

Dynamic Analysis of Rectangular Isotropic Plates Using Weak-Form Variational Principle

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ABSTRACT: This paper applies weak-form variational principle to determine the natural frequencies of thin rectangular isotropic plate with various boundary conditions. Ten boundary conditions of rectangular plates were investigated. The energy functional was formulated using weak-form variational statement on the integral function of the plate problem. The displacement functions were developed based on static deflection configurations of orthogonal beam network. The stiffness and mass matrices were developed from the expression of the formulated energy functional. The first four natural frequencies were numerically computed; and the variation of fundamental frequencies with aspect ratios was determined. The numerical values of the natural frequencies were compared with the results from previous works found in literature; and there was satisfactory convergence with a mean absolute percentage error of 6.76. Conclusively, weak-form variational principle is a satisfactory approximation technique to vibration analysis of thin rectangular plates.

KEYWORDS: Algebraic polynomial, Shape functions, Boundary conditions, Rectangular plate, Weak-form, Variational principle, Natural frequency

I. INTRODUCTION

Rectangular isotropic plates (RIP) are versatile structural components which are being used in different fields of engineering. In most of their engineering applications, coupled with their small thicknesses vis-à-vis other typical dimensions, RIP are often subjected to oscillatory problems by external disturbances. And the dynamic effect that may result from oscillatory problem is always a source of worry to engineers. Dynamic analysis of RIP starts with the determination of the natural frequencies; but unfortunately there are virtually no closed-form solutions to RIP [1, 2]. Consequent upon the dearth of closed-form solutions to RIP, most of the research works are carried out using approximation techniques such as the finite element method, the finite difference method; and the energy methods like the Hamilton energy principle, the direct variational principle in Rayleigh-Ritz and the residual energy principle in Galerkin method.

In energy techniques, the ability to develop approximation functions which are admissible to the true solutions is a challenging task of its own. Certainly the quality of the shape functions applied in a particular approximation technique influences the rate of convergence to the actual solution to the plate problem [3]. Trigonometric functions in form of the Fourier series have been widely used by many researchers as approximation functions. But trigonometric functions can only be used successfully in limited number of rectangular plate boundary conditions.

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The primary objective of this paper is to determine the natural frequencies of vibration of rectangular isotropic plate with various boundary conditions using weak-form variational principle technique. The efficacy of the application of weak-form variational principle in dynamic analysis of rectangular plates has not been exploited in the previous works found in literature. Furthermore, the secondary objective is to develop algebraic polynomial shape functions based on static deflection configurations of orthogonal beam network that simulate the deflection of rectangular plate with various boundary conditions as alternative to trigonometric shape functions.

II. RELATED WORK

In related work, Leissa [4, 5] compiled and published reviews of research works done on plate vibrations from 1981 to 1985. Nevertheless, many other researchers further carried out noble works on the vibrations of rectangular plates. Leissa et al. [1] studied vibrations of continuous systems in which free vibrations of rectangular isotropic plates were investigated using trigonometric functions in Rayleigh-Ritz method. Chakraverty [6] employed boundary characteristic orthogonal polynomials in free vibration analysis of rectangular plate with various boundary conditions. Other techniques employed in the approximation include the finite element method [7, 8, 9]; the modal method of analysis [10, 11]; and the finite difference method [12]. Furthermore, Phamova et al. [13] employed energy approach based on Hamilton principle to determine the vibration modes for all edges clamped and all edges simply supported rectangular plates. Generalized differential quadrature method was used to study free vibration of monoclinic rectangular plates [2]. In another development, Njoku et al. [14], and Ezeh et al. [15] exploited the use of polynomial shape functions in Galerkin functional to determine the fundamental frequencies of certain set of boundary conditions of rectangular plates.

III. ANALYTICAL FORMULATION

Formulation of Energy Functional

Fig. 1 shows an elastic isotropic and homogeneous thin rectangular plate of uniform thickness, t with arbitrary edge constraints. The plate is assumed to be completely flat. The plate parameters are invariants in Young modulus, E and mass density, ρ ; and the governing differential equation of motion for free vibration is given by Leissa et al.[1] as expressed in Eq. (1).

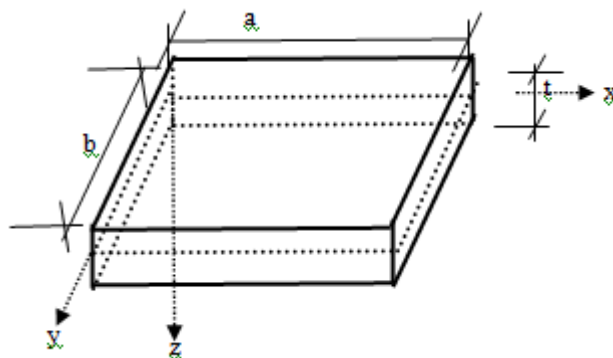


Fig. 1: Isotropic Rectangular Plate showing Coordinate Dimensions

$$\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} - \frac{\rho t \omega^2}{D} w = 0 \quad (1)$$

Where w is displacement function; ω is cyclic frequency of free vibrations; and D is the flexural rigidity of the plate as defined in Eq. (2).

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$$D = \frac{E * t^3}{12(1 - \mu^2)} \quad (2)$$

By defining the weighting function as $V(x, y)$, the weak-form variational statement is formulated using Eq. (1); and is given as expressed in Eq. (3):

$$0 = \int_0^a \int_0^b V(x, y) \left[\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} - \frac{\rho t \omega^2}{D} w \right] dx dy \quad (3)$$

Integration by parts was carried out on Eq. (3) to trade differentiation from deflection w to weighting function V . Thus:

$$\iint_R V \frac{\partial^4 w}{\partial x^4} dx dy = \iint_R \frac{\partial^2 V}{\partial x^2} \frac{\partial^2 w}{\partial x^2} dx dy + \int_0^b V \frac{\partial^3 w}{\partial x^3} \Big|_0^a dy - \int_0^b \frac{\partial V}{\partial x} \frac{\partial^2 w}{\partial x^2} \Big|_0^a dy \quad (4)$$

Also

$$2 \iint_R V \frac{\partial^4 w}{\partial x^2 \partial y^2} dx dy = 2 \iint_R \frac{\partial^2 V}{\partial x \partial y} \frac{\partial^2 w}{\partial x \partial y} dx dy + 2 \int_0^b V \frac{\partial^3 w}{\partial x \partial y^2} \Big|_0^a dy - 2 \int_0^a \frac{\partial V}{\partial x} \frac{\partial^2 w}{\partial x \partial y} \Big|_0^b dx \quad (5)$$

And

$$\iint_R V \frac{\partial^4 w}{\partial y^4} dx dy = \iint_R \frac{\partial^2 V}{\partial y^2} \frac{\partial^2 w}{\partial y^2} dx dy + \int_0^a V \frac{\partial^3 w}{\partial y^3} \Big|_0^b dx - \int_0^a \frac{\partial V}{\partial y} \frac{\partial^2 w}{\partial y^2} \Big|_0^b dx \quad (6)$$

$$- \iint_R V \frac{\rho t \omega^2}{D} w dx dy = - \frac{\rho t \omega^2}{D} \iint_R V w dx dy \quad (7)$$

By substituting Eqs. (4) through (7) into Eq. (3), the expression for energy functional and the natural boundary conditions is obtained as given by Eq. (8).

$$0 = \int_0^a \int_0^b \left[\frac{\partial^2 V}{\partial x^2} \frac{\partial^2 w}{\partial x^2} + 2 \frac{\partial^2 V}{\partial x \partial y} \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 V}{\partial y^2} \frac{\partial^2 w}{\partial y^2} - \omega^2 MKVw \right] dx dy \\ + \int_0^a \left\{ \left[-2 \frac{\partial V}{\partial x} \frac{\partial^2 w}{\partial x \partial y} + V \frac{\partial^3 w}{\partial x \partial y^2} - \frac{\partial V}{\partial y} \frac{\partial^2 w}{\partial y^2} \right]_0^b \right\} dx \\ + \int_0^b \left\{ \left[V \frac{\partial^3 w}{\partial x^3} - \frac{\partial V}{\partial x} \frac{\partial^2 w}{\partial x^2} + 2V \frac{\partial^3 w}{\partial x \partial y^2} \right]_0^a \right\} dy \quad (8)$$

The last two integrands in curl brackets of Eq. (8) are the natural boundary conditions; and they are identically satisfied at the boundaries. The first integrand of Eq. (8) is the functional of the plate problem. Therefore the energy functional formulated using weak-form variational statement is as given in Eq. (9).

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$$0 = \int_0^a \int_0^b \left[\frac{\partial^2 V}{\partial x^2} \frac{\partial^2 w}{\partial x^2} + 2 \frac{\partial^2 V}{\partial x \partial y} \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 V}{\partial y^2} \frac{\partial^2 w}{\partial y^2} - \omega^2 MKVw \right] dx dy \quad (9)$$

Where V is weighting function; M is the mass density of the plate materials.

$$M = \rho * t \quad (10)$$

$$K = \frac{1}{D} \quad (11)$$

Let Eq. (9) be transformed in the form as shown in Eq. (12):

$$\int_0^a \int_0^b \left[\frac{\partial^2 V}{\partial x^2} \frac{\partial^2 w}{\partial x^2} + 2 \frac{\partial^2 V}{\partial x \partial y} \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 V}{\partial y^2} \frac{\partial^2 w}{\partial y^2} \right] dx dy = \omega^2 MK \int_0^a \int_0^b Vw dx dy \quad (12)$$

Then by substituting appropriate displacement function and weighting function into the left hand side (L.H.S.) of Eq. (12) and integrating appropriately yields the stiffness matrix. Similarly, by substituting the same displacement function and weighting function into the right hand side (R.H.S.) of Eq. (12) and integrating, yields the mass matrix.

Development of Algebraic Polynomial Displacement Functions

Fig. 2 shows a rectangular plate which can be idealized as a system consisting of series of orthogonal beam network. The deflection configuration of the plate can be approximated by a series of orthogonal beam deflection configurations. It is assumed that each beam element running along any coordinate direction is a good representative of other series of beams that run along the same direction.

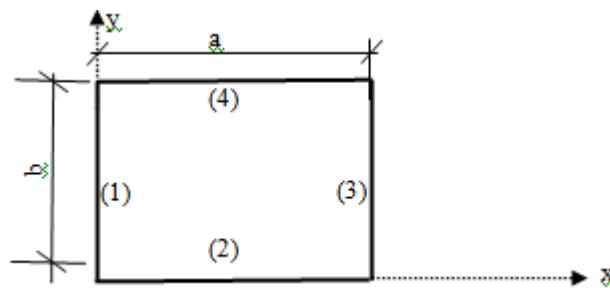


Fig. 2: A Rectangular Plate with Arbitrary Edge Constraints, showing Coordinate Dimensions a and b

The constraint at each edge can be clamped, or simply supported or free; and a minimum of two boundary conditions must be satisfied on each edge.

Thus, for clamped edge condition, the boundary conditions to be satisfied are:

$$w(\cdot, \cdot) = w'(\cdot, \cdot) = 0 \quad (13)$$

For simply supported edge condition, the boundary conditions to be satisfied are:

$$w(\cdot, \cdot) = w''(\cdot, \cdot) = 0 \quad (14)$$

For free edge condition, the boundary conditions to be satisfied are:

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$$w'(\cdot, \cdot) = w''(\cdot, \cdot) = w'''(\cdot, \cdot) = 0 \tag{15}$$

Where $w(\cdot, \cdot)$ is the deflection; $w'(\cdot, \cdot)$, $w''(\cdot, \cdot)$ and $w'''(\cdot, \cdot)$ are first, second and third partial derivatives with respect to coordinate axis respectively.

Then let the five-term static polynomial deflection configurations in x- and y-directions be defined as expressed in Eqs. (16) and (17) respectively:

$$w(x) = \sum_{i=0}^4 c_i x^i \tag{16}$$

$$w(y) = \sum_{j=0}^4 d_j y^j \tag{17}$$

Where c_i and d_j are undetermined coefficients.

For demonstration, suppose Fig. 2 is a rectangular plate with all its edges clamped as replicated in Fig. 3

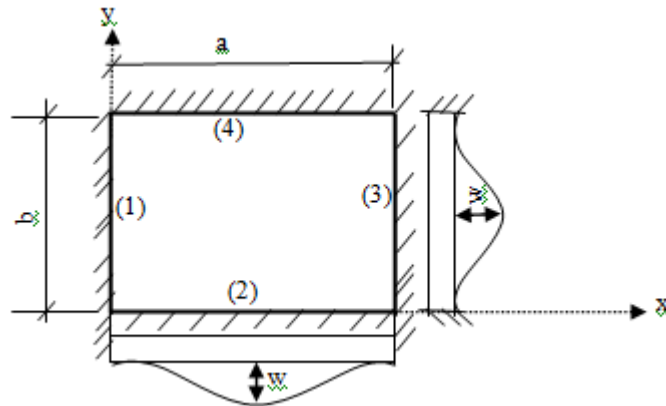


Fig. 3: A Rectangular Plate with all Edges Clamped; showing Orthogonal Beam Network Deflection Configurations.

By applying boundary conditions of Eq. (13) on Edges (1) and (3) at $x = 0$ and $x = a$ respectively, Eq. (16) yields an expression as expressed in Eq. (18):

$$w(x) = c_4 * a^4 \left[\left(\frac{x}{a}\right)^2 - 2 \left(\frac{x}{a}\right)^3 + \left(\frac{x}{a}\right)^4 \right] \tag{18}$$

To take care of series of beams running in the same direction, Eq. (18) is transformed as expressed in Eq. (19).

$$w(x) = c_4 * a^4 \left[\left(\frac{x}{a}\right)^{m+1} - 2 \left(\frac{x}{a}\right)^{m+2} + \left(\frac{x}{a}\right)^{m+3} \right] \tag{19}$$

For $m = 1, 2, 3, \dots$

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Similarly, by applying boundary conditions of Eq. (16) on Edges (2) and (4) at $y = 0$ and $y = b$ respectively, Eq. (17) yields an expression as expressed in Eq. (20).

$$w(y) = d_4 * b^4 \left[\left(\frac{y}{b}\right)^2 - 2\left(\frac{y}{b}\right)^3 + \left(\frac{y}{b}\right)^4 \right] \quad (20)$$

To take care of series of beams running in the same direction, Eq. (20) is transformed as expressed in Eq. (21).

$$w(y) = d_4 * b^4 \left[\left(\frac{y}{b}\right)^{n+1} - 2\left(\frac{y}{b}\right)^{n+2} + \left(\frac{y}{b}\right)^{n+3} \right] \quad (21)$$

For $n = 1, 2, 3, \dots$

Thus, the algebraic orthogonal polynomial displacement functions for all clamped edges (CCCC) rectangular plate based on static deflection configuration are as expressed in Eq. (22).

$$w(x, y) = A_{mn} \left[\left(\frac{x}{a}\right)^{m+1} - 2\left(\frac{x}{a}\right)^{m+2} + \left(\frac{x}{a}\right)^{m+3} \right] \left[\left(\frac{y}{b}\right)^{n+1} - 2\left(\frac{y}{b}\right)^{n+2} + \left(\frac{y}{b}\right)^{n+3} \right] \quad (22)$$

Where A_{mn} is the amplitude of deflection, defined as expressed in Eq. (23).

$$A_{mn} = (c_4 * a^4) * (d_4 * b^4) \quad (23)$$

Thus by using the same procedure, the algebraic orthogonal polynomial displacement functions for any set of rectangular plate boundary conditions can be developed; and they are as shown in Table 1.

Table 1: Algebraic Orthogonal Polynomial Shape Functions for Rectangular Plate with various Boundary Conditions

Boundary Conditions	Algebraic Orthogonal Polynomial Shape Functions, $w(x,y)$
CCCC	$A_{mn} \left[\left(\frac{x}{a}\right)^{m+1} - 2\left(\frac{x}{a}\right)^{m+2} + \left(\frac{x}{a}\right)^{m+3} \right] \left[\left(\frac{y}{b}\right)^{n+1} - 2\left(\frac{y}{b}\right)^{n+2} + \left(\frac{y}{b}\right)^{n+3} \right]$
SSSS	$A_{mn} \left[\left(\frac{x}{a}\right)^m - 2\left(\frac{x}{a}\right)^{m+2} + \left(\frac{x}{a}\right)^{m+3} \right] \left[\left(\frac{y}{b}\right)^n - 2\left(\frac{y}{b}\right)^{n+2} + \left(\frac{y}{b}\right)^{n+3} \right]$
CSCS	$A_{mn} \left[\left(\frac{x}{a}\right)^{m+1} - 2\left(\frac{x}{a}\right)^{m+2} + \left(\frac{x}{a}\right)^{m+3} \right] \left[\left(\frac{y}{b}\right)^n - 2\left(\frac{y}{b}\right)^{n+2} + \left(\frac{y}{b}\right)^{n+3} \right]$
CSSS	$A_{mn} \left[\frac{3}{2}\left(\frac{x}{a}\right)^{m+1} - \frac{5}{2}\left(\frac{x}{a}\right)^{m+2} + \left(\frac{x}{a}\right)^{m+3} \right] \left[\left(\frac{y}{b}\right)^n - 2\left(\frac{y}{b}\right)^{n+2} + \left(\frac{y}{b}\right)^{n+3} \right]$
CCSS	$A_{mn} \left[\frac{3}{2}\left(\frac{x}{a}\right)^{m+1} - \frac{5}{2}\left(\frac{x}{a}\right)^{m+2} + \left(\frac{x}{a}\right)^{m+3} \right] \left[\frac{3}{2}\left(\frac{y}{b}\right)^{n+1} - \frac{5}{2}\left(\frac{y}{b}\right)^{n+2} + \left(\frac{y}{b}\right)^{n+3} \right]$
CSFS	$A_{mn} \left[6\left(\frac{x}{a}\right)^{m+1} - 4\left(\frac{x}{a}\right)^{m+2} + \left(\frac{x}{a}\right)^{m+3} \right] \left[\left(\frac{y}{b}\right)^n - 2\left(\frac{y}{b}\right)^{n+2} + \left(\frac{y}{b}\right)^{n+3} \right]$
CCCS	$A_{mn} \left[\left(\frac{x}{a}\right)^{m+1} - 2\left(\frac{x}{a}\right)^{m+2} + \left(\frac{x}{a}\right)^{m+3} \right] \left[\frac{3}{2}\left(\frac{y}{b}\right)^{n+1} - \frac{5}{2}\left(\frac{y}{b}\right)^{n+2} + \left(\frac{y}{b}\right)^{n+3} \right]$
CCCF	$A_{mn} \left[\left(\frac{x}{a}\right)^{m+1} - 2\left(\frac{x}{a}\right)^{m+2} + \left(\frac{x}{a}\right)^{m+3} \right] \left[6\left(\frac{y}{b}\right)^{n+1} - 4\left(\frac{y}{b}\right)^{n+2} + \left(\frac{y}{b}\right)^{n+3} \right]$
SSSF	$A_{mn} \left[\left(\frac{x}{a}\right)^m - 2\left(\frac{x}{a}\right)^{m+2} + \left(\frac{x}{a}\right)^{m+3} \right] \left[8\left(\frac{y}{b}\right)^n - 4\left(\frac{y}{b}\right)^{n+2} + \left(\frac{y}{b}\right)^{n+3} \right]$
CCSF	$A_{mn} \left[\frac{3}{2}\left(\frac{x}{a}\right)^{m+1} - \frac{5}{2}\left(\frac{x}{a}\right)^{m+2} + \left(\frac{x}{a}\right)^{m+3} \right] \left[6\left(\frac{y}{b}\right)^{n+1} - 4\left(\frac{y}{b}\right)^{n+2} + \left(\frac{y}{b}\right)^{n+3} \right]$

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IV. NUMERICAL SOLUTION AND DISCUSSION OF RESULTS

Determination of Natural Frequencies

The plate parameters adopted in this work are:

$$E = 10.92 * 10^6 Nm^{-2}; \rho = 100Kgm^{-3}; \mu = 0.3; a = 1.0m; t = 0.01m; b = open; \beta = \frac{b}{a}$$

With reference to Fig. 3 and Eq. (22), the algebraic orthogonal polynomial shape functions for all edges clamped rectangular plate are as expressed in Eq. (24).

$$w(x,y) = A_{mn} \left[\left(\frac{x}{a}\right)^{m+1} - 2\left(\frac{x}{a}\right)^{m+2} + \left(\frac{x}{a}\right)^{m+3} \right] \left[\left(\frac{y}{b}\right)^{n+1} - 2\left(\frac{y}{b}\right)^{n+2} + \left(\frac{y}{b}\right)^{n+3} \right] \quad (24)$$

Where

$$m, n = 1, 2, 3, \dots$$

A_{mn} = amplitude of deflection.

For small deflection theory, the parameter A_{mn} will cancel out in Eq. (12) of the energy equation; and as such there is no loss in accuracy if A_{mn} is taken to be unity. The values of m and n in Eq. (24) identify and define the modes for any given displacement function as shown in Eqs. (25) through (28).

$$w(1,1) = \left[\left(\frac{x}{a}\right)^2 - 2\left(\frac{x}{a}\right)^3 + \left(\frac{x}{a}\right)^4 \right] \left[\left(\frac{y}{b}\right)^2 - 2\left(\frac{y}{b}\right)^3 + \left(\frac{y}{b}\right)^4 \right] \quad (25)$$

$$w(1,2) = \left[\left(\frac{x}{a}\right)^2 - 2\left(\frac{x}{a}\right)^3 + \left(\frac{x}{a}\right)^4 \right] \left[\left(\frac{y}{b}\right)^3 - 2\left(\frac{y}{b}\right)^4 + \left(\frac{y}{b}\right)^5 \right] \quad (26)$$

$$w(2,1) = \left[\left(\frac{x}{a}\right)^3 - 2\left(\frac{x}{a}\right)^4 + \left(\frac{x}{a}\right)^5 \right] \left[\left(\frac{y}{b}\right)^2 - 2\left(\frac{y}{b}\right)^3 + \left(\frac{y}{b}\right)^4 \right] \quad (27)$$

$$w(2,2) = \left[\left(\frac{x}{a}\right)^3 - 2\left(\frac{x}{a}\right)^4 + \left(\frac{x}{a}\right)^5 \right] \left[\left(\frac{y}{b}\right)^3 - 2\left(\frac{y}{b}\right)^4 + \left(\frac{y}{b}\right)^5 \right] \quad (28)$$

Where $w(1,1)$ is the first mode, $w(1,2)$ is the second mode, $w(2,1)$ is the third mode and $w(2,2)$ is the fourth mode.

To calculate multiple natural frequencies of a system, the effect of modal interaction must be taken into account. In this work, the participations of the functions V and w in the energy equation of Eq. (12) were taken to represent the modal interaction at any given modal point. Equally, from small deflection theory, there exists symmetry in the stiffness distribution so that $k_{ij} = k_{ji}$. Therefore to determine the stiffness k_{11} , let $V = w = w(1,1)$; to determine $k_{21} = k_{12}$, let $V = w(1,2)$ and let $w = w(1,1)$; to determine $k_{31} = k_{13}$, let $V = w(2,1)$ and let $w = w(1,1)$; to determine $k_{41} = k_{14}$, let $V = w(2,2)$ and let $w = w(1,1)$ and so on. In this work, all the complex computations were carried with the aid of Mathematica® commercial software.

Thus

$$\int_0^a \int_0^b \left[\left(\frac{\partial^2 w(1,1)}{\partial x^2} \right)^2 + 2 \left(\frac{\partial^2 w(1,1)}{\partial x \partial y} \right)^2 + \left(\frac{\partial^2 w(1,1)}{\partial y^2} \right)^2 \right] dx dy = k_{11} \quad (29)$$

$$\int_0^a \int_0^b \left[\frac{\partial^2 w(1,2)}{\partial x^2} \frac{\partial^2 w(1,1)}{\partial x^2} + 2 \frac{\partial^2 w(1,2)}{\partial x \partial y} \frac{\partial^2 w(1,1)}{\partial x \partial y} + \frac{\partial^2 w(1,2)}{\partial y^2} \frac{\partial^2 w(1,1)}{\partial y^2} \right] dx dy = k_{21} = k_{12} \quad (30)$$

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$$\int_0^a \int_0^b \left[\frac{\partial^2 w(2,1)}{\partial x^2} \frac{\partial^2 w(1,1)}{\partial x^2} + 2 \frac{\partial^2 w(2,1)}{\partial x \partial y} \frac{\partial^2 w(1,1)}{\partial x \partial y} + \frac{\partial^2 w(2,1)}{\partial y^2} \frac{\partial^2 w(1,1)}{\partial y^2} \right] dx dy = k_{31} = k_{13} \quad (31)$$

⋮

$$\int_0^a \int_0^b \left[\left(\frac{\partial^2 w(2,2)}{\partial x^2} \right)^2 + 2 \left(\frac{\partial^2 w(2,2)}{\partial x \partial y} \right)^2 + \left(\frac{\partial^2 w(2,2)}{\partial y^2} \right)^2 \right] dx dy = k_{44} \quad (32)$$

The resulting stiffness matrix is given as expressed in Eq. (33).

$$[K] = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \quad (33)$$

Using the plate parameters as adopted in this work for CCCC plate, then Eq. (33) in numerical values is expressed as in Eq. (34):

$$[K] = \begin{bmatrix} 3.26531 & 1.63265 & 1.63265 & 0.81633 \\ 1.63265 & 1.1324 & 0.81633 & 0.56621 \\ 1.63265 & 0.81633 & 1.13241 & 0.56621 \\ 0.81633 & 0.56621 & 0.56621 & 0.37747 \end{bmatrix} 10^{-3} \quad (34)$$

The same procedure that was used for developing the stiffness matrix was also applied to the right hand side of Eq. (12) to develop the mass matrix.

Thus

$$\omega^2 MK \int_0^a \int_0^b (w(1,1))^2 dx dy = b_{11} \quad (35)$$

$$\omega^2 MK \int_0^a \int_0^b w(1,2) * w(1,1) dx dy = b_{21} = b_{12} \quad (36)$$

$$\omega^2 MK \int_0^a \int_0^b w(2,1) * w(1,1) dx dy = b_{31} = b_{13} \quad (37)$$

⋮

$$\omega^2 MK \int_0^a \int_0^b (w(2,2))^2 dx dy = b_{44} \quad (38)$$

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The resulting mass matrix is as expressed in Eq. (39):

$$[\mathbf{B}] = \omega^2 \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix} MK \quad (39)$$

Also using the plate parameters as defined, the numerical values of Eq. (39) with respect to CCCC plate is as expressed in Eq. (40):

$$[\mathbf{B}] = \omega^2 \begin{bmatrix} 2.51953 & 1.25976 & 1.25976 & 0.62988 \\ 1.25976 & 0.68714 & 0.62988 & 0.34357 \\ 1.25976 & 0.62988 & 0.68714 & 0.34357 \\ 0.62988 & 0.34357 & 0.34357 & 0.18740 \end{bmatrix} 10^{-6} \quad (40)$$

Suppose the difference between $[\mathbf{K}]$ and $[\mathbf{B}]$ is $[\mathbf{A}]$, then

$$[\mathbf{A}] = [\mathbf{K}] - [\mathbf{B}] \quad (41)$$

The determinant of $[\mathbf{A}]$ is thus:

$$\det[\mathbf{A}] = \det([\mathbf{K}] - [\mathbf{B}]) \quad (42)$$

For nontrivial solution, Eq. (42) is set equal to zero. That is

$$\det([\mathbf{K}] - [\mathbf{B}]) = 0 \quad (43)$$

Eq. (43) can be numerically simplified by pre-multiplying Eq. (43) by the inverse of Eq. (40) as shown in Eq. (44).

$$\det([\mathbf{B}]^{-1}[\mathbf{K}] - [\mathbf{B}]^{-1}[\mathbf{B}]) = 0 \quad (44)$$

The result of the manipulation of Eq. (44) gives an expression as shown in Eq. (45).

$$\det \begin{bmatrix} 1296.79 - \omega & -2109.99 & -2110.44 & 514.06 \\ -1.61178 & 5514.9 - \omega & -3.0878 & -3140.29 \\ -1.61178 & -5.0098 & 5517.0 - \omega & -3140.29 \\ 3.2754 & 9.8258 & 5.9816 & 11800.9 - \omega \end{bmatrix} = 0 \quad (45)$$

The solution of the resulting quartic algebraic equation in terms of ω gives the natural frequencies of vibration of the rectangular plate at aspect ratio equal to one. Therefore by carrying out the same procedure for any set of boundary conditions of rectangular plate yields the desired natural frequencies for the rectangular plate up to first four lowest natural frequencies. However, in the present study, ten boundary conditions of all edges clamped, (CCCC), all edges simply supported, (SSSS), clamped-simply supported-clamped-simply supported, (CSCS), clamped-simply-simply-simply supported, (CSSS), and clamped-clamped-simply-simply supported, (CCSS). Others are clamped-simply supported-free-simply supported, (CSFS), clamped-clamped-clamped-simply supported, (CCCS), clamped-clamped-clamped-free, (CCCCF), simply-simply-simply-free edges, (SSSF) and clamped-clamped-simply supported-free edge, (CCSF) rectangular plates have been selected for investigation under free vibrations using energy

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functional formulated using weak-form variational principle; and Table 2 presents the results of the investigation along side with the results from previous works found in literature.

Table 2: First Four lowest Natural Frequencies of Rectangular Plate with various Boundary Conditions, at $\beta = 1$

Boundary conditions	Mode	Present work [rad/s]	Chakraverty (2009) [rad/s]	Percentage Error [%]	Leissa and Qatu (2011) [rad/s]	Percentage Error [%]
CCCC	ω_{11}	36.00	35.988	-0.330	35.990	-0.028
	ω_{12}	74.2967	73.398	-1.224	73.410	-1.208
	ω_{21}	74.2967	73.398	-1.224	73.410	-1.208
	ω_{22}	108.591	108.260	-0.306	***	***
SSSS	ω_{11}	19.7476	19.7390	-0.044	197392	-0.043
	ω_{12}	53.2413	49.3480	-7.889	49.3480	-7.889
	ω_{21}	53.2413	49.3480	-7.889	49.3480	-7.889
	ω_{22}	84.6470	79.400	-6.608	***	***
CSCS	ω_{11}	28.9560	28.950	-0.021	28.9509	-0.018
	ω_{12}	58.4283	54.873	-6.479	69.3270	15.721
	ω_{21}	70.1868	69.327	-1.240	54.7431	-28.211
	ω_{22}	97.2977	94.703	-2.740	***	***
CSSS	ω_{11}	23.6515	23.646	-0.023	23.6463	-0.022
	ω_{12}	55.4824	51.813	-7.082	58.6464	5.395
	ω_{21}	62.2737	58.650	-6.179	51.6743	-20.512
	ω_{22}	91.8503	86.252	-6.491	***	***
CCSS	ω_{11}	27.0618	27.055	-0.025	27.06	-0.007
	ω_{12}	64.1546	60.550	-5.953	60.79	-5.535
	ω_{21}	64.3702	60.550	-5.953	60.54	-6.327
	ω_{22}	98.6751	92.914	-6.200	***	***
CSFS	ω_{11}	13.4904	12.687	-6.332	12.6874	-6.329
	ω_{12}	40.7537	33.067	-23.246	33.0651	-23.253
	ω_{21}	46.9548	41.714	-12.564	41.7019	-12.596
	ω_{22}	79.3170	63.260	-25.383	***	***
CCCS	ω_{11}	31.8363	31.827	-0.029	31.83	-0.020
	ω_{12}	66.8770	63.348	-5.571	63.35	-5.567
	ω_{21}	71.9549	71.083	-1.227	71.08	-1.231
	ω_{22}	103.7510	100.830	-2.897	***	***
CCCF	ω_{11}	24.5354	23.960	-2.402	24.02	-2.146
	ω_{12}	47.5407	40.021	-18.789	40.04	-18.733
	ω_{21}	65.1823	63.291	-2.988	63.49	-2.665
	ω_{22}	92.8811	78.139	-18.867	***	***
SSSF	ω_{11}	12.3053	11.684	-5.318	11.6845	-5.313
	ω_{12}	30.0426	27.757	-8.234	27.7563	-8.237
	ω_{21}	46.2809	41.220	-12.278	41.1967	-12.341
	ω_{22}	65.8793	59.360	-10.983	***	***
CCSF	ω_{11}	18.2369	17.555	-3.884	17.62	-3.501
	ω_{12}	43.7587	36.036	-21.431	36.05	-21.383
	ω_{21}	56.5625	51.862	-9.063	52.06	-8.649
	ω_{22}	87.1799	71.218	-22.413	***	***

The triple star symbol, *** indicates that there is no result with respect 4th lowest natural frequency.

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Furthermore in this study, the variation of fundamental frequencies with aspect ratios (Beta) was investigated, and the results of the investigation were graphically presented as shown in Figure 4.

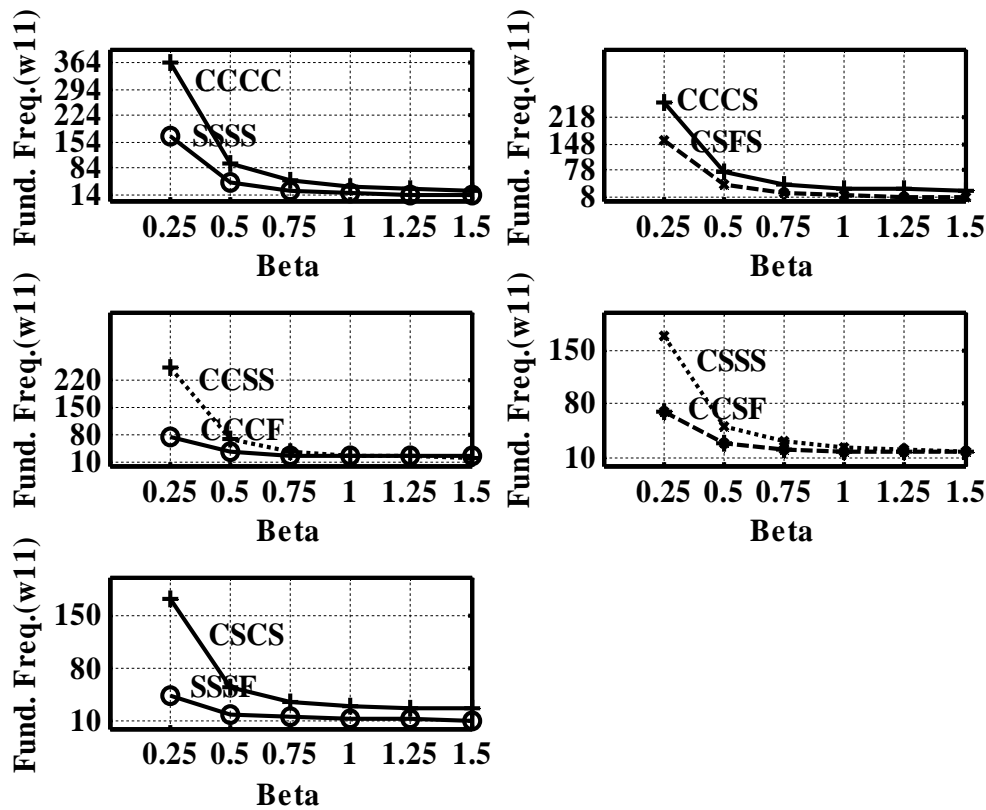


Fig. 4 Variation of Fundamental Frequencies with Aspect Ratios

Statistical Evaluation

In order to properly compare the present study's results with the results from previous works found in literature, statistical analysis was carried out. The mean absolute percentage error values had to be evaluated. Also, the coefficient of determination and the correlation coefficient had to be evaluated as measures to determine the variation of the present study's results with the results from previous works found in literature. Thus:

$$\bar{\mu}\% = \frac{|\sum_{i=1}^N (\omega_i^* - \omega_i)|}{N} \% \tag{44}$$

$$\bar{\omega}^* = \frac{\sum_{i=1}^N \omega_i^*}{N} \tag{45}$$

$$r^2 = \frac{\sum_{i=1}^N (\omega_i - \bar{\omega}^*)^2}{\sum_{i=1}^N (\omega_i^* - \bar{\omega}^*)^2} \tag{46}$$

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Where $\bar{\mu}$ is the mean of the error; ω_i is the natural frequency of the present work at mode i ; ω_i^* is the corresponding natural frequency of the previous work found in literature at mode i ; $\bar{\omega}^*$ is the mean of the natural frequencies of the previous work found in literature; r is the correlation coefficient; and r^2 is the coefficient of determination. The values in Table 2 were used to calculate the required parameters of Eqs. (44), (45) and (46) respectively.

Discussion of Results

From Table 2, it was discovered that the present study's results with respect to fundamental frequencies are excellently stable when compared with the results from previous works found in literature for all the boundary conditions investigated. Also, the present work's results were found to be on upper bounds. From statistical evaluation on the aggregation of natural frequencies computed for all boundary conditions, it was observed that 82.5% of them give good convergence to the results from previous works found in literature. Also from Table 2, and with reference to column 5, the mean absolute error between the present work's results and the results from previous work found in [6] is 7.188%; and the coefficient of determination and correlation coefficient evaluated using the previous results found in [6] as standard were 0.99998 and 99.999% respectively. This implies that 99.998% of the variation between the present study's results and those of the previous work done by [6] has been explained. Also with reference to column 7, the mean absolute error between the present study's results and the results from previous work found in [1] is 6.325%. Similarly, the coefficient of determination and correlation coefficient evaluated using the previous results done by [1] as standard were 0.99996 and 99.998% respectively, which implies that 99.996% of the variation between the present study's results and those of the previous work done by [1] has been explained. From the statistical analysis carried out, the present work's results are acceptable for any engineering precisions. Furthermore, from Fig. 4, the variation of fundamental frequencies with plate aspect ratios exhibited soft-spring type. This indicates that at very low plate aspect ratio, the fundamental natural frequencies approach equivalent beam fundamental natural frequencies.

V. CONCLUSION

This paper applies weak-form variational principle and algebraic orthogonal polynomial displacement functions to determine the natural frequencies of rectangular isotropic plate with various boundary conditions. The natural frequencies computed herein in this work, in view of statistical interpretation, are in good agreement with results from previous works found in literature. Therefore, it is here-under concluded that weak-form variational principle has high competitive capability with other approximation techniques in dealing with problems of dynamic analysis of thin rectangular isotropic plates. Furthermore, it is here concluded that the application of algebraic polynomial displacement functions is capable and stable in yielding satisfactory approximations to any set of boundary conditions of rectangular plates.

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