

Theory of Convergence

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ABSTRACT: **Convergence**, in [mathematics](#), property (exhibited by certain [infinite series](#) and [functions](#)) of approaching a [limit](#) more and more closely as an argument (variable) of the function increases or decreases or as the number of terms of the series increases. For example, the function $y = 1/x$ [converges](#) to zero as x increases. Although no finite value of x will cause the value of y to actually become zero, the limiting value of y is zero because y can be made as small as desired by choosing x large enough. The line $y = 0$ (the x -axis) is called an asymptote of the function. Similarly, for any value of x between (but not including) -1 and $+1$, the series $1 + x + x^2 + \dots + x^n$ converges toward the limit $1/(1 - x)$ as n , the number of terms, increases. The interval $-1 < x < 1$ is called the range of convergence of the series; for [values](#) of x outside this range, the series is said to diverge.

KEYWORDS: convergence, mathematics, functions, variable, function, asymptote, infinite

I. INTRODUCTION

Mathematical [concept](#) based on the idea of closeness, used primarily to assign values to certain [functions](#) at points where no values are defined, in such a way as to be consistent with nearby values. For example, the [function](#) $(x^2 - 1)/(x - 1)$ is not defined when x is 1, because division by zero is not a valid mathematical operation. For any other value of x , the numerator can be factored and divided by the $(x - 1)$, giving $x + 1$. Thus, this quotient is equal to $x + 1$ for all values of x except 1, which has no value. However, 2 can be assigned to the function $(x^2 - 1)/(x - 1)$ not as its value when x equals 1 but as its limit when x approaches. The following more-basic definition of limit, independent of

$$\lim_{x \rightarrow x_0} f(x) = L$$

the concept of [continuity](#), can also be given: if, for any desired degree of closeness ϵ , one can find an interval around x_0 so that all values of $f(x)$ calculated here differ from L by an amount less than ϵ (i.e., if $|x - x_0| < \delta$, then $|f(x) - L| < \epsilon$). This last definition can be used to determine whether or not a given number is in fact a limit. The calculation of limits, especially of quotients, usually involves manipulations of the function so that it can be written in a form in which the limit is more obvious, as in the above example of $(x^2 - 1)/(x - 1)$. Limits are the method by which the [derivative](#), or rate of change, of a function is calculated, and they are used throughout analysis as a way of making approximations into exact quantities, as when the area inside a curved region is defined to be the limit of approximations by rectangles. **Geometric series**, in [mathematics](#), an [infinite series](#) of the form $a + ar + ar^2 + ar^3 + \dots$, where r is known as the common ratio.¹ A simple example is the geometric series for $a = 1$ and $r = 1/2$, or $1 + 1/2 + 1/4 + 1/8 + \dots$, which [converges](#) to a sum of 2 (or 1 if the first term is excluded). The [Achilles paradox](#) is an example of the difficulty that ancient Greek mathematicians had with the idea that an [infinite series](#) could produce a finite sum. The confusion around [infinity](#) did not abate until the 18th century, when mathematicians developed [analysis](#) and the concept of [limits](#).

The sum of the first n terms of a geometric series is equal to $a(1 - r^n)/(1 - r)$. If the absolute value of r is less than 1, the series converges to $a/(1 - r)$. For any other value of r , the series diverges.

Uniform convergence, in [analysis](#), property involving the [convergence](#) of a sequence of [continuous functions](#)— $f_1(x), f_2(x), f_3(x), \dots$ —to a [function](#) $f(x)$ for all x in some interval (a, b) . In particular, for any positive number $\epsilon > 0$ there exists a positive integer N for which $|f_n(x) - f(x)| \leq \epsilon$ for all $n \geq N$ and all x in (a, b) . In pointwise convergence, N depends on both the closeness of ϵ and the particular point x . An [infinite series](#) $f_1(x) + f_2(x) + f_3(x) + \dots$ [converges](#) uniformly on an interval if the sequence of partial sums converges uniformly on the interval.² Many mathematical tests for uniform convergence have been devised. Among the most widely used are a variant of [Abel's test](#), devised by Norwegian mathematician [Niels Henrik Abel](#) (1802–29), and the Weierstrass M-test, devised by German mathematician [Karl Weierstrass](#) (1815–97). The literature already contains several theories of limits which have great generality (10; 12; 1; 5; 6; 13; 3; 14; 4, p. 34). Nevertheless, the intrinsic importance and frequent use of the concept may justify the publication of another variant, provided that it has advantages in ease of application without sacrifice of generality. The theory of convergence studied in this note includes the other theories and their applications in a smooth way, without artifice.³



II.DISCUSSION

In [mathematics](#), [computer science](#) and [logic](#), **convergence** is the idea that different sequences of transformations come to a conclusion in a finite amount of time (the transformations are [terminating](#)), and that the conclusion reached is independent of the path taken to get to it (they are [confluent](#)).

More formally, a [preordered](#) set of [term rewriting](#) transformations are said to be **convergent** if they are [confluent](#) and [terminating](#).⁴

A **series** is the sum of a **sequence**, which is a list of numbers that follows a pattern. An **infinite series** is the sum of an infinite number of terms in a sequence, such as 2 + 4 + 6 + 8 + ... Infinite series are subject to convergence and divergence.

In Zeno's Paradox of the Dichotomy, a runner must always reach the halfway point between her current position and the finish line. Those distances will become half as large each time. Thus, there will always be some distance left to go, and so she will never reach her destination. This is similar to a car that reduces its speed by half each second so that it will travel half of the prior second's distance. Like the runner, the car will never reach its destination. In both of these cases, the distances traveled become smaller by half each time, and so the series of distances traveled converges to a set limit (the total distance of the trip).⁵



The amount of distance left until the finish line, point F, is equal to the last distance traveled, which is half of the prior distance traveled.

In contrast, saving one penny on Day 1 and then doubling the amount saved each day would result in infinite savings. Thus, the series of money saved each day does not converge to a set limit, and so it diverges. The first question to understand is: what does "converge" mean? **Converge** means to approach a set limit, such as 2 or -350.25. In the context of geometric series, the absolute values of the terms will get smaller, and so, as with the runner, there will always be some distance left between the limit and the current sum of the terms. So, for those wondering, "What is a convergent series; what is a divergent series?" A convergent series is simply a series that approaches a set limit, and a divergent series is a series that does not approach a set limit. **Diverge** means to fail to approach a set limit. In the context of geometric series with $|r| > 1$, the absolute values of the terms will get larger, causing the series to approach infinity or negative infinity.⁶

While many divergent series becomes infinitely large, not all divergent series will do so. All that is required for a series to be divergent is for it to fail to converge. For example, the terms of $\sum_{n=1}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + \dots$ alternate between -1 and 1, so the sum alternates between -1 and 0. Therefore, the series diverges since it does not approach a set limit.) There are a variety of convergence theorems and tests for a variety of series types. In essence, any convergence theorem states that if a series approaches a high or low limit, it converges. Convergence tests are used to determine whether a series will converge. The particular test to use varies based on the characteristics of the series being analyzed, but, in general, the tests determine whether the sum of the terms approaches a finite limit. Some tests do this by determining whether the absolute values of the terms get smaller, while others use limits and integrals from calculus.⁷

III.RESULTS

For many people — myself included — mathematics can be the most intimidating of the fundamental sciences: an unfathomable language accessible only to a select few with the ability to master its complex myriad of symbols and rules. It is often overshadowed by its close relative physics, as the unpredictable nature of the quantum world, the enigma of a black hole or even just an episode of Star Trek never fails to capture the public’s imagination. It seems that mathematics never receives the credit it deserves. Enter *The Universe Speaks in Numbers* by Graham Farmelo, which aims to bring mathematics to the forefront of the reader’s mind and convince them that mathematics has and will continue to play an indispensable role in the future of physics.⁸ From the writings of the ancient Greeks through to the

latest developments at the Large Hadron Collider, the book aims to highlight the invaluable contribution of mathematics to the most important advances scientists have made in uncovering the fundamental laws that govern the Universe. By immersing us in historic details the book narrates the circumstances of key discoveries and the individuals behind them. It begins with Newton's Principia and the foundation of classical mechanics, Maxwell's work on electromagnetism, Einstein's theory of general relativity through to quantum mechanics⁹. Each field is exemplified to showcase how the research, although firmly embedded in the world of physics, was also based on foundations of profound mathematical insight. A wide range of well-known scientists is brought in to provide a narrative of the surrounding circumstances, differing views, seminal lectures and publications, as well as to illuminate how the relationship between theoretical physics and mathematics has evolved through the centuries.¹⁰

Farmelo laments the fact that, despite joint successes, mathematicians and theoretical physicists have tended to take separate paths, sometimes pouring scorn on each other along the way, and that too few scientists, even now, have sought to establish themselves in both fields. The book describes how influential figures — and even a clandestine, masonic-like mathematical society publishing under the pseudonym of Nicolas Bourbaki — aimed to drive a wedge between the two fields. This was an era described as the 'long divorce' in an interview with the renowned theoretical physicist and mathematician Freeman Dyson. According to Dyson, despite fruitful collaborations between the two fields, the two began to diverge approaching the Second World War. Physicists became content adopting the mathematical tools already available for new theories and mathematicians preferred to focus on the beauty of their equations with little care for any potential relevance to the real world.

This period came to an end during the 1970s with the inception of string theory: a theoretical framework that some believe is the strongest candidate for a unified theory that encompasses all four fundamental forces. However, to test this in an experiment, string theory requires energies and distances on a scale that today's researchers are more likely to see in Star Trek than in the most advanced particle accelerator. Here, Farmelo argues that 'orthodox' methods such as observation and experiment cannot be relied upon, and so instead theoretical physicists should "use the torch of mathematics to help light the way forward" until, eventually, technology catches up to test theories experimentally.¹¹

This touches upon one of the biggest dilemmas facing the current generation of theoretical physicists: that the flow of experimental data in need of explanatory theories has slowed. The vast amount of data collected at the Large Hadron Collider has not discriminated between theories in the way that particle physicists would have hoped for and in such circumstances Farmelo argues that one has to be more creative, rely on pure thought and adapt abstract mathematical concepts to develop new theories. This is a methodology termed 'physical mathematics' and described as: "A fusion of mathematics and theoretical physics that aims to shed light on new mathematics, on the fundamental laws of Nature, and on the relationship between them." This suggestion is not without controversy, since some see this subfield as an unhelpful distraction for physicists who could be swallowed up in a mire of mathematical trivialities.¹²

IV. CONCLUSIONS

In line with the research focus, it is aimed to reveal the approaches of pre-service elementary mathematics teachers on the convergence and divergence of the series with the help of the real-life problem in the context of theory and application by using the concept of harmonic series. The research carried out the qualitative research method is designed according to the case study. The data of the research have consisted of two written questions and one problem and four interview questions which were formed by the researchers and their validity and reliability were ensured. The study concluded that pre-service teachers have theoretical knowledge on the convergence of series and harmonic series. Also, their perception of series, harmonic series, convergence and divergence concepts changed in the application process and they performed different approaches in problem-solving¹³. As qualitative research is inherently based on observation and interpretation, the role of researchers is essential in this process. The observations and experience of researchers were influential on the selection of the participants of the study. The study is structured on the experiences of the first researcher during his undergraduate education and on his belief that he felt inadequate in application of the concepts of series and harmonic series to real-life during his undergraduate education as well as the second researchers' experiences, opinions and guidance on the matter. Moreover, the time spent together with the participants in the courses and extracurricular environments and the various comments on series during natural and short conversations made us think that various approaches about the series can be evaluated. In this study, the knowledge of pre-service elementary mathematics teachers on convergence and divergence of series and how they apply this knowledge to the problem

enriched with real-life context were examined. Findings have revealed that pre-service teachers performed better in explaining theoretical aspect of the convergence of series than in explaining the divergence of series. One of the reasons for this situation may be the fact that the concepts such as theorems, results and criteria that the pre-service teachers have learned are mostly built on the concept of “convergence” in textbooks. The fact that the convergence and divergence concepts are used as opposite may arise from the fact that these two are dichotomous concepts.¹⁴

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