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Calculus of variations and It's Applications

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ABSTRACT:The purpose of this paper is to introduce a powerful technique for the solution of problems of applied mathematics the calculation of variation. We look at how optimization problems are resolved with the Euler-Lagrange equation. Functions that content this equation and the prescribed boundary conditions, need also fulfill Legendre's condition. At last we explore applications of the multi-dimensional Euler-Lagrange equations.

KEYWORDS: optimization, functional, Euler-Lagrange equation. Legendre's equation Introduction.

I. INTRODUCTION

The calculus of variations was founded in the eighteenth century by L. Euler and J.L. Lagrange and extensively developed by the other mathematicians in the nineteenth century. At present, it is one of the greatest significant branches of applied mathematics and is a powerful technique considering the clarification of problems in dynamics and statics of rigid bodies, the concept of plates and shells, vibration problems, optimization of orbits etc. The subject is primarily concerned with the study of certain optimisation problems which cannot be treated by the techniques of elementary calculus.

II. FUNCTIONAL

Look at the problem of conclusion a curve through two points

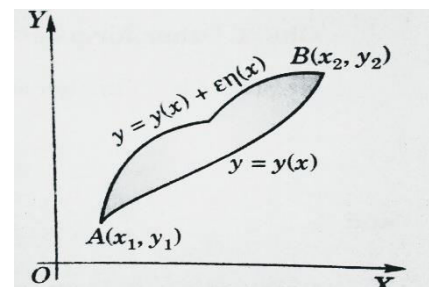
(x_1, y_1) and (x_2, y_2) whose length is a minimum. It is same as determining the curve $y = y(x)$ for which $y(x_1) = y_1, y(x_2) = y_2$

such that $\int_{x_1}^{x_2} \sqrt{(1 + y'^2)} dx$ is a minimum. In general terms, $y = y(x)$

where $y(x_1) = y_1$ and $y(x_2) = y_2$ such that for a given function

$$F(x, y, y'),$$

$$\int_{x_1}^{x_2} F(x, y, y') dx \text{ is a stationery value or an extremum} \quad (1)$$



An integral such as (1), which assumes a definite value for functions of the type $y = y(x)$ is called a functional.

In diff. calculus, we deal with the problems of optimization of functions. The calculus of variations is however, concerned with optimizing functional.



III. EULER-LAGRANGE EQUATION

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) = 0$$

The indicated is the essential condition for $I = \int_{x_1}^{x_2} F(x, y, y') dx$ to be maximum or minimum.

Proof: Let $y=y(x)$ be the curve AB which makes the given function is an extremum.

Consider a neighbouring curve

$$Y = y(x) + \beta \cdot \eta(x) \dots\dots\dots(1)$$

β is a parameter and $\eta(x)$ Is An Uninformed Differentiable Function.

Closing the point A and B , $\eta(x_1) = \eta(x_2) = 0$

When $\beta = 0$ neighbouring, curves become $y = y(x)$, which is extremal. The family of neighbouring curves is called the family of comparison functions.

If in the functional $\int_{x_1}^{x_2} F(x, y, y') dx$ we replace y by Y, we get

$$I(\beta) = \int_{x_1}^{x_2} F(x, Y, Y') dx = \int_{x_1}^{x_2} F(x, y(x) + \beta \cdot \eta(x), y'(x) + \beta \cdot \eta'(x)) dx$$

Which is a function of β , say $I(\beta)$

$$\therefore I(\beta) = \int_{x_1}^{x_2} F(x, Y, Y') dx$$

For $\beta = 0$, the neighbouring curves become the extremal, an extremum for $\beta = 0$.

The essential condition for this is $I'(\beta) = 0 \dots\dots\dots(2)$

Differentiating I under the integral sign by Leibnitz's rule, we have

$$I'(\beta) = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial x} \cdot \frac{\partial x}{\partial \beta} + \frac{\partial F}{\partial Y} \cdot \frac{\partial Y}{\partial \beta} + \frac{\partial F}{\partial Y'} \cdot \frac{\partial Y'}{\partial \beta} \right) dx$$

$$I'(\beta) = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial Y} \cdot \frac{\partial Y}{\partial \beta} + \frac{\partial F}{\partial Y'} \cdot \frac{\partial Y'}{\partial \beta} \right) dx \left(\frac{\partial x}{\partial \beta} = 0 \right)$$

as β is not dependent of x.-----(3)

On differentiating (1), w.r.to. 'x', we get $Y' = y'(x) + \beta \eta'(x)$

Again, differentiating w.r.to β , we get $\frac{\partial Y'}{\partial \beta} = \eta'(x)$

Differentiating (1), w.r.to β , we get $\frac{\partial Y}{\partial \beta} = \eta(x)$

Now (3) becomes $I'(\beta) = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial Y} \cdot \eta(x) + \frac{\partial F}{\partial Y'} \cdot \eta'(x) \right) dx$

Integrating the second term on the right by parts, we get

$$= \int_{x_1}^{x_2} \frac{\partial F}{\partial Y} \cdot \eta(x) dx + \left[\left\{ \frac{\partial F}{\partial Y'} \cdot \eta(x) \right\}_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx} \left(\frac{\partial F}{\partial Y'} \right) \eta(x) dx \right]$$



$$\begin{aligned}
 &= \int_{x_1}^{x_2} \frac{\partial F}{\partial Y} \cdot \eta(x) dx + \left[\left\{ \frac{\partial F}{\partial Y'} \cdot \eta(x_2) - \frac{\partial F}{\partial Y'} \cdot \eta(x_1) \right\} \right] - \int_{x_1}^{x_2} \frac{d}{dx} \left(\frac{\partial F}{\partial Y'} \right) \eta(x) dx \\
 &= \int_{x_1}^{x_2} \frac{\partial F}{\partial Y} \cdot \eta(x) dx + 0 - \int_{x_1}^{x_2} \frac{d}{dx} \left(\frac{\partial F}{\partial Y'} \right) \eta(x) dx \quad [\because \eta(x_1) = \eta(x_2) = 0] \\
 &= \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial Y} - \frac{d}{dx} \left(\frac{\partial F}{\partial Y'} \right) \right] \eta(x) dx
 \end{aligned}$$

$\eta(x)$ is an arbitrary continuous function.

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) = 0 \quad \text{which is a required Euler's equation.}$$

Note: Other forms of Euler's equation

$$1. \frac{d}{dx} F(x, y, y') = \frac{\partial F}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dx} + \frac{\partial F}{\partial y'} \cdot \frac{dy'}{dx}$$

$$\text{OR } \frac{dF}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} y' + \frac{\partial F}{\partial y'} y'' \quad \text{-----(4)}$$

$$\text{But } \frac{d}{dx} \left(y' \frac{\partial F}{\partial y'} \right) = y' \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) + \frac{\partial F}{\partial y'} \cdot y'' \quad \text{-----(5)}$$

On subtracting (5) from (4)

$$\begin{aligned}
 \frac{dF}{dx} - \frac{d}{dx} \left(y' \frac{\partial F}{\partial y'} \right) &= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} y' - y' \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) \\
 \frac{d}{dx} \left[F - y' \frac{\partial F}{\partial y'} \right] - \frac{\partial F}{\partial x} &= y' \left[\frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) \right] = (y')(0) = 0
 \end{aligned}$$

$$\text{Hence } \frac{d}{dx} \left[F - y' \frac{\partial F}{\partial y'} \right] - \frac{\partial F}{\partial x} = 0$$

which is another form of Euler's equation.

2. We know that $\frac{\partial F}{\partial y}$ is also a function of x, y, y' say $\phi(x, y, y')$.

$$\begin{aligned}
 \frac{d}{dx} \left(\frac{\partial F}{\partial y} \right) &= \frac{\partial \phi}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial \phi}{\partial y} \cdot \frac{dy}{dx} + \frac{\partial \phi}{\partial y'} \cdot \frac{dy'}{dx} = \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} y' + \frac{\partial \phi}{\partial y'} y'' \\
 \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial y} \right) + \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial y} \right) y' + \frac{\partial}{\partial y'} \left(\frac{\partial F}{\partial y} \right) y'' &= \frac{\partial^2 F}{\partial x \partial y} + y' \frac{\partial^2 F}{\partial y \partial y'} + y'' \frac{\partial^2 F}{\partial y'^2}
 \end{aligned}$$

Putting the value of $\frac{d}{dx} \left(\frac{\partial F}{\partial y} \right)$ in Euler's equation, we get

$$\frac{\partial F}{\partial y} - \frac{\partial^2 F}{\partial x \partial y} - y' \frac{\partial^2 F}{\partial y \partial y'} - y'' \frac{\partial^2 F}{\partial y'^2} = 0$$

This is the third form of Euler's equation.

Extremal:

Any function which satisfies the Euler's equation is known Extremal. It is obtained by solving the Euler's equation.

Case-I: When F does not contain x explicitly i.e., $\frac{\partial F}{\partial x} = 0$

On substituting the value of $\frac{\partial F}{\partial x} = 0$ in (6), we have $\frac{d}{dx} \left[F - y' \frac{\partial F}{\partial y'} \right] = 0$

Integrating, we get $F - y' \frac{\partial F}{\partial y'} = \text{constant}$

Case -II: When F does not contain y explicitly i.e., $\frac{\partial F}{\partial y} = 0$

Putting the value of $\frac{\partial F}{\partial y}$ in Euler's equation, we get

$\frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) = 0$, Integrating we get $\frac{\partial F}{\partial y'} = \text{constant}$

Case-III: If F does not contain y' explicitly i.e. $\frac{\partial F}{\partial y'} = 0$, on substituting the value of $\frac{\partial F}{\partial y'}$ in Euler's equation, we get

$$\frac{\partial F}{\partial y} = 0$$

Case-IV: If F does not contain x and y explicitly,

We have $\frac{\partial F}{\partial x} = 0$ and $\frac{\partial F}{\partial y} = 0$ or $\frac{\partial^2 F}{\partial x \partial y'} = 0$ and $\frac{\partial^2 F}{\partial y \partial y'} = 0$

Putting these value in Euler's equation (7), we have $y'' \frac{\partial^2 F}{\partial y'^2} = 0$

If $\frac{\partial^2 F}{\partial y'^2} \neq 0$ then $y'' = 0$ whose solution is $y = ax + b$.

IV. APPLICATION OF THE EULERS- LAGRANGES EQUATION

1. (Minimum Surface Area Problem):

The curve passing through the points (x_1, y_1) and (x_2, y_2) which when rotated about the x-axis gives a minimum surface area.

Solution. We know that the surface area is given by

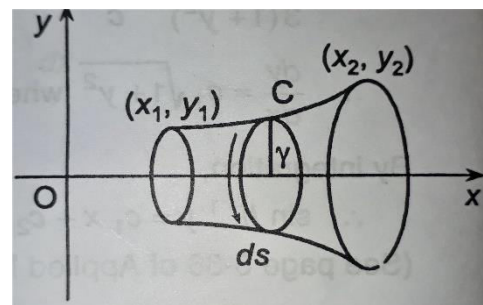
$$s = \int_{x_1}^{x_2} 2\pi y ds = 2\pi \int_{x_1}^{x_2} y \sqrt{1 + y'^2} .dx .\text{This has to be minimum}$$

We have $F = y\sqrt{1 + y'^2}$

Since F is independent of x, therefore,

Euler's equation reduces to $F - y' \frac{\partial F}{\partial y'} = c$

$$y\sqrt{1 + y'^2} - y' \frac{\partial}{\partial y'} \left\{ y\sqrt{1 + y'^2} \right\} = c$$



$$\text{i.e. } y\sqrt{1+y'^2} - y\left\{\frac{y}{2}(1+y'^2)^{-1/2} \cdot 2y'\right\} = c \therefore \frac{y}{\sqrt{1+y'^2}} = c$$

$$\therefore y^2 = c^2 + c^2 y'^2 \therefore c^2 y'^2 = y^2 - c^2$$

$$y' = \sqrt{\frac{y^2 - c^2}{c^2}} \therefore \frac{dy}{\sqrt{y^2 - c^2}} = \frac{dx}{c}$$

$$\therefore \cosh^{-1}\left(\frac{y}{c}\right) = \frac{x}{c} + c' = \frac{x + c_1}{c}$$

$$\therefore y = c \cosh\left(\frac{x + c_1}{c}\right)$$

which is catenary. The constants c and c_1 are determined from the points (x_1, y_1) and (x_2, y_2) .

2.Solid Of Revolution With Least Resistance:

The shape of the solid of revolution moving in a flow of gas with least resistance if the total resistance experienced by a body is given by

$$R = 4\pi\rho v^2 \int_0^L yy'^3 dx \text{ Where } \rho \text{ is the density of gas, } v \text{ is the relative velocity of the gas with respect to the body, with}$$

boundary conditions $y(0)=0$ and $y(L)=R$

Solution: Since $F = yy'^3 \therefore \frac{\partial F}{\partial y'} = 3yy'^2$

$$\text{by, } F - y' \cdot \frac{\partial F}{\partial y'} = c$$

$$\therefore yy'^3 - y' \cdot 3yy'^2 = c \therefore -2yy'^3 = c \therefore yy'^3 = \frac{c}{-2} = c'$$

$$\therefore y' = \frac{dy}{dx} = \sqrt[3]{\frac{c'}{y}} \therefore \sqrt[3]{y} dy = \sqrt[3]{c'} dx = c_1 dx$$

By integration, we get

$$\frac{y^{4/3}}{4/3} = c_1 x + c_2 \therefore \frac{y^{4/3}}{4/3} = \frac{4}{3} c_1 x + \frac{4}{3} c_2$$

$$\therefore y^{4/3} = c_3 x + c_4 \therefore y = (c_3 x + c_4)^{3/4}$$

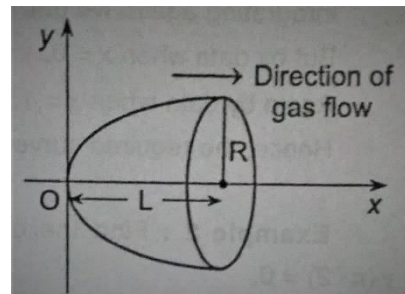
But by data $x=0, y=0 \therefore c_4 = 0 \therefore y = (c_3 x)^{3/4}$

When $x=L, Y=R, \therefore R = (c_3 L)^{3/4} \therefore \frac{R^{4/3}}{L} = c_3$

Putting this value of c_3 in $y = (c_3 x)^{3/4}$ we get

$$\therefore y = \left(\frac{R^{4/3}}{L} x\right)^{3/4} \therefore y = R \left(\frac{x}{L}\right)^{3/4}$$

This is the required relation of x and y in terms of R and L .





3. Applications of the multi-dimensional Euler-Lagrange equations.

This is the required relation of x and y in terms of R and L.

In order to find the Euler-Lagrange equations for the multi-dimensional vibrational problems, we need the basic objective functional. First we review the Euler-Lagrange equations for functional depending on different variables and derivatives.

F depending on	Euler-Lagrange equation
(x, y, y') One independent variable One dependent variable	$\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'}$
(x, y, u, u_x, u_y) Two independent variables One dependent variable	$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial u_x} \right) - \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial u_y} \right)$
$(x_1, \dots, x_m, u, u_{x_1}, \dots, u_{x_m})$ Several independent variables One dependent variable	$\frac{\partial F}{\partial u} - \sum_{i=1}^m \frac{\partial}{\partial x_i} \left(\frac{\partial F}{\partial u_{x_i}} \right)$
(x, y, y') One independent variable Several dependent variables	$\frac{\partial F}{\partial y_i} - \frac{d}{dx} \left(\frac{\partial F}{\partial y_i'} \right)$
(x, y, y', y'') Second order derivative One independent variable One dependent variable	$\frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) + \frac{d^2}{dx^2} \left(\frac{\partial F}{\partial y''} \right)$
$(x, y, y', \dots, y^{(n)})$ Higher order derivatives One independent variable One dependent variable	$\frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) + \frac{d^2}{dx^2} \left(\frac{\partial F}{\partial y''} \right) - \dots + (-1)^n \frac{d^n}{dx^n} \left(\frac{\partial F}{\partial y^{(n)}} \right)$

Problem: Show that the curve which extremizes the functional $I = \int_0^{\pi/4} (y''^2 - y^2 + x^2) dx$ under the conditions

$y(0)=0, y'(0)=1, y(\pi/4) = y'(\pi/4) = 1/\sqrt{2}$ is $y = \sin x$.

Solution: Since $F = y''^2 - y^2 + x^2$

Then we will apply here $\frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) + \frac{d^2}{dx^2} \left(\frac{\partial F}{\partial y''} \right)$

Now $\frac{\partial F}{\partial y} = -2y, \frac{\partial F}{\partial y'} = 0, \frac{\partial F}{\partial y''} = 2y''$

$\therefore -2y + \frac{d^2}{dx^2} (2y'') = 0 \therefore y^{iv} - y = 0$ or $(D^4 - 1)y = 0$

Its solution is $y(x) = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$

$y'(x) = c_1 e^x - c_2 e^{-x} - c_3 \sin x + c_4 \cos x$

Applying the boundary conditions, we get,

$0 = y(0) = c_1 + c_2 + c_3, 1 = y'(0) = c_1 - c_2 + c_4$

$\frac{1}{\sqrt{2}} = y(\pi/4) = c_1 e^{\pi/4} + c_2 e^{-\pi/4} + \frac{1}{\sqrt{2}} c_3 + \frac{1}{\sqrt{2}} c_4$



$$\frac{1}{\sqrt{2}} = y'(\pi/4) = c_1 e^{\pi/4} - c_2 e^{-\pi/4} - \frac{1}{\sqrt{2}} c_3 + \frac{1}{\sqrt{2}} c_4$$

After solving the above equation, we get $c_1 = c_2 = c_3 = 0, c_4 = 1$

Then the curve will be $y = \sin x$.

V. CONCLUSION

In this paper, we have introduced the Euler-Lagrange equation which is a powerful equation for solving optimization problems and studied its applications on minimum surface area, solid of revolution with least resistance. Also, we have determined the curve when the functional is given under the boundary conditions.

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