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Allied Complete Hausdorffness in Topology

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ABSTRACT: In 2013 & 2015, the functionally Hausdorff spaces were defined and studied their various properties. In 1968, R.M. Stephenson Jr. defined and studied the concept of Completely Hausdorff spaces in connection with to study the Stone-Weierstrass theorem. Again in 1970, S. Willard has defined the same concept of Completely Hausdorff spaces in topology. In this paper, we introduce and study the concepts of completely α -Hausdorff spaces, completely s -Hausdorff spaces, completely p -Hausdorff spaces and completely β -Hausdorff spaces in topology analogous to the definition due to S. Willard. Also, we study some basic characterizations of completely Hausdorff spaces in this paper.

KEYWORDS: semiopen sets, preopen sets, α -open sets, β -open sets, semicontinuity, precontinuity, α -continuity, β -continuity, CH-spaces, FH-spaces, τ_α -topology and semi-regular spaces

I. INTRODUCTION

In 1968, R.M. Stephenson Jr. [50] defined the concept of completely Hausdorff spaces (we denote them as CH-spaces) and studied the Stone-Weierstrass theorem by defining CH-closed spaces and shown that a CH-space is Stone-Weierstrass iff it is CH-closed space. In 1970, S. Willard [54] introduced the same concept of CH-spaces. In 1971, Asit B. Raha [45] introduced and studied the some more properties of CH-closed spaces and CH-closed extensions. In 1979, R.A. Herrmann [21] studied the CH-spaces and shown that a CH-closed space is H-closed iff it is nearly compact iff it is almost completely regular. In 1984, Garg & Sivaraj [20] studied some properties of CH-spaces. In 1985, M. Mrsevic et al [34] have studied many semiregularization properties of FH-spaces. In 1986, C. Dorsett [13] studied some basic properties of CH-spaces and shown that CH-space satisfy semitopological property, also CH-spaces generalized to weakly CH-spaces. Latter, in 2013 and 2015, Sami Lazaar [24] and A. Dehghani et al [12] defined the concept of functionally Hausdorff spaces (we denote them as FH-spaces) and studied their various properties in topological spaces. In [12&24], respectively, among other results obtained that: (i) Characterize topological spaces for which FH-reflection is a spectral space; (ii) A Tychonoff space implies FH-space implies Hausdorff space & (iii) Every locally connected FH-space is c -resolvable. In the literature of topological spaces, notions of CH-spaces and FH-spaces are treated as same concepts. The purpose of this paper is to introduce and study the concepts of $C\alpha$ H-spaces, Cs H-spaces, Cp H-spaces and $C\beta$ H-spaces in topology analogous to the definition due to S. Willard [54]. Also, we study some basic characterizations of CH-spaces in this paper.

II. PRELIMINARIES

Throughout this paper, X and Y always mean topological spaces (or simply spaces) on which no separation axiom is assumed unless explicitly stated. The closure and the interior of a set A is denoted by $Cl(A)$ and $Int(A)$, respectively. R and I denotes respectively, the set of reals and the unit interval $[0,1]$.

Definition 2.1: A subset A of a space X is said to be:

- (i) semiopen [25] set if $A \subset Cl Int(A)$.
- (ii) preopen [31] set if $A \subset Int Cl(A)$.
- (iii) α -open [41] set if $A \subset Int Cl Int(A)$.
- (iv) β -open [1] set if $A \subset Cl Int Cl(A)$.

The complement of a semiopen (resp. preopen, α -open, β -open) set is called a semiclosed [8] (resp. preclosed [15], α -closed [32], β -closed [2]) set of X .

Definition 2.2: A function $f: X \rightarrow Y$ is said to be:

- (i) α -continuous [32] if for each open set V in Y , $f^{-1}(V)$ is α -open in X .
- (ii) semi-continuous [25] if for each open set V in Y , $f^{-1}(V)$ is semi-open in X .

- (iii) pre-continuous [31] if for each open set V in Y , $f^{-1}(V)$ is pre-open in X .
- (iv) β -continuous [1] if for each open set V in Y , $f^{-1}(V)$ is β -open in X .
- (v) pre- α -open [11] if for each α -open set U in X , $f(U)$ is α -open set in Y .
- (vi) pre- α -closed [11] if for each α -closed set F in X , $f(F)$ is α -closed set in Y .
- (vii) pre- β -open [26] if for each β -open set U in X , $f(U)$ is β -open set in Y .
- (viii) pre- β -closed [26] if for each β -closed set F in X , $f(F)$ is β -closed set in Y .
- (ix) presemiopen [10] if for each semiopen set U in X , $f(U)$ is semiopen set in Y .
- (vii) presemiclosed [19] if for each semiclosed set F in X , $f(F)$ is semiclosed set in Y .
- (x) M-preopen [33] if for each preopen set U in X , $f(U)$ is preopen set in Y .
- (xi) M-preclosed [33] if for each preclosed set U in X , $f(U)$ is preclosed set in Y .

DEFINITION 2.3 : A space X is said to be :

- (i) p -completely regular [29] (resp. s -completely regular [28] , α -completely regular [37], β -completely regular [36]) if for each closed set F and each point $x \in (X \setminus F)$, there exists a precontinuous (resp. a semicontinuous , an α -continuous , a β -continuous) function $f : X \rightarrow [0,1]$ such that $f(x) = 0$ and $f(y) = 1$ for each $y \in F$.
- (ii) submaximal [9] if every dense subset of it is open (i.e. if $PO(X) = \tau$ [46]).
- (iii) an extremally disconnected (E.D.) [55] if closure of each open set is open in it (i.e. if $A \in \tau$ for each $A \in RF(X)$ [23]).
- (iv) PS-space [4] if every preopen set of X is semiopen in X .
- (v) α -space [16] if every α -open set of X is open in X (i.e. $\tau = \alpha O(X)$).

Lemma 2.4 [3] : If (X, τ) is submaximal and E.D. space , then $\tau = \beta O(X, \tau)$.

Definition 2.5 [55] : A space X is completely -Hausdorff (CH) if and only if whenever $x \neq y$ in X , there is a continuous $f : X \rightarrow [0,1]$ with $f(x) = 0$ and $f(y) = 1$.

Definition 2.6 [24] : A space X is functionally -Hausdorff (FH) if for any two distinct points x and y in X , there is a Urysohn function for $\{x\}$ and $\{y\}$ (we say that x and y are functionally separated or completely separated)

Definition 2.7 [48] . A space X is called almost completely regular if for each regular closed set closed set F in X and for a point $x \in X - F$, there exists a continuous function $f : X \rightarrow [0,1]$ such that $f(x) = 0$ and $f(y) = 1$ for each $y \in F$.

Definition 2.8 [53] : A Hausdorff space X is called H-closed if every open cover of X has a proximate subcover.

Definition 2.9 [49] : A space X is called nearly compact if every open cover of X has a finite subfamily , the interiors of the closures of whose members cover X .

Lemma 2.10 [30] : If A is preopen set in X and B is an α -open set in X , then $A \cap B$ is an α -open set in the subspace (A, τ_A) .

Lemma 2. 11 [47] : If A is semiopen set in X and B is an α -open set in X , then $A \cap B$ is an α -open set in the subspace (A, τ_A) .

Result 2.12 [1]: Let X be a space and A and U be subsets of X , If $U \in \alpha O(X)$ and $A \in \beta O(X)$ then $A \cap U \in \beta O(U)$.

Lemma 2.13 [4] : For a space X the following conditions are equivalent :

- (i) X is E.D.-space
- (ii) every semiopen subset is preopen
- (iii) every semiopen subset is α -open
- (iv) every β -open subset is preopen

Lemma 2.14[4] :For a space X the following conditions are equivalent :

- (i) X is PS-space
- (ii) every preopen subset is semiopen
- (iii) every β -open subset is semiopen
- (iv) every preopen subset is α -open

Theorem 2.15 [22 & 35]: In an E.D.-space and submaximal-space X , then
 $\tau = \alpha O(X) = SO(X) = PO(X) = \beta O(X)$.

Lemma 2.16 [5 & 7] : Let X be a topological space .The following properties are satisfied .

- (i) $SO(X, \tau) = SO(X, \tau_\alpha)$; $SF(X, \tau) = SF(X, \tau_\alpha)$;
- (ii) $PO(X, \tau) = PO(X, \tau_\alpha)$; $PF(X, \tau) = PF(X, \tau_\alpha)$;
- (iii) $RO(X, \tau) = RO(X, \tau_\alpha)$; $RF(X, \tau) = RF(X, \tau_\alpha)$.

Lemma 2.17[6 & 18] : Let X be a topological space .The following properties are satisfied .

- (i) $SPO(X, \tau)$ (i.e $\beta O(X, \tau) = SPO(X, \tau_\alpha)$ (i.e. $\beta O(X, \tau_\alpha)$) ;
- (ii) $SPF(X, \tau)$ (i.e $\beta F(X, \tau) = SPF(X, \tau_\alpha)$ (i.e. $\beta F(X, \tau_\alpha)$) ;

Lemma 2.18 [41] : Let (X, τ) be a topological space. Then,

- (i) τ_α is a topology on X and $\tau \subset \tau_\alpha$;
- (ii) $(\tau_\alpha)_\alpha = \tau_\alpha$;
- (iii) Topological spaces (X, τ) and (X, τ_α) have the same class of semiopen sets ;
- (iv) Let σ be any topology on X such that $SO(X, \sigma) = SO(X, \tau)$. Then , $\sigma \subset \tau_\alpha$.

III.CH - SPACES

We , follow the [55, p.99 14G] definition :

Definition 3.1[55] : A space X is called completely -Hausdorff (in brief , CH) if and only if whenever $x \neq y$ in X , there is a continuous $f: X \rightarrow [0,1]$ with $f(x) = 0$ and $f(y) = 1$.

Lemma 3.2 : Every CH-space is Hausdorff space.

Proof : Let $p, q \in X$ with $p \neq q$. Since X is CH-space, then there is a continuous function $f: X \rightarrow [0,1]$ such that $f(p) = 0$ and $f(q) = 1$.

As $f^{-1}([0, \frac{1}{2})) = U$ and $f^{-1}((\frac{1}{2}, 1]) = V$ are open sets in X such that $p \in U$ and $q \in V$. So , X is Hausdorff space.

A space X is called a completely regular if for each closed set A of X and a point $p \notin A$, there is a continuous function $f: X \rightarrow [0,1]$ such that $f(p) = 0$ and $f(q) = 1$ for each $q \in A$.

It is clear in view of Ex.113 in [54] that if a space X is Hausdorff E.D.-space then it is CH-space.

It is well known that a completely regular T_1 -space is called Tychonoff space and every singleton set is closed in a T_1 -space.

Next , we prove the following.

Lemma 3.3 : Every Tychonoff space is CH- space.

Proof : Let $p, q \in X$ with $p \neq q$ and X is Tychonoff space. . As X is T_1 -space , singleton set $\{q\}$ is closed for each $q \in X$. Hence , $\{q\} = A$ (say) and $p \notin A$. Also, X is completely regular space, there is a continuous function $f: X \rightarrow [0,1]$ such that $f(p) = 0$ and $f(q) = 1$ for each $q \in A$ i.e., $f(q) = 1$. Hence , X is CH space.

Remark 3.4 :In fact , if $f: X \rightarrow Y$ is continuous and A is a subspace of X , then $f|_A : A \rightarrow Y$ is also continuous.

So, we state the following in view of. above Remark-3.4 :

Theorem 3.5 : Every subspace of a CH- space X is CH.

In 1984 , Garg & Sivaraj [20] proved that a space (X, τ) is CH iff (X, τ_α) is CH.

In a space (X, τ) , a set A is called regular open if $A = \text{Int Cl}(A)$ and regular closed if $A = \text{Cl Int}(A)$. The family of all regular open sets of X is denoted by $\text{RO}(X)$. Since the intersection of two regular open sets is regular open, so the family of regular open sets forms a base for a smaller topology τ_s on X as $\tau_s \subset \tau$, called the semi-regularization of τ . The space X is called (X, τ) is said to be semi-regular if $\tau_s = \tau$. Any regular space is semi-regular and $I = [0, 1]$ is regular. See, M.Mrsevic et al [34] and S.Psyk [44] both studied for semi-regularization topologies in the literature.

In 1985, Mrsevic et al [34] proved that (X, τ) is FH-space iff (X, τ_s) is FH.

In 1986, C.Dorsett [13] proved that the CH-space is both a semi-topological property and a r-property. Also, generalized CH-space to weakly CH-space. Thus, we have the following:

Definition 3.6 [13]: A space X is called weakly completely Hausdorff (in brief, wCH) if and only if for each pair $x, y \in X$ with $\text{Cl}\{x\} \neq \text{Cl}\{y\}$, there is a continuous function $f: X \rightarrow [0, 1]$ such that $f(\text{Cl}\{x\}) = 0$ and $f(\text{Cl}\{y\}) = 1$.

Remark 3.7 [13]: (i) Every CH-space implies wCH,

(ii) If (X, τ) is wCH-space then (X, τ_s) is wCH.

In 1961, Z.Frolik has proved the following result.

Lemma 3.8[17]: If $f: X \rightarrow Y$ is an open, closed surjection map and $\phi: X \rightarrow [0, 1]$ is a continuous map, then the map $\psi: Y \rightarrow [0, 1]$ given by $\psi(y) = \text{Sup}\{\phi(x) \mid x \in f^{-1}(y)\}$ for each $y \in Y$, is continuous. See also p.96 Ex.16 in [14] & p.142 Lemma to Th.4.4.2 in [27]

Theorem 3.9: If $f: X \rightarrow Y$ be closed, open bijective function from a CH-space X onto a space Y , then Y is CH.

Proof: Let $f: X \rightarrow Y$ be open closed bijective function and X is a CH-space. Let $p, q \in Y$ be two distinct points. Let $x, y \in X$ be two points such that $f(x) = p$ and $f(y) = q$. Then, $x \in f^{-1}(p)$ and $y \in f^{-1}(q)$ implies that $x \neq y$ since p and q are distinct points in Y . Since X is CH-space, there is a continuous function $\phi: X \rightarrow [0, 1]$ such that $\phi(x) = 0$ and $\phi(y) = 1$. Now, let $\psi: Y \rightarrow [0, 1]$ be a function defined by $\psi(p) = \text{Sup}\{\phi(x) \mid x \in f^{-1}(p)\}$ for every $p \in Y$. Then, ψ is continuous function by Lemma-3.8. Clearly, $\psi(p) = 0$, since $\phi(x) = 0$ & $x \in f^{-1}(p)$ and as $f(y) = q$ and $\phi(y) = 1$ hence $\psi(q) = \text{Sup}\{\phi(y) \mid y \in f^{-1}(q)\}$ for every $q \in Y$, implies that $\psi(q) = 1$. Therefore, Y is CH.

IV. C α H-SPACES

We define the following.

Definition 4.1: A space X is called α -Hausdorff if for any two distinct points x, y of X , there exist disjoint α -open sets U and V such that $x \in U$ and $y \in V$.

Definition 4.2: A space X is called completely α -Hausdorff (in brief, C α H) if for every pair of distinct points x and y of X , there exists a α -continuous function $f: X \rightarrow [0, 1]$ such that $f(x) = 0$ and $f(y) = 1$.

Clearly, every CH-space is C α H.

Lemma 4.3: Every C α H-space is α -Hausdorff space.

Proof: Let $p, q \in X$ with $p \neq q$. Since X is C α H-space, then there is a α -continuous function $f: X \rightarrow [0, 1]$ such that $f(p) = 0$ and $f(q) = 1$.

As $f^{-1}([0, \frac{1}{2})) = U$ and $f^{-1}((\frac{1}{2}, 1]) = V$ are α -open sets in X such that $p \in U$ and $q \in V$. So, X is α -Hausdorff space.

In view of Lemma-2.18 above, we state the following.

Theorem 4.3: A space (X, τ_α) is C α H iff $(X, \tau_{\alpha\alpha})$ is C α H.

We, define the following analogous to Def. -3.6 above.

Definition 4.4: A space X is called weakly completely α -Hausdorff (in brief, wC α H) if only if for each pair $x, y \in X$ with $\text{Cl}\{x\} \neq \text{Cl}\{y\}$, there is a α -continuous function $f: X \rightarrow [0, 1]$ such that $f(\text{Cl}\{x\}) = 0$ and $f(\text{Cl}\{y\}) = 1$.

Clearly, every wCH -space is $wC\alpha H$ and every $C\alpha H$ -space is $wC\alpha H$ -space.

Theorem 4.5 : Every preopen subspace of a $C\alpha H$ -space X is $C\alpha H$.

We define the following.

Definition 4.6 : A space X is called α -Tychonoff if it is α -completely regular T_1 -space.

Lemma 4.7 : Every α -Tychonoff space is $C\alpha H$ -space.

Proof : Let $p, q \in X$ with $p \neq q$ and X is α -Tychonoff space. As X is T_1 -space, singleton set $\{q\}$ is closed for each $q \in X$. Hence, $\{q\} = A$ (say) and $p \notin A$. Also, X is α -completely regular space, there is a α -continuous function $f : X \rightarrow [0,1]$ such that $f(p) = 0$ and $f(q) = 1$ for each $q \in A$ i.e., $f(q) = 1$. Hence, X is $C\alpha H$ space.

The following three results were proved in [40].

Lemma 4.8 : A function $f : X \rightarrow Y$ from a space X into a space Y is α -continuous iff the inverse image of each member of a basis for the topology of Y is α -open in X .

Lemma 4.9 : Let X be an α -space. Then a function $f : X \rightarrow Y$ is α -continuous iff the inverse image of each member of subbasis for the topology of Y is α -open in X .

Lemma 4.10 : If $f : X \rightarrow Y$ be a pre- α -open, pre- α -closed surjection from a space X onto an α -space Y . Let $\phi : X \rightarrow [0,1]$ is a α -continuous function. If $\psi : Y \rightarrow [0,1]$ is a function defined by $\psi(y) = \text{Sup}\{\phi(x) \mid x \in f^{-1}(y)\}$ for each $y \in Y$, then ψ is α -continuous.

In view of Lemma-4.10 above, we prove the following.

Theorem 4.11 : If $f : X \rightarrow Y$ be pre- α -closed, pre- α -open bijective function from a $C\alpha H$ -space X onto an α -space Y , then Y is $C\alpha H$.

Proof : Let $f : X \rightarrow Y$ be pre- α -closed, pre- α -open bijective function and X is a $C\alpha H$ -space. Let $p, q \in Y$ be two distinct points. Let $x, y \in X$ be two points such that $f(x) = p$ and $f(y) = q$. Then, $x \in f^{-1}(p)$ and $y \in f^{-1}(q)$ implies that $x \neq y$ since p and q are distinct points in Y . Since X is $C\alpha H$ -space, there is a α -continuous function $\phi : X \rightarrow [0,1]$ such that $\phi(x) = 0$ and $\phi(y) = 1$. Now, let $\psi : Y \rightarrow [0,1]$ be a function defined by $\psi(p) = \text{Sup}\{\phi(x) \mid x \in f^{-1}(p)\}$ for every $p \in Y$. Then, ψ is α -continuous function by Lemma-4.10. Clearly, $\psi(p) = 0$, since $\phi(x) = 0$ & $x \in f^{-1}(p)$ and as $f(y) = q$ and $\phi(y) = 1$ hence $\psi(q) = \text{Sup}\{\phi(y) \mid y \in f^{-1}(q)\}$ for every $q \in Y$, implies that $\psi(q) = 1$. Therefore, Y is $C\alpha H$.

V. CsH - SPACES

We define the following.

Definition 5.1: A space X is called s -Hausdorff if for any two distinct points x, y of X , there exist disjoint s -open sets U and V such that $x \in U$ and $y \in V$.

In view of Lemma -2.16 (i) one can prove the following.

Theorem 5.2 : (X, τ) is s -Hausdorff space if and only if (X, τ_α) is s -Hausdorff.

We define the following.

Definition 5.3 : A space X is called completely s -Hausdorff (in brief, CsH) if for every pair of distinct points x and y of X , there exists a semicontinuous function $f : X \rightarrow [0,1]$ such that $f(x) = 0$ and $f(y) = 1$.

Clearly, every CH -space is CsH .

Lemma 5.4 : Every CsH -space is s -Hausdorff space.

Proof : Let $p, q \in X$ with $p \neq q$. Since X is CsH -space, then there is a semicontinuous function $f : X \rightarrow [0,1]$ such that $f(p) = 0$ and $f(q) = 1$.

As $f^{-1}([0, \frac{1}{2})) = U$ and $f^{-1}((\frac{1}{2}, 1]) = V$ are semiopen sets in X such that $p \in U$ and $q \in V$. So, X is s -Hausdorff space.

We define the following.

Definition 5.4 : A space X is called s -Tychonoff if it is s -completely regular T_1 -space.

Lemma 5.5 : Every s -Tychonoff space is CsH- space.

Proof : Let $p, q \in X$ with $p \neq q$ and X is s -Tychonoff space. As X is T_1 -space, singleton set $\{q\}$ is closed for each $q \in X$. Hence, $\{q\} = A$ (say) and $p \notin A$. Also, X is s -completely regular space, there is a semicontinuous function $f : X \rightarrow [0, 1]$ such that $f(p) = 0$ and $f(q) = 1$ for each $q \in A$ i.e., $f(q) = 1$. Hence, X is CsH space.

We, define the following analogous to Def. -3.6 above.

Definition 5.6: A space X is called weakly completely s - Hausdorff (in brief, $wCsH$) if only if for each pair $x, y \in X$ with $Cl\{x\} \neq Cl\{y\}$, there is a semi-continuous function $f : X \rightarrow [0, 1]$ such that $f(Cl\{x\}) = 0$ and $f(Cl\{y\}) = 1$.

Clearly, every wCH - space is $wCsH$ and every CsH-space is $wCsH$ -space.

Theorem 5.7 : Every open subspace of a CsH- space X is CsH.

Proof : Let X be a CsH- space and Y be an open subspace of X . Let p and q be two distinct points of Y . Since $Y \subset X$, so $p, q \in X$ with $p \neq q$. As X is CsH- space, there exists a semicontinuous function $f : X \rightarrow [0, 1]$ with $f(p) = 0$ and $f(q) = 1$. Since the restriction of a semicontinuous function to an open subspace is semicontinuous by Th.3 of [42], so it follows that the function $f/Y : Y \rightarrow [0, 1]$ is semicontinuous such that $(f/Y)(p) = 0$ and $(f/Y)(q) = 1$. Hence, Y is completely s -Hausdorff space.

The following three results were proved in [28].

Lemma 5.8 : A function $f : X \rightarrow Y$ from a space X into a space Y is semicontinuous iff the inverse image of each member of a basis for the topology of Y is α -open in X .

Lemma 5.9 : Let X be an E.D.-space. Then a function $f : X \rightarrow Y$ is semicontinuous iff the inverse image of each member of subbasis for the topology of Y is semi-open in X .

Lemma 5.10 : If $f : X \rightarrow Y$ be a presemiopen and presemiclosed function from a space X onto an E.D. space Y . Let $\phi : X \rightarrow [0, 1]$ be a semicontinuous function. If $\psi : Y \rightarrow [0, 1]$ be a function defined by $\psi(y) = \text{Sup}\{\phi(x) \mid x \in f^{-1}(y)\}$ for each $y \in Y$, then ψ is semicontinuous.

We, state the following in view of Lemma-5.10 above.

Theorem 5.11 : If $f : X \rightarrow Y$ be presemiclosed presemiopen bijective function from a CsH- space X onto an E.D. space Y , then Y is CsH.

VI. pH - SPACES

We define the following.

Definition 6.1 : A space X is called p -Hausdorff if for any two distinct points x, y of X , there exist disjoint p -open sets U and V such that $x \in U$ and $y \in V$.

In view of Lemma -2.16(ii), one can prove the following.

Theorem 6.2 : (X, τ) is p -Hausdorff space if and only if (X, τ_α) is p -Hausdorff.

We define the following.

Definition 6.3 : A space X is called completely p -Hausdorff (in brief, CpH) if for every pair of distinct points x and y of X , there exists a precontinuous function $f : X \rightarrow [0, 1]$ such that $f(x) = 0$ and $f(y) = 1$.

Clearly, every CH-space is CpH .

Lemma 6.4 : Every CpH-space is p-Hausdorff space.

Proof is on similar lines of Lemma-5.4

We recall the following.

Lemma 6.4 [T.Noiri 1998] : If Y is an open subspace of a space X and $A \in PO(X)$, then $A \cap Y \in PO(Y)$.

We prove the following.

Lemma 6.5 : If $f : X \rightarrow Y$ is precontinuous and U is an open subspace of X , then the restriction $f/U : U \rightarrow Y$ is precontinuous.

Proof : Let $V \subset Y$ be an open set, so $f^{-1}(V) \in PO(X)$. Since U is an open subspace of X , we have by Lemma-6.4 , $f^{-1}(V) \cap U = (f/U)^{-1}(V) \in PO(U)$. Therefore , $f/U : U \rightarrow Y$ is precontinuous.

Theorem 6.6 : Every open subspace of a CpH- space X is CpH.

Proof : Let X be a CpH-space and U be an open subspace of X . Let p and q be two distinct points of U . Since $U \subset X$, so $p, q \in X$ with $p \neq q$. As X is CpH-space , there exists a precontinuous function $f : X \rightarrow [0,1]$ with $f(p) = 1$ and $f(q) = 0$. Since the restriction of a precontinuous function to an open subspace is precontinuous by Lemma-6.5 above , so it follows that the function $f/U : U \rightarrow [0,1]$ is precontinuous such that $(f/U)(p) = 1$ and $(f/U)(q) = 0$. Hence , U is CpH-space.

We recall the following.

Lemma 6.7 [30] : If $V \in PO(X)$ and $U \in SO(X)$, then $U \cap V \in PO(U)$.

In view of Th.2.1 of [30], one can prove the following.

Theorem 6.8 : Every semiopen subspace of a CpH-space is CpH.

We define the following.

Definition 6.9 : A space X is called p-Tychonoff if it is p-completely regular T_1 -space.

Lemma 6.10 : Every p-Tychonoff space is CpH- space.

Proof : Let $p, q \in X$ with $p \neq q$ and X is p-Tychonoff space. . As X is T_1 -space , singleton set $\{q\}$ is closed set for each $q \in X$. Hence , $\{q\} = A$ (say) and $p \notin A$. Also, X is p-completely regular space, there is a precontinuous function $f : X \rightarrow [0,1]$ such that $f(p) = 0$ and $f(q) = 1$ for each $q \in A$ i.e., $f(q) = 1$. Hence , X is CpH space.

We , define the following analogous to Def. -3.6 above.

Definition 6.11 : A space X is called weakly completely p- Hausdorff (in brief, wCpH) if only if for each pair $x, y \in X$ with $Cl\{x\} \neq Cl\{y\}$, there is a pre-continuous function $f : X \rightarrow [0,1]$ such that $f(Cl\{x\}) = 0$ and $f(Cl\{y\}) = 1$.

Clearly, every wCH- space is wCpH and every CpH-space is wCpH-space.

The following three results were proved in [29].

Lemma 6.12 : A function $f : X \rightarrow Y$ from a space X into a space Y is precontinuous iff the inverse image of each member of a basis for the topology of Y is preopen in X .

Lemma 6.13: Let X be an submaximal.-space .Then a function $f : X \rightarrow Y$ is precontinuous iff the inverse image of each member of subbasis for the topology of Y is preopen in X .

Lemma 6.14 : Let $f: X \rightarrow Y$ be an M -preopen and M -preclosed surjective function from a space X onto an submaximal space Y . Let $\phi: X \rightarrow [0,1]$ be a pre-continuous function. If $\psi: Y \rightarrow [0,1]$ is a function defined by $\psi(y) = \text{Sup}\{\phi(x) \mid x \in f^{-1}(y)\}$ for each $y \in Y$, then ψ is precontinuous.

We, state the following in view of Lemma-6.14 above.

Theorem 6.15 : If $f: X \rightarrow Y$ be M -preclosed, M -preopen bijective function from a C_pH -space X onto an submaximal space Y , then Y is C_pH .

VII. $C\beta H$ - SPACES

We define the following.

Definition 7.1 : A space X is called β -Hausdorff if for any two distinct points x, y of X , there exist disjoint β -open sets U and V such that $x \in U$ and $y \in V$.

In view of Lemma – 2.17, one can prove the following.

Theorem 7.2 : (X, τ) is β -Hausdorff space if and only if (X, τ_α) is β -Hausdorff.

Definition 7.3 : A space X is called completely β -Hausdorff (in brief, $C\beta H$) if for every pair of distinct points x and y of X , there exists a β -continuous function $f: X \rightarrow [0,1]$ such that $f(x) = 0$ and $f(y) = 1$.

Clearly, every CH -space is $C\beta H$.

Lemma 7.4 : Every $C\beta H$ -space is β -Hausdorff space.

Proof is on similar lines of Lemma-5.4

Theorem 7.4 : Every α -open subspace of a $C\beta H$ - space X is $C\beta H$.

We define the following.

Definition 7.5 : A space X is called β -Tychonoff if it is β -completely regular T_1 -space.

Lemma 7.6: Every β -Tychonoff space is $C\beta H$ - space.

Proof : Let $p, q \in X$ with $p \neq q$ and X is β -Tychonoff space. As X is T_1 -space, singleton set $\{q\}$ is closed set for each $q \in X$. Hence, $\{q\} = A$ (say) and $p \notin A$. Also, X is β -completely regular space, there is a β -continuous function $f: X \rightarrow [0,1]$ such that $f(p) = 0$ and $f(q) = 1$ for each $q \in A$ i.e., $f(q) = 1$. Hence, X is $C\beta H$ space.

We, define the following analogous to Def. -3.6 above.

Definition 7.7 : A space X is called weakly completely β - Hausdorff (in brief, $wC\beta H$) if for each pair $x, y \in X$ with $Cl\{x\} \neq Cl\{y\}$, there is a β -continuous function $f: X \rightarrow [0,1]$ such that $f(Cl\{x\}) = 0$ and $f(Cl\{y\}) = 1$.

Clearly, every wCH - space is $wC\beta H$ and every $C\beta H$ -space is $wC\beta H$ -space.

The following three results were proved in [39].

Lemma 7.8 : A function $f: X \rightarrow Y$ from a space X into a space Y is β -continuous iff the inverse image of each member of a basis for the topology of Y is β -open in X .

Lemma 7.9 : Let X be an submaximal & $E.D.$ -space. Then a function $f: X \rightarrow Y$ is β -continuous iff the inverse image of each member of subbasis for the topology of Y is β -open in X .



Lemma 7.10 : Let $f: X \rightarrow Y$ be an pre- β -open and pre- β -closed surjective function from a space X onto an submaximal & E.D.- space Y . Let $\phi : X \rightarrow [0,1]$ be a β -continuous function .If $\psi : Y \rightarrow [0,1]$ is a function defined by $\psi(y) = \text{Sup}\{\phi(x) \mid x \in f^{-1}(y)\}$ for each $y \in Y$, then ψ is β -continuous .

We , state the following in view of Lemma-7.10 above .

Theorem 7.11 : If $f : X \rightarrow Y$ be pre- β -closed, pre- β -open bijective function from a $C\beta H$ -space X onto an submaximal & E.D.-space Y , then Y is $C\beta H$.

VIII. CONCLUSIONS

Remark 8.1 : In view of Def. 2.3 (v), we have

CH -spaces & $C\alpha H$ -spaces are coincide each other in an α -space .

Remark 8.2 : In view Def. 2.3 ((ii) , we have :

CH - spaces & CpH -spaces are coincide each other in an submaximal space,

Remark 8.3 : In view of Lemma -2.4 above , we have :

CH -spaces and $C\beta H$ -spaces are coincide each other in an E.D and submaximal space. ,

Remark 8.4 : Since every α -open set \rightarrow preopen set $\rightarrow \beta$ -open set , so we have :

$C\alpha H$ - space $\rightarrow CpH$ -space $\rightarrow C\beta H$ -space.

Remark 8.5 : Since every α -open set \rightarrow semiopen set $\rightarrow \beta$ -open set , so we have :

$C\alpha H$ - space $\rightarrow CsH$ -space $\rightarrow C\beta H$ -space.

Remark 8.6 : In view of Lemma -2.13 above, we have :

(i) Every CsH -space implies CpH .

(ii) Every CsH -space implies $C\alpha H$.

(iii) Every $C\beta H$ -space implies CpH .

Remark 8.7 : In view of Lemma-2.14 above , we have :

(i) Every CpH -space implies CsH .

(ii) Every $C\beta H$ -space implies CsH .

(iii) Every CpH -space implies $C\alpha H$.

Remark 8.8 : In view of above Lemma – 2.15 , we have : CH - space X , $C\alpha H$ - space X , CsH - space X , CpH - space X and $C\beta H$ - space X coincide each other.

Finally, we state that there is a lot of scope to work on each of the concepts like - $wC\alpha H$ -spaces, $wCsH$ -spaces , $wCpH$ -spaces and $wC\beta H$ -spaces.

REFERENCES

- [1]. M.E.Abd El-Monsef, S.N.El-Deeb and R.A.Mahmoud , β -open sets and β -continuous mappings , Bull.Fac.Sci.Assiut Univ., 12(1983), 77-90.
- [2]. M.E.Abd El-Monsef, R.A.Mahmoud and E.R.Lashin , β -closure and β - interior, Journal Fac.Ed.Ain Shams Univ.,10(1986),235-245.
- [3]. M.E.Abd El-Monsef and A.M.Kozae , On extremally disconnectedness ,ro-equivalence and properties of some maximal topologies , Proc.4th Conference Oper. Res.Math. Method., (Alex Univ. 1988) .
- [4]. T.Aho and T.Nieminen, Spaces in which preopen sets are semiopen , Ricerche Mat., 43,no.1(1994) , 45-59.
- [5]. D.Andrijevic , Some properties of the topology of α -sets, Mat.Vesnik, 36(1984),1-10.
- [6]. D.Andrijevic , Semi-preopen sets , Mat.Vesnik, 38(1986), 24-32.
- [7]. D.Andrijevic,A note on α -equivalent topologies, Mat.Vesnik, 43(1993),65-69.
- [8]. N.Biswas , On Characterizations of semicontinuous functions, Atti.Accad.Naz.Lincei Rend. Cl.Sci.Fis.Mat.Natur. (8) 48 (1979), 399-400.
- [9]. N.Bourbaki,General Topology , Addison-Wesley,Mass (1966).
- [10]. S.G.Crossley and S.K.Hildeband , Semitopological properties, Fund. Math., 74(1972),233-254.
- [11]. R.Devi , K.Balachandran and H.Maki, Generalized α -closed maps and α -generalized closed maps ,Indian J.Pure Appl.Math., 29(1) (Jan-1998),37-49.
- [12]. A.Dehghani and M.Karavan , Every locally connected functionally Hausdorff space is c-resolvable, Topology and its Applications, 183 (2015), 86-89.

- [13]. Charles Dorsett , Semitopological properties , Completely Hausdorff spaces and Generalized completely Hausdorff spaces, Bull.Malaysian Math.Soc. (9) 2 (1986) , 71-80.
- [14]. James Dugundji, Topology , Allyn & Becon (1966).
- [15]. S.N.El-Deeb , I.A.Hasanein , A.S.Mashhour and T.Noiri , On p-regular spaces, Bull. Math. R.S.Roumanie , N.S. , 27(75) (1983) , pp. 311-315.
- [16]. Giovanni Lo Faro, On strongly α -irresolute mappings, Indian J.Pure Appl.Math.,18(1) (Feb-1987),146-151.
- [17]. Z.Frolík, Applications of complete families of continuous functions to the theory of Q-spaces, Czech Math.J., 11(1961),115-123.
- [18]. M.Ganster and D.Andrijevic, Some Questions concerning semireopen sets, Jour.Inst.Math .& Comp.Sci., (Math.Ser.)Vol.1 ,No.2 (1988),65-75.
- [19]. G.L.Garg and D.Shivaraj , Presemiclosed mappings, Periodica Math.Hungar.,19(2),1988, 99-106.
- [20]. G.L.Garg and D.Sivaraj , Semitopological properties, Mat.Vesnik , 36(1984), 137-142.
- [21]. R.A.Herrmann,Point Monads and P-closed spaces, Notre Dame Journal of Formal Logic,Vol.XX , No.2 , (April 1979), 395- 400.
- [22]. D.S.Jankovic,On locally irreducible spaces, Ann.Soc. Sci.Bruxelles Ser.I (97) (1983) , 36-41.
- [23]. D.S.Jankovic, A note on mappings of Extremely Disconnected space, Acta Math. Hungar., 46(1-2)(1985), 83-92.
- [24]. Sami Lazaar , On Functionally Hausdorff spaces, Missouri Jour. of Mathematics, Vol.25, No.1 (2013) ,88-97.
- [25]. N.Levine , Semiopen sets and semicontinuity in topological spaces, Amer. Math, Monthly, 70(1963) ,36-41.
- [26]. R.A.Mahmoud and M.E.Abd El-Monsef, β -irresolute and β - topological invariant, Proc.Pakistan Acad. Sci., 27(3)(1990), 285—296.
- [27]. S.R.Malghan ,General Topology , Serials Publications , New Delhi (2013)(India).
- [28]. S.R.Malghan , S.S.Benchalli and Govindappa Navalagi , s-completely regular spaces, Journal Tripura Math.Soc., 6(2004) , pp.67-74.
- [29]. S.R.Malghan and G.B.Navalagi, Almost p-regular , p-completely regular and almost p-completely regular spaces, Bull. Math.R.S.Roumanie N.S., 32(82) , No. 4 (1990), pp. 317-326.
- [30]. A.S.Mashhour et al , A note on semicontinuity and precontinuity, Indian J. pure appl. Math., 13(10), Oct- 1982, 1119-1123.
- [31]. A.S.Mashhour , M.E.Abd El-Monsef and S.N.El-Deeb , On precontinuous and weak precontinuous mappings, Proc. Math. Phys. Soc. Egypt , 53(1982), pp. 42-52.
- [32]. A.S.Mashhour , I.A.Hasanein and S.N.El-Deeb ,On α -continuous and α -open mapping, Acta Math.Hungar., 41(1983),213-218.
- [33]. A.S.Mashhour , M.E.Abd El-Monsef , I.A.Hasanein and T.Noiri , Strongly compact spaces, Delta J. Sci., 8 (1984) , 30-46.
- [34]. M.Mrsevic , I.L.Reilly and M.K.Vamanamurthy, On semi-regularization topologies, J.Austra.Math.Soc. (Series A), 38 (1985) , 40-54.
- [35]. A.A.Nasef and T.Noiri , Strong forms of faint continuity, Mem.Fac.Sci. Kochi Univ. Ser.A Math.,19 (1998),21-28.
- [36]. Govindappa Navalagi , β -zero sets and their properties in topology , IJERD , Vol.16,Issue 9(Sept-2020),20-27 .
- [37]. Govindappa Navalagi, α -zero sets and their properties in topology, World Jour. of Engineering Research and Technology, Vol.6 ,Issue 5 (Sept. 2020) ,
- [38]. Govindappa Navalagi , Pre-zero sets and their properties , International Journal of General Topology , Vol.3 ,Nos.1-2 (Jan-Dec. - 2010),37-43.
- [39]. Govindappa Navalagi, A note on β -completely regular spaces in topology (Submitted).
- [40]. Govindappa Navalagi, A note on α -completely regular spaces in topology (Submitted).
- [41]. O.Njastad , On some classes of Nearly open set, Pacific J.Math., 15 (1965) ,961-970.
- [42]. T.Noiri , On semicontinuous mappings, Atti. Accad. Naz. Del Linci.Rendiconti , LIV(1973),210-214.
- [43]. T.Noiri , Almost p-regular spaces and some functions, Acta Math. Hungar., 79(3) (1998),207-216.
- [44]. S.Psyk , Quasi-continuity with respect to semiregularization topology, Bull.Austr.Math.Soc., 37(1988), 423-428.
- [45]. Asit Baran Raha , On completely Hausdorff –completion of a completely Hausdorff space, Pacific J.of Mathematics, Vol.38,No.1 (1971).
- [46]. I.L.Reilly and M.K.Vamanamurthy, On some Questions concerning preopen sets,Kyungpook Math.J., 30(1990),87-93.
- [47]. I.L.Reilly and M.K.Vamanamurthy, Connectedness and strong semicontinuity, Casopis Pro Pestovani,Matematematik ,roc ,109(1984), Praha 261-265.
- [48]. M.K.Singal and S.P.Arya , Almost normal and almost completely regular spaces,Glasnik Mat., 5(25) (1970), 141-152.



- [49]. M.K.Singal and Asha Mathur, On nearly compact spaces, Boll.Un.Mat.Ital.,(4) 2 (1969), 702-710.
- [50]. R.M.Stephenson Jr., Spaces for which Stone-Weierstrass Theorem holds, T.A.M.S., 133(1968),537-546.
- [51]. R.M.Stephenson Jr., Minimal First Countable Hausdorff spaces , Pacific J. of Mathematics, Vol.36, No.3 (1971), 819-825.
- [52]. M.H.Stone ,Appl. Of theory of Boolean rings to general topological spaces,T.A.M.S., 41(1937), 375-381.
- [53]. N.V.Velicko, H-closed spaces, Mat.Sb., 70(112)(1966),98-121.
- [54]. A.Wilasky , Topology for Analysis, Ginn & Company USA (1970).
- [55]. S.Willard , General Topology , Addison-Wesley,Mass(1970).



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