

Types of Colourings and Types of Families of Disjoint Sets

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ABSTRACT: In this paper we prove minimum number of families of disjoint sets with known degree of intersection (minimum number of colours) and we introduce another proof for maximum number of families of disjoint sets (maximum number of colours). Also in this paper we introduce the concept types of families of disjoint sets (types of colouring) and prove some results related to this concept, and we introduce the concept restricted families of disjoint sets (restricted colouring), restricted colouring is colouring in which we add a condition. Suppose we have small identical cubes, we want to colour these cubes such that any two adjacent small cubes coloured by two different colours, and any big cube consists of twenty seven small identical cubes coloured by twenty seven different colours, how many colours will be used. This problem of cubes can be solved by using restricted colouring.

KEYWORDS: restricted colouring, restricted degree of intersection, restricted families, types of families of disjoint sets, types of colouring.

I. INTRODUCTION

In paper [1] we introduce the concept of family of disjoint sets as topic of set theory. A family of nonempty sets is called family of disjoint sets if every two sets, belong to this family, are disjoint. Throughout this paper we assume any family of disjoint sets can contain finite or infinite sets. We also introduce definitions of degree of intersection, meeting and standard of meeting, minimum number of families of disjoint sets and maximum number of families of disjoint sets. In this paper we show minimum families of disjoint sets with degree of intersection n equal $n + 1$, and introduce other proof showing maximum families of disjoint sets with degree of intersection n equal $2n - 1$.

In paper [3] we introduce some concepts in graph theory, like colouring at vertex, colouring at cycle, degree of colouring, standard of colouring, to introduce two new methods for colouring edges and vertices of graph. First method related to vertices of the graph and second method related to cycles. These two methods depend on the theory in paper [1] with suitable changes.

We have two branches of families of disjoint sets, or we can say two branches of colouring (by colouring we mean colouring of families of disjoint sets), first one is familiar families of disjoint sets (familiar colouring), second is restricted families of disjoint sets, which introduced in this paper with some examples. In this paper we defined types of families of disjoint sets and introduced some related results.

II. RELATED WORK

Definition 2.1. Let $\chi_1, \chi_2, \chi_3, \dots, \chi_m$ be m families of disjoint sets and $X_{i_1}, X_{i_2}, X_{i_3}, \dots, X_{i_n}$ be n sets such that any two sets belong to two different families of disjoint sets, and X is nonempty set, $X = X_{i_1} \cap X_{i_2} \cap X_{i_3} \cap \dots \cap X_{i_n}$,

$X = \{x : x \in X_{ij}, 1 \leq i \leq r, 1 \leq j \leq n, n \leq r, X_{ij} \in \chi_i\}$ then the positive integer n is called degree of intersection of X [1].

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Definition2.2. Let $\chi_1, \chi_2, \chi_3, \dots, \chi_r$ be r families of disjoint sets and X is nonempty set, $X = \chi_1 \cap \chi_2 \cap \chi_3 \cap \dots \cap \chi_r$, then the intersection X is called represent the families $\chi_1, \chi_2, \chi_3, \dots, \chi_r$, and we used the notation $X = \chi_1 \cap \chi_2 \cap \chi_3 \cap \dots \cap \chi_r$, or $X = (\chi_1, \chi_2, \chi_3, \dots, \chi_r)$ [1].

Proposition2.3. Let $\chi_1, \chi_2, \chi_3, \dots, \chi_r$ be r families of disjoint sets and $X_1, X_2, X_3, \dots, X_w$ be w nonempty intersections, the family of disjoint sets χ_i appear w_i times at w intersections and the family of disjoint sets χ_j appear w_j time at w intersections, then we say the family of disjoint sets χ_i meet the family of disjoint sets χ_j at these w intersections if and only if $w + 1 \leq w_i + w_j$.

Definition2.4. Let $\chi_1, \chi_2, \chi_3, \dots, \chi_m$ be m families of disjoint sets, $X_1, X_2, X_3, \dots, X_w$ each of degree n , and w be the minimum number of nonempty intersections, such that any two families of disjoint sets meet each other at these w intersections, then these w intersections called standard of meeting for these m families of disjoint sets [1].

Definition2.5. Let $X_1, X_2, X_3, \dots, X_w$ be standard of meeting for m families of disjoint sets $\chi_1, \chi_2, \chi_3, \dots, \chi_m$ with degree of intersection n , and let $Y_1, Y_2, Y_3, \dots, Y_w$ be standard of meeting for the same m families of disjoint sets $\chi_1, \chi_2, \chi_3, \dots, \chi_m$, with degree of intersection n , then we say standard of meeting $X_1, X_2, X_3, \dots, X_w$ is congruent to standard of meeting $Y_1, Y_2, Y_3, \dots, Y_w$. This means two standards of meeting are congruent if they satisfy three conditions: They are both intersections of the same m families of disjoint sets, they have same number of intersections with same degree of intersection, but appearance of families of disjoint sets can be different at the two standards of meeting [1].

Example2.6. Let $\chi_1, \chi_2, \chi_3, \dots, \chi_6$ be six families of disjoint sets and X_1, X_2, X_3, X_4, X_5 be five intersections each with degree of intersection four, $X_1 = (1,2,3,4)$, $X_2 = (1,2,3,5)$, $X_3 = (1,3,4,6)$, $X_4 = (1,4,5,6)$, $X_5 = (1,2,5,6)$. These five intersections X_1, X_2, X_3, X_4, X_5 is standard of meeting for families of disjoint sets $\chi_1, \chi_2, \chi_3, \dots, \chi_6$. Let Y_1, Y_2, Y_3, Y_4, Y_5 be five intersections each with degree of intersection four, $Y_1 = (1,2,3,4)$, $Y_2 = (1,2,6,5)$, $Y_3 = (1,3,4,6)$, $Y_4 = (2,3,4,5)$, $Y_5 = (1,2,5,6)$. These five intersections Y_1, Y_2, Y_3, Y_4, Y_5 is standard of meeting for families of disjoint sets $\chi_1, \chi_2, \chi_3, \dots, \chi_6$. These five intersections X_1, X_2, X_3, X_4, X_5 is congruent to the five intersections Y_1, Y_2, Y_3, Y_4, Y_5 [1].

Theorem2.7. Let w intersections each with degree of intersection equal n , m benumber of local maximum families of disjoint sets, meet each other at w intersections, then we have $m(\frac{w+1}{2}) \leq wn$ for w odd, and $m(\frac{w}{2} + 1) \leq wn$ for w even [1].

Definition2.8. Let $\chi_1, \chi_2, \chi_3, \dots, \chi_m$ be m families of disjoint sets, meet each other at $X_1, X_2, X_3, \dots, X_w$ intersections each of degree n , then m is called local maximum families of disjoint sets at w intersections, if any two families of disjoint sets meet each other at these w intersections, and any family of disjoint sets has appearance either equal $\frac{w+1}{2}$ or $\frac{w+3}{2}$ for w odd such that not all families of disjoint sets has appearance equal $\frac{w+3}{2}$, and for w even we have three types, first all families of disjoint sets has appearance equal $\frac{w}{2} + 1$, second appearance equal $\frac{w}{2} + 1$ and $\frac{w}{2} + 2$, not all $\frac{w}{2} + 2$ but at least has one appearance equal $\frac{w}{2} + 2$, third all families of disjoint sets has appearance equal $\frac{w}{2} + 1$, except only one family of disjoint sets has appearance equal $\frac{w}{2}$. Maximum of locals maximum families of disjoint sets called maximum families of disjoint sets [1].

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Definition 2.9. Let $\chi_1, \chi_2, \chi_3, \dots, \chi_m$ be m families of disjoint sets, $X_1, X_2, X_3, \dots, X_w$ be w nonempty intersections each of degree n , then m families of disjoint sets is called local minimum families of disjoint sets at w intersections, if $m(w-1) \leq nw$ such that $m \neq n$. When $m=n$, then m families of disjoint sets is called local minimum families of disjoint sets at w if $m(w-k) \leq nw < m(w-k+1)$ such that $m \neq n$, where $k=2,3,4, \dots$. Minimum of local minimum families of disjoint sets called minimum families of disjoint sets [1].

Remark 2.10. If n is degree of intersection, m is number of families of disjoint sets then $n+1 \leq m \leq 2n-1$ [1].

Remark 2.11. If n is degree of intersection, w is standard of meeting for m families of disjoint sets number of families of disjoint sets then $3 \leq w \leq 2n-1$ [1].

Proposition 2.12. Let $\chi_1, \chi_2, \chi_3, \dots, \chi_m$ be m families of disjoint sets, $X_1, X_2, X_3, \dots, X_w$ are nonempty different intersections each of degree n , and t_i number of times the family of disjoint sets χ_i appear at these w intersections, then we have $\sum_{i=1}^m t_i = wn$.

Remark 2.13. If n is degree of intersection, w is standard of meeting for m families of disjoint sets number of families of disjoint sets, both w and m depend on n , also m depend on w [1].

Remark 2.14. Let $X_1, X_2, X_3, \dots, X_w$ be standard of meeting for m families of disjoint sets $\chi_1, \chi_2, \chi_3, \dots, \chi_m$

With degree of intersection n , it is trivial case $w=1$. When $w=1$ we have $m=n$. Also standard of meeting consist of X_1 and X_2 where $X_1 = (1,2,3, \dots, n-1, n)$ and $X_2 = (1,2,3, \dots, n-2, n-1, n+1)$, X_1, X_2 is not trivial. For $w=2$, if standard of meeting for $n+1$ families of disjoint sets $\chi_1, \chi_2, \chi_3, \dots, \chi_n, \chi_{n+1}$, because X_1 does not meet X_2 . If X_1 is standard of meeting for n families of disjoint sets $\chi_1, \chi_2, \chi_3, \dots, \chi_n$, also X_1, X_2 is standard of meeting for n families of disjoint sets $\chi_1, \chi_2, \chi_3, \dots, \chi_n$. Since $X_1 = X_2$, not allowed, two intersections have same representation (repeated), then standard of meeting X_1, X_2 means standard of meeting X_1 , and standard of meeting X_1, X_2 is trivial [1].

Remark 2.15. Trivial case when standard of meeting w equal one, we have two trivial cases, first degree of intersection n equal one and second $n=m$ where m is number of families of disjoint sets. When $n=1$ this leads to $w=1$, and when $n=m$ this also leads to $w=1$. Let standard of meeting w equal two, X_1, X_2 each of degree n , for families of disjoint sets $\chi_1, \chi_2, \chi_3, \dots, \chi_n, \chi_{n+1}$. If $X_1 = (1,2,3, \dots, n-1, n)$ and $X_2 = (1,2,3, \dots, n-1, n+1)$, family of disjoint sets χ_n and family of disjoint sets χ_{n+1} not meet each other at X_1, X_2 , $w=2$ is trivial because $X_1 = X_2$, and we can't find $w=2$ [1].

Definition 2.16. Let $\chi_1, \chi_2, \chi_3, \dots, \chi_m$ be m families of disjoint sets, meet each other at $X_1, X_2, X_3, \dots, X_w$ intersections each of degree n , then m is called local maximum families of disjoint sets at w intersections, if any two families of disjoint sets meet each other at these w intersections, and any family of disjoint sets has appearance either equal $\frac{w+1}{2}$ or $\frac{w+3}{2}$ for w odd such that not all families of disjoint sets has appearance equal $\frac{w+3}{2}$, and for w even we have three types, first all families of disjoint sets has appearance equal $\frac{w}{2}+1$, second appearance equal $\frac{w}{2}+1$ and $\frac{w}{2}+2$, not all $\frac{w}{2}+2$ but at least has one appearance equal $\frac{w}{2}+2$, third all families of disjoint sets has appearance equal $\frac{w}{2}+1$, except only one

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family of disjoint sets has appearance equal $\frac{w}{2}$. Maximum of locals maximum families of disjoint sets called maximum families of disjoint sets [1].

Definition 2.17. Let $\chi_1, \chi_2, \chi_3, \dots, \chi_m$ be m families of disjoint sets, $X_1, X_2, X_3, \dots, X_w$ be w nonempty intersections each of degree n , then m families of disjoint sets is called local minimum families of disjoint sets at w intersections with t families of disjoint sets appear at each intersection, if $m(w-1) + t = nw$ where $0 \leq t < w$ such that $m \neq n$. When $m = n$, then m families of disjoint sets is called local minimum families of disjoint sets at w intersections with t families of disjoint sets appear at each intersection if $m(w-k) + t = nw$ where $0 \leq t < w, k = 2, 3, 4, \dots$ and $m(w-k) \leq nw < m(w-k+1)$ such that $m \neq n$. Minimum of all locals minimum families of disjoint sets called minimum families of disjoint sets [2].

Remark 2.18. By colouring m families of disjoint sets we mean an assignment of colour to each family of disjoint sets so that no two different families of disjoint sets have the same colour, this means all sets belong to one family of disjoint sets will be in same colour. Two different families of disjoint sets intersect each other, so they have two different colours. The idea of colouring families of disjoint sets have been used in paper [1] and any family of disjoint sets labelled by number or colour.

Example 2.19. Let degree of intersection n equal three, regions of a map are sets with degree of intersection three. Let for m be number families of disjoint sets, (number of colours used to draw map), using the rule $m \leq 2n - 1$, substitute $n = 3$, we have $m \leq 5$, this means maximum number of colour to draw a map is 5. If you used six or more than six colours, this means by small effort you can reduce the number of colours to five [1].

Definition 2.20. Colouring edges at vertex is defined by assignment one of m colours to each edge so that adjacent edges have not same colour [3].

Definition 2.21. Degree of colouring edges at vertex is maximum number of edges have different colours at vertex [3].

Definition 2.22. Colouring of vertices at vertex is defined by assignment one of m colours to each vertex so that adjacent vertices have different colours and vertices jointed adjacent edges have different colours [3].

Definition 2.23. Degree of colouring vertices at vertex is maximum number of vertices have different colours at vertex [3].

Definition 2.24. Colouring of edges at cycle of length n is defined by assignment one of m colours to each edge of graph G so that adjacent edges have different colours edges of any cycle of length n have different colours (or cycle of length less than n edges and path of n edges have different colours) [3].

Definition 2.25. Degree of colouring at cycle of length n for edges is maximum number of edges have different colours at cycle of length n [3].

Definition 4.26. Colouring of vertices at cycle of length n is defined by assignment one of m colours to each vertex of graph G so that adjacent vertices have different colours and vertices of any cycle of length n vertices have different colours, (or length less than n vertices and any path of n vertices have different colours) n [3].

Definition 4.27. Degree of colouring at cycle of length n for vertices is maximum number of vertices have different colours at cycle of length n [3].

Definition 2.28. A point is a corner of a map if and only if it belongs to the closures of at least three regions [11].

Definition 2.29. Two regions of a map are adjacent if their respective closures have a common point that is not a corner of the map [11].

III. MINIMUM FAMILIES OF DISJOINT SETS

In paper [1] we show maximum families of disjoint sets with degree of intersection n equal $2n - 1$. Here we want to show minimum families of disjoint sets with degree of intersection n equal $n + 1$, and introduce other proof showing maximum families of disjoint sets with degree of intersection n equal $2n - 1$.

Theorem 3.1. Minimum number of families of disjoint sets with degree of intersection n equal $n + 1$.

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Proof: Let m be number of families of disjoint sets with degree of intersection n , since $m = n$ is trivial case, and minimum of all local minimum families of disjoint sets called minimum families of disjoint sets, it is enough to show there exist m local minimum families of disjoint sets such that $m = n + 1$ for all value of n , where $n \geq 2$. From Remark 2.11. we have $3 \leq w \leq 2n-1$, let this local minimum families of disjoint sets m at $w = 3$ and from Proposition 2.12. and definition of local minimum families of disjoint sets we have $X_1 = (1,2,3,\dots,n-1,n)$, $X_2 = (1,2,3,\dots,n-1,n+1)$, $X_3 = (1,2,3,\dots,n-2,n,n+1)$. Since $n \geq 2$, for $n = 2$ we have $X_1 = (1,2)$, $X_2 = (1,3)$, $X_3 = (2,3)$, therefore from X_1, X_2, X_3 we have m number of local minimum families of disjoint sets and $3 = m = n + 1 = 2 + 1$. For $n = 3$ we have $X_1 = (1,2,3)$, $X_2 = (1,2,4)$, $X_3 = (1,3,4)$, therefore from X_1, X_2, X_3 we have m number of local minimum families of disjoint sets and $4 = m = n + 1 = 3 + 1$. For $n = 4$ we have $X_1 = (1,2,3,4)$, $X_2 = (1,2,3,5)$, $X_3 = (1,2,4,5)$, therefore from X_1, X_2, X_3 we have m number of local minimum families of disjoint sets and $5 = m = n + 1 = 4 + 1$. Continue in this process, for $n = k$ we have $X_1 = (1,2,3,\dots,k)$, $X_2 = (1,2,3,\dots,k-1,k+1)$, $X_3 = (1,2,3,\dots,k-2,k,k+1)$, therefore from X_1, X_2, X_3 we have m number of local minimum families of disjoint sets and $m = k + 1$. Therefore for all $n = 2, 3, 4, \dots$ any families of disjoint sets at X_1, X_2, X_3 appear three times except $\chi_{n-1}, \chi_n, \chi_{n+1}$ each appear twice, therefore $n + 1$ number of local minimum families of disjoint sets where $m = n + 1$, since $m = n + 1$ is minimum of all local minimum families of disjoint sets, therefore $m = n + 1$, minimum families of disjoint sets.

Theorem 3.2. Maximum number of families of disjoint sets with degree of intersection n equal $2n - 1$.

Proof: Let m be number of families of disjoint sets with degree of intersection n , since maximum of all local maximum families of disjoint sets called maximum families of disjoint sets, it is enough to show there exist m local maximum families of disjoint sets such that $m = 2n - 1$ for all value of n , where $n \geq 2$. From Remark 2.11. we have $3 \leq w \leq 2n-1$, let this local maximum families of disjoint sets m at $w = 2n - 1$ and from Proposition 2.12. and definition of local maximum families of disjoint sets we have $m \binom{w+1}{2} \leq wn$. Since $w = 2n - 1$, we can write $m \binom{w+1}{2} = mn \leq wn = n(2n - 1)$, therefore $m = 2n - 1$ is local maximum families of disjoint sets. Since $m = 2n - 1$ is maximum of all local maximum families of disjoint sets, therefore $m = 2n - 1$ is maximum families of disjoint sets.

IV. DIFFERENT TYPES OF FAMILIES OF DISJOINT SETS DEPEND ON DEGREE OF INTERSECTION

Remark 4.1. In this paper we mean by different families of disjoint sets, different values of m from relation $\sum_{i=1}^m t_i = wn$,

where n is constant and both w and t_i are variables, and we mean by different types of families of disjoint sets,

different values of m from relation $\sum_{i=1}^m t_i = wn$, where n is variable, both w and t_i can be variables or constant.

Definition 4.2. Let $\bigcup_i S_i$ be infinite sets, $\chi_1, \chi_2, \chi_3, \dots, \chi_m$ be m families of disjoint sets, $\bigcup_i S_i = \chi_1, \chi_2, \chi_3, \dots, \chi_m$ and $X_{i1}, X_{i2}, X_{i3}, \dots, X_{in}$ be n sets such that any two sets belong to two different families of disjoint sets, and X is nonempty set, $X = X_{i1} \cap X_{i2} \cap X_{i3} \cap \dots \cap X_{in}$, $X = \{x : x \in X_{ij}, 1 \leq i \leq m, 1 \leq j \leq n, n < m, X_{ij} \in \chi_i\}$ the positive integer n is degree of intersection of X , then we say $\bigcup_i S_i$ partitioned into families $\chi_1, \chi_2, \chi_3, \dots, \chi_m$ with degree of intersection n .

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Definition 4.3. Let $\bigcup_i S_i$ be infinite sets partition into families $\chi_1, \chi_2, \chi_3, \dots, \chi_{m_1}$ with degree of intersection n_1 and also $\bigcup_i S_i$ partition into families $\chi_1, \chi_2, \chi_3, \dots, \chi_{m_2}$ with degree of intersection n_2 then we call $\chi_1, \chi_2, \chi_3, \dots, \chi_{m_1}$ and $\chi_1, \chi_2, \chi_3, \dots, \chi_{m_2}$ are two different types of families of disjoint sets.

Definition 4.4. Let $\bigcup_i S_i$ be infinite sets partition into families $\chi_1, \chi_2, \chi_3, \dots, \chi_{m_1}$ with degree of colouring n_1 and also $\bigcup_i S_i$ partition into families $\chi_1, \chi_2, \chi_3, \dots, \chi_{m_2}$ with degree of colouring n_2 then we call $\chi_1, \chi_2, \chi_3, \dots, \chi_{m_1}$ and $\chi_1, \chi_2, \chi_3, \dots, \chi_{m_2}$ are two different types of colouring.

Remark 4.5. Colouring of families of disjoint sets technique depend on degree of intersection (degree of colouring), number of families of disjoint sets can be number of colours used. Here we can colour same sets using different types of colouring and each type give different number of colour from others. Colouring of cubes in the following example can be example of different types of colouring.

Example 4.6. Suppose we have infinite number of identical cubes arranged so that any two adjacent cubes their faces congruent and so their edges and vertices. In case two faces of two cubes congruent we consider the intersection of two cubes faces. Also we consider intersection of four cubes is an edge and intersection of eight cubes is a vertex. We have three types of families of disjoint sets, first with degree of intersection two, second with degree of intersection four and third with degree of intersection eight. If we want to colour these cubes we have three types of colouring, first with degree of colouring two, second with degree of colouring four and third with degree of colouring eight.

Example 4.7. Two sets be in different colours if they intersect each other or belong to two different families of disjoint sets, if we using Theorem 2.7. in Example 4.6. we have maximum number to colour cubes is three (degree of intersection two) when we define two adjacent cubes as two cubes have two faces congruent, maximum number to colour cubes is seven (degree of intersection four) when we define two adjacent cubes as two cubes have two edges congruent, and maximum number to colour cubes is fifteen (degree of intersection eight) when we define two adjacent cubes as two cubes have two vertex congruent. Here we have three types of cubes (three types of families of disjoint sets), or we can say three types of colourings cubes.

Example 4.8. Using Theorem 2.7. in Example 4.6. we have maximum number to colour faces of cubes is three (degree of intersection two) when we define two faces of cubes as adjacent when they are congruent, we have maximum number to colour faces of cubes is seven (degree of intersection four) when we define two adjacent faces of cubes as two faces have two edges congruent, and we have maximum number to colour faces of cubes is fifteen (degree of intersection eight) when we define two adjacent faces of cubes as two faces have two vertices congruent. Here we have three types of faces (three types of families of disjoint sets), or we can say three types of colourings faces.

Example 4.9. Using Theorem 2.7. in Example 3.6. we have maximum number to colour edges of cubes is seven (degree of intersection four) when we define two edges of cubes as adjacent when they are congruent, and we have maximum number to colour edges of cubes is fifteen (degree of intersection eight) when we define two adjacent faces of cubes as two edges have two vertices congruent. Here we have two types of edges (two types of families of disjoint sets), or we can say two types of colourings edges.

Example 4.10. Using Theorem 2.7. in Example 4.6. we have maximum number to colour vertices of cubes is fifteen (degree of intersection eight) when we define two vertices of cubes as adjacent when they are congruent. Here we have one type of vertices (one type of families of disjoint sets), or we can say one type of colourings vertices of cubes.

Remark 4.11. From Example 4.6. any face can be boundary between cubes, it is boundary of degree two. Any edge can be boundary between cubes, it is boundary of degree four, we can consider any edge is boundary between faces, it is boundary of degree four. Any vertex can be boundary between cubes, it is boundary of degree eight, we can consider any vertex is boundary between faces, it is boundary of degree eight. Also we can consider any vertex can be boundary between edges, it is boundary of degree eight.

Remark 4.12. Let $\chi_1, \chi_2, \chi_3, \dots, \chi_{m_1}$ be m_1 families of disjoint sets, maximum bipartition of infinite sets S_1, S_2, S_3, \dots with degree of intersection n_1 , and $\chi_1, \chi_2, \chi_3, \dots, \chi_{m_2}$ be m_2 families of disjoint sets bipartition of infinite sets S_1, S_2, S_3, \dots with

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degree of intersection n_2 , we continue partitioning of infinite sets S_1, S_2, S_3, \dots till $\chi_1, \chi_2, \chi_3, \dots, \chi_{m_k}$ be m_k families of disjoint sets be partition of infinite sets S_1, S_2, S_3, \dots with degree of intersection n_k ,

Remark 4.13. From Definition 2.1. the degree of intersection n can be finite or infinite.

Proposition 4.14. If we have infinite number of sets with finite degree of intersection n and there is no restriction any of degree of intersection so that any value of $2, 3, 4, \dots, n-1, n$ can represent a type of families of disjoint sets (type of colourings), then we have $n-1$ types of families of disjoint sets (types of colourings).

Proof: Using definition of type of families of disjoint sets for $n=2$ this is first type and for $n=3$ this is second type, therefore we have $n-1$ types of families of disjoint sets (types of colourings).

Remark 4.15. Using Proposition 4.14. if we have infinite number of sets with infinite degree of intersection and there is no restriction any of degree of intersection so that any value of $2, 3, 4, \dots$ can represent a type of families of disjoint sets (type of colourings), then we have infinite types of families of disjoint sets (types of colourings).

Remark 4.16. From Proposition 4.14 any degree of intersection n represents a type of families of disjoint sets (a type of colourings).

Proposition 4.17. For any type of families of disjoint sets, values of number of families of disjoint sets can not exceed $n-1$ values.

Proof: For any degree of intersection n represents a type of families of disjoint sets, values of families of disjoint sets will be with minimum $n+1$ and maximum $2n-1$ and suppose it include all values between minimum and maximum. If we arrange all families of disjoint sets of this type according its values. First one will be minimum families of disjoint sets with degree of intersection n has $n+1$ families of disjoint sets, second families of disjoint sets with degree of intersection n has $n+2$ families of disjoint sets, third families of disjoint sets with degree of intersection n has $n+3$ families of disjoint sets and last one maximum families of disjoint sets with degree of intersection n has $2n-1$ families of disjoint sets. Since $2n-1 = n + (n-1)$ therefore values of families of disjoint sets can not exceed $n-1$ values.

Remark 4.18. Proposition 4.17. can be states as: For any type of families of disjoint sets, values of number of colours used to colour families of disjoint sets can not exceed $n-1$ values.

Example 4.19. In maps of countries/regions of a map are sets with degree of intersection three, using Proposition 4.17. and Remark 4.18. we have $n=3, n-1=2$, two values means four families of disjoint sets and five families of disjoint sets or means four colours and five colours can be used to draw maps of countries.

Example 4.20. In Example 4.6. we have three types of colouring, first with degree of colouring two, second with degree of colouring four and third with degree of colouring eight, using Proposition 4.17. and Remark 4.18. first we have $n=2, n-1=1$, one value means one family of disjoint sets or one colour (minimum and maximum have the same value). This is value equal three, at $n=2$ we have $m=2n-1=n+1=3$. Second we have $n=4, n-1=3$, these three values are $m=5, m=6$ and $m=7$. Third we have $n=8, n-1=7$, these seven values are $m=9, m=10, m=11, m=12, m=13, m=14$, and $m=15$.

Proposition 4.21. For any type of families of disjoint sets, we have $2n-3$ local minimum families of disjoint sets and $2n-3$ local maximum families of disjoint sets.

Proof: For any degree of intersection n represents a type of families of disjoint sets, at each intersections $X_1, X_2, X_3, \dots, X_w$ each of degree n , there a local minimum families of disjoint sets and a local maximum families of disjoint sets, where $3 \leq w \leq 2n-1$. Since the variable w is has $2n-3$ values and at each w intersection we have local minimum families of disjoint sets and local maximum families of disjoint sets.

called local maximum families, therefore we have $2n-3$ local minimum families of disjoint sets and $2n-3$ local maximum families of disjoint sets.

Example 4.22. Type of families of disjoint sets (type of colouring) with degree of intersections $n=5$, we have $2n-3=7$, this type have seven local minimum families of disjoint sets and seven local maximum families of disjoint sets, Table I explain this.

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w intersections	value of local minimum	value of local maximum
3	7	7
4	6	6
5	6	8
6	6	7
7	6	8
8	6	8
9	6	9

Table I

V. FAMILIES OF DISJOINT SETS WITH RESTRICTED DEGREE OF INTERSECTION

There are two branches of families of disjoint sets, first one families of disjoint sets with degree of intersection equal n , and second is families of disjoint sets with restricted degree of intersection equal k . In paper [1] we study families of disjoint sets with degree of intersection equal n . In paper [3] we introduced colouring of edges at cycle and colouring of vertices at cycle, it is example of families of disjoint sets with restricted degree of intersection equal k . In this section we introduced families of disjoint sets with restricted degree of intersection equal k with more details and examples.

Definition 5.1. Let $\chi_1, \chi_2, \chi_3, \dots, \chi_m$ be m families of disjoint sets with degree of intersection equal k , and let n be positive integer such that $n > k$ and trivial case when $n = m$ with a condition, then non trivial families of disjoint sets called families of disjoint sets with restricted degree of intersection equal n . Definition 5.2. Let m_k be number of colours with degree of colouring equal k , and let n be positive integer such that $n > k$ and trivial case when $n = m$ with a condition, then non trivial number of colours called restricted number of colours with restricted degree of colouring equal n .

Proposition 5. 3. Let $\chi_1, \chi_2, \chi_3, \dots, \chi_m$ be m families of disjoint sets, with restricted degree of intersection equal n . $X_1, X_2, X_3, \dots, X_w$ be nonempty different sets each related to n elements (or each set labelled by n numbers of $1, 2, 3, \dots, m$ where $n < m$), and t_i number of times the family of disjoint sets χ_i appear at these w sets (t_i number of times the number i appear at these w sets), then we have $\sum_{i=1}^m t_i = wn$.

Proof: Since the appearance for m families of disjoint sets at w sets $X_1, X_2, X_3, \dots, X_w$ each set labelled by n numbers of $1, 2, 3, \dots, m$, then summation of all appearance for all m families of disjoint sets equal wn therefore $\sum_{i=1}^m t_i = wn$.

Remark 5.4. As we seen in Proposition 5. 2. The concept of restricted degree of intersection satisfies all results in paper [1] if this concept put instead of degree of intersection.

Remark 5.5. In colouring sets (families of disjoint sets) so that any two families of disjoint sets have two different colours we need to know degree of intersection only, in case of colouring sets (families of disjoint sets) so that any two families of disjoint sets have two different colours, and we add other condition we need to know restricted degree of intersection.

Remark 5.6. In colouring edges (or vertices) of a graph so that any two edges (or vertices) have two different colours we need to know degree of colouring only, in case of colouring edges (or vertices) so that any two edges (or vertices) have two different colours, and we add other condition we need to know restricted degree of colouring.

Example 5.7. Suppose we have a big cube consists of twenty seven small identical cubes arrange so that two cubes congruent by faces, four cubes congruent by edges and eight cubes congruent by vertices. Each two adjacent cubes into two different colours, two cubes defined adjacent if they congruent by vertices, we have eight cubes congruent by

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vertices at each vertex. Here we introduce an example of families of disjoint sets, with restricted degree of intersection, with condition every twenty seven small identical cubes formed the big cube are into twenty seven colours, degree of intersection equal $k = 8$, and n be positive integer such that $n > k$ and trivial case when $n = m$, in this case $n = 27$.

If we want to colour small cubes such that each two adjacent cubes into two different colours minimum number of colours equal $k + 1 = 8 + 1 = 9$ and maximum number of colours equal $2k - 1 = 16 - 1 = 15$. If we want to colour all small cubes formed the big cube into twenty seven different colours minimum number of colours equal $n + 1 = 27 + 1 = 28$ and maximum number of colours equal $2n - 1 = 54 - 1 = 53$.

Example 5.8. Suppose we have a graph G so that adjacent vertices have different colours and maximum degree of vertex equal k , we have the condition: Colouring cycle of length n will be by n different colours. This is an example of families of disjoint sets, with restricted degree of intersection, with condition Colouring cycle of length n will be by n different colours. If we put $k=5, n=9$, and we want to colour vertices of the graph G , minimum number of colours used to colour vertices equal $k + 1 = 5 + 1 = 6$ and maximum number of colours equal $2k - 1 = 10 - 1 = 9$. Minimum number of colours used to colour vertices of all cycles of length not more than nine with condition: all vertices of a cycle have different colours equal $n + 1 = 9 + 1 = 10$, and maximum number of colours used to colour vertices of all cycles of length not more than nine with condition: all vertices of a cycle have different colours equal $2n - 1 = 18 - 1 = 17$.

Example 5.9. Suppose we have a graph G so that adjacent edges have different colours and maximum degree of vertex equal k , we have the condition: Colouring edges of a cycle of length n will be by n different colours. This is an example of families of disjoint sets, with restricted degree of intersection, with condition Colouring edges of a cycle of length n will be by n different colours. If we put $k=7, n=15$, and we want to colour edges of the graph G , minimum number of colours used to colour edges equal $k + 1 = 7 + 1 = 8$, and maximum number of colours equal $2k - 1 = 14 - 1 = 13$. Minimum number of colours used to colour edges of all cycles of length not more than fifteen with condition: all edges of a cycle have different colours equal $n + 1 = 15 + 1 = 16$, and maximum number of colours used to colour edges of all cycles of length not more than fifteen with condition: all edges of a cycle have different colours equal $2n - 1 = 30 - 1 = 29$.

Example 5.10. Suppose we have a map of countries, and we want to colour this map with degree of intersection (degree of colouring) equal $k = 3$, minimum number of colours used is four and number of maximum colours used is five. If we want to colour this map with the condition: For any six countries such that any one of the six countries (sets) intersect at least one of the five countries (sets) shall be coloured by six different colours, this means with restricted degree of intersection (restricted degree of colouring) equal n , where $n = 6$. Minimum colour with restricted degree of colouring equal $n + 1 = 6 + 1 = 7$, and maximum number of colours used equal $2n - 1 = 12 - 1 = 11$.

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