n$ ); in spherical problems, one obtains half-integer orders ( $\alpha = n + 1/2$ ). [11] For example:

- Electromagnetic waves in a cylindrical waveguide
- Heat conduction in a cylindrical object
- Diffusion problems on a lattice
- Solutions to the radial Schrödinger equation (in spherical and cylindrical coordinates) for a free particle

## II. DISCUSSION

In many areas of applied mathematics, physics and chemistry in physical situation with a spherical geometry, determination of wave functions of electrons in the orbits of an atom. These functions are developed only for special cases of equations. So that the names were given special functions. It is easy to eliminate the discretization procedure, which may take the solution of the perturbed equation outside the region of well-posedness by using Laguerre polynomials. Bessel function is used for finding the solution of radial part Schrodinger equation for hydrogen atom. Associated Legendre function is used for solving azimuthal part of Schrodinger equation for hydrogen atom.

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