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# Study of Inverse Fractional Trigonometric Functions

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**ABSTRACT:** The purpose of this paper is to study inverse fractional trigonometric functions. Based on Jumarie's modified Riemann-Liouville (R-L) fractional calculus and a new multiplication of fractional analytic functions, some fractional derivative and integral formulas of inverse fractional trigonometric functions are obtained. These results we obtained are the same as those in classical calculus. The chain rule for fractional derivatives plays an important role in this study. On the other hand, the new multiplication we defined is a natural operation of fractional analytic functions.

**KEYWORDS:** Inverse fractional trigonometric functions, Jumarie's modified R-L fractional calculus, New multiplication, Fractional analytic functions, Chain rule for fractional derivatives.

## I. INTRODUCTION

Fractional calculus is a research hotspot in recent years. The application of fractional calculus in many fields such as numerical analysis, physics and engineering has aroused great interest [1-11]. In the first half of the 19th century, Abel, Liouville and Riemann correctly introduced fractional integral and derivative in the analysis. However, the use of generalized differential and integral operators became more familiar in the last decades of the 19th century because of the symbolic calculus of Heaviside and the work of mathematicians such as Hadamard, Hardy and Littlewood, M. Riesz, and H. Weyl.

In this paper, the properties of inverse fractional trigonometric functions are studied. Based on Jumarie type of modified R-L fractional calculus and a new multiplication of fractional analytic functions, we obtain several fractional derivative and fractional integral formulas of inverse fractional trigonometric functions. The main method used in this paper is the chain rule for fractional derivatives. In fact, our results are similar to those in traditional calculus. Moreover, the new multiplication is a natural operation of fractional analytic functions.

## II. DEFINITIONS AND PROPERTIES

In the following, we introduce the fractional calculus used in this article.

**Definition 2.1** ([12]): Let  $0 < \alpha \leq 1$ , and  $a$  be a real number. The Jumarie's modified Riemann-Liouville  $\alpha$ -fractional derivative is defined by

$$\left( {}_a D_{\theta}^{\alpha} \right) [f(\theta)] = \frac{1}{\Gamma(1-\alpha)} \frac{d}{d\theta} \int_a^{\theta} \frac{f(x)-f(a)}{(\theta-x)^{\alpha}} dx, \quad (1)$$

and  $\alpha$ -fractional integral is defined by

$$\left( {}_a I_{\theta}^{\alpha} \right) [f(\theta)] = \frac{1}{\Gamma(\alpha)} \int_a^{\theta} \frac{f(x)}{(\theta-x)^{1-\alpha}} dx, \quad (2)$$

where  $\Gamma(y) = \int_0^{+\infty} t^{y-1} e^{-t} dt$  is the gamma function.



**Proposition 2.2** ([13]): If  $\alpha, \beta, C$  are real numbers and  $\beta \geq \alpha > 0$ , then

$$\left( {}^0D_{\theta}^{\alpha} \right) \left[ \theta^{\beta} \right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} \theta^{\beta-\alpha}, \tag{3}$$

and

$$\left( {}^0D_{\theta}^{\alpha} \right) [C] = 0. \tag{4}$$

Next, the class of fractional analytic function is introduced.

**Definition 2.3** ([14]): If  $\theta, \theta_0$ , and  $a_k$  are real numbers for all  $k$ ,  $\theta_0 \in (a, b)$ , and  $0 < \alpha \leq 1$ . If the function  $f_{\alpha}: [a, b] \rightarrow R$  can be expressed as an  $\alpha$ -fractional power series, i.e.,  $f_{\alpha}(\theta^{\alpha}) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)} (\theta - \theta_0)^{k\alpha}$  on some open interval  $(\theta_0 - s, \theta_0 + s)$ , then we say that  $f_{\alpha}(\theta^{\alpha})$  is  $\alpha$ -fractional analytic at  $\theta_0$ , where  $s$  is the radius of convergence about  $\theta_0$ . Moreover, if  $f_{\alpha}: [a, b] \rightarrow R$  is continuous on closed interval  $[a, b]$  and it is  $\alpha$ -fractional analytic at every point in open interval  $(a, b)$ , then  $f_{\alpha}$  is called an  $\alpha$ -fractional analytic function on  $[a, b]$ .

**Definition 2.4:**  $\left( f_{\alpha}(\theta^{\alpha}) \right)^{\otimes n} = f_{\alpha}(\theta^{\alpha}) \otimes \dots \otimes f_{\alpha}(\theta^{\alpha})$  is called the  $n$ -th power of the  $\alpha$ -fractional analytic function  $f_{\alpha}(\theta^{\alpha})$ . On the other hand, if  $g_{\alpha}(\theta^{\alpha})$  is also an  $\alpha$ -fractional analytic function such that  $f_{\alpha}(\theta^{\alpha}) \otimes g_{\alpha}(\theta^{\alpha}) = 1$ , then we say that  $g_{\alpha}(\theta^{\alpha})$  is the  $\otimes$  reciprocal of  $f_{\alpha}(\theta^{\alpha})$ , and is denoted by  $\left[ f_{\alpha}(\theta^{\alpha}) \right]^{\otimes -1}$ .

In the following, we define a new multiplication of fractional analytic functions.

**Definition 2.5** ([12]): Suppose that  $0 < \alpha \leq 1$ , and  $\theta_0$  is a real number. If  $f_{\alpha}(\theta^{\alpha})$  and  $g_{\alpha}(\theta^{\alpha})$  are two  $\alpha$ -fractional analytic functions,

$$f_{\alpha}(\theta^{\alpha}) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)} (\theta - \theta_0)^{k\alpha} = \sum_{k=0}^{\infty} \frac{a_k}{k!} \left( \frac{1}{\Gamma(\alpha+1)} (\theta - \theta_0)^{\alpha} \right)^{\otimes k}, \tag{5}$$

$$g_{\alpha}(\theta^{\alpha}) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)} (\theta - \theta_0)^{k\alpha} = \sum_{k=0}^{\infty} \frac{b_k}{k!} \left( \frac{1}{\Gamma(\alpha+1)} (\theta - \theta_0)^{\alpha} \right)^{\otimes k}. \tag{6}$$

Then we define

$$\begin{aligned} & f_{\alpha}(\theta^{\alpha}) \otimes g_{\alpha}(\theta^{\alpha}) \\ &= \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)} (\theta - \theta_0)^{k\alpha} \otimes \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)} (\theta - \theta_0)^{k\alpha} \\ &= \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)} \left( \sum_{m=0}^k \binom{k}{m} a_{k-m} b_m \right) (\theta - \theta_0)^{k\alpha}. \end{aligned} \tag{7}$$

In other words,

$$\begin{aligned} & f_{\alpha}(\theta^{\alpha}) \otimes g_{\alpha}(\theta^{\alpha}) \\ &= \sum_{k=0}^{\infty} \frac{a_k}{k!} \left( \frac{1}{\Gamma(\alpha+1)} (\theta - \theta_0)^{\alpha} \right)^{\otimes k} \otimes \sum_{k=0}^{\infty} \frac{b_k}{k!} \left( \frac{1}{\Gamma(\alpha+1)} (\theta - \theta_0)^{\alpha} \right)^{\otimes k} \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \left( \sum_{m=0}^k \binom{k}{m} a_{k-m} b_m \right) \left( \frac{1}{\Gamma(\alpha+1)} (\theta - \theta_0)^{\alpha} \right)^{\otimes k}. \end{aligned} \tag{8}$$



**Definition 2.6** ([15]): Let  $0 < \alpha \leq 1$ , and  $f_\alpha(\theta^\alpha), g_\alpha(\theta^\alpha)$  be  $\alpha$ -fractional analytic functions,

$$f_\alpha(\theta^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)} (\theta - \theta_0)^{k\alpha} = \sum_{k=0}^{\infty} \frac{a_k}{k!} \left( \frac{1}{\Gamma(\alpha+1)} (\theta - \theta_0)^\alpha \right)^{\otimes k}, \tag{9}$$

$$g_\alpha(\theta^\alpha) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)} (\theta - \theta_0)^{k\alpha} = \sum_{k=0}^{\infty} \frac{b_k}{k!} \left( \frac{1}{\Gamma(\alpha+1)} (\theta - \theta_0)^\alpha \right)^{\otimes k}. \tag{10}$$

The compositions of  $f_\alpha(\theta^\alpha)$  and  $g_\alpha(\theta^\alpha)$  are defined as follows:

$$(f_\alpha \circ g_\alpha)(\theta^\alpha) = f_\alpha(g_\alpha(\theta^\alpha)) = \sum_{k=0}^{\infty} \frac{a_k}{k!} (g_\alpha(\theta^\alpha))^{\otimes k}, \tag{11}$$

and

$$(g_\alpha \circ f_\alpha)(\theta^\alpha) = g_\alpha(f_\alpha(\theta^\alpha)) = \sum_{k=0}^{\infty} \frac{b_k}{k!} (f_\alpha(\theta^\alpha))^{\otimes k}. \tag{12}$$

**Definition 2.7** ([15]): Suppose that  $0 < \alpha \leq 1$ . If  $f_\alpha(\theta^\alpha), g_\alpha(\theta^\alpha)$  are two  $\alpha$ -fractional analytic functions satisfies

$$(f_\alpha \circ g_\alpha)(\theta^\alpha) = (g_\alpha \circ f_\alpha)(\theta^\alpha) = \frac{1}{\Gamma(\alpha+1)} \theta^\alpha. \tag{13}$$

Then we say these two fractional analytic functions are inverse to each other.

**Theorem 2.8**(chain rule for fractional derivatives)([15]): Suppose that  $0 < \alpha \leq 1, \theta_0$  is a real number, and let  $f_\alpha(\theta^\alpha), g_\alpha(\theta^\alpha)$  be  $\alpha$ -fractional analytic functions. Then

$$\left( \theta_0 D_\theta^\alpha \right) \left[ f_\alpha(g_\alpha(\theta^\alpha)) \right] = \left( \theta_0 D_\theta^\alpha \right) \left[ f_\alpha(\theta^\alpha) \right] (g_\alpha(\theta^\alpha)) \otimes \left( \theta_0 D_\theta^\alpha \right) \left[ g_\alpha(\theta^\alpha) \right]. \tag{14}$$

The followings are some fractional analytic functions

**Definition 2.9**([15]): Assume that  $0 < \alpha \leq 1$ , and  $\theta$  is a real variable. The  $\alpha$ -fractional exponential function is defined by

$$E_\alpha(\theta^\alpha) = \sum_{k=0}^{\infty} \frac{\theta^{k\alpha}}{\Gamma(k\alpha+1)} = \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{1}{\Gamma(\alpha+1)} \theta^\alpha \right)^{\otimes k}. \tag{15}$$

The  $\alpha$ -fractional logarithmic function  $Ln_\alpha(\theta^\alpha)$  is the inverse function of the  $E_\alpha(\theta^\alpha)$ . Furthermore, the  $\alpha$ -fractional sine and cosine function are defined as follows:

$$\sin_\alpha(\theta^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)}, \tag{16}$$

and

$$\cos_\alpha(\theta^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k\alpha}}{\Gamma(2k\alpha+1)}. \tag{17}$$

Moreover,

$$\sec_\alpha(\theta^\alpha) = (\cos_\alpha(\theta^\alpha))^{\otimes -1} \tag{18}$$

is called the  $\alpha$ -fractional secant function.

$$\csc_\alpha(\theta^\alpha) = (\sin_\alpha(\theta^\alpha))^{\otimes -1} \tag{19}$$

is the  $\alpha$ -fractional cosecant function.

$$\tan_\alpha(\theta^\alpha) = \sin_\alpha(\theta^\alpha) \otimes \sec_\alpha(\theta^\alpha) \tag{20}$$

is the  $\alpha$ -fractional tangent function. And

$$\cot_\alpha(\theta^\alpha) = \cos_\alpha(\theta^\alpha) \otimes \csc_\alpha(\theta^\alpha) \tag{21}$$



is the  $\alpha$ -fractional cotangent function.

**Proposition 2.10**(fractional Euler’s formula)[17]: Let  $0 < \alpha \leq 1$ , then

$$E_\alpha(i\theta^\alpha) = \cos_\alpha(\theta^\alpha) + i\sin_\alpha(\theta^\alpha). \tag{22}$$

**Remark 2.11:** If  $\alpha = 1$ , we obtain the classical Euler’s formula  $e^{i\theta} = \cos\theta + i\sin\theta$ . On the other hand, the smallest positive real number  $T_\alpha$  such that  $E_\alpha(iT_\alpha) = 1$ , is called the period of  $E_\alpha(i\theta^\alpha)$ .

**Theorem 2.12** ([16]): Suppose that  $0 < \alpha \leq 1$ , then

$$[\sin_\alpha(\theta^\alpha)]^{\otimes 2} + [\cos_\alpha(\theta^\alpha)]^{\otimes 2} = 1, \tag{23}$$

$$1 + [\tan_\alpha(\theta^\alpha)]^{\otimes 2} = [\sec_\alpha(\theta^\alpha)]^{\otimes 2}, \tag{24}$$

$$1 + [\cot_\alpha(\theta^\alpha)]^{\otimes 2} = [\csc_\alpha(\theta^\alpha)]^{\otimes 2}. \tag{25}$$

**Theorem 2.13** ([18]): If  $0 < \alpha \leq 1$ , then

$$({}^0D_\theta^\alpha) \left[ \sin_\alpha(\theta^\alpha) \right] = \cos_\alpha(\theta^\alpha), \tag{26}$$

$$({}^0D_\theta^\alpha) \left[ \cos_\alpha(\theta^\alpha) \right] = -\sin_\alpha(\theta^\alpha), \tag{27}$$

$$({}^0D_\theta^\alpha) \left[ \tan_\alpha(\theta^\alpha) \right] = (\sec_\alpha(\theta^\alpha))^{\otimes 2}, \tag{28}$$

$$({}^0D_\theta^\alpha) \left[ \cot_\alpha(\theta^\alpha) \right] = -(\csc_\alpha(\theta^\alpha))^{\otimes 2}. \tag{29}$$

$$({}^0D_\theta^\alpha) \left[ \sec_\alpha(\theta^\alpha) \right] = \sec_\alpha(\theta^\alpha) \otimes \tan_\alpha(\theta^\alpha), \tag{30}$$

$$({}^0D_\theta^\alpha) \left[ \csc_\alpha(\theta^\alpha) \right] = -\csc_\alpha(\theta^\alpha) \otimes \cot_\alpha(\theta^\alpha). \tag{31}$$

In the following, we introduce inverse fractional trigonometric functions.

**Definition 2.14:** Let  $0 < \alpha \leq 1$ . Then  $\arcsin_\alpha(\theta^\alpha)$  is the inverse function of  $\sin_\alpha(\theta^\alpha)$ , and it is called inverse  $\alpha$ -fractional sine function.  $\arccos_\alpha(\theta^\alpha)$  is the inverse function of  $\cos_\alpha(\theta^\alpha)$ , and we say that it is the inverse  $\alpha$ -fractional cosine function. On the other hand,  $\arctan_\alpha(\theta^\alpha)$  is the inverse function of  $\tan_\alpha(\theta^\alpha)$ , and it is called the inverse  $\alpha$ -fractional tangent function.  $\text{arccotan}_\alpha(\theta^\alpha)$  is the inverse function of  $\cot_\alpha(\theta^\alpha)$ , and we say that it is the inverse  $\alpha$ -fractional cotangent function.  $\text{arcsec}_\alpha(\theta^\alpha)$  is the inverse function of  $\sec_\alpha(\theta^\alpha)$ , and we say that it is the inverse  $\alpha$ -fractional secant function.  $\text{arccsc}_\alpha(\theta^\alpha)$  is the inverse function of  $\csc_\alpha(\theta^\alpha)$ , and it is called the inverse  $\alpha$ -fractional cosecant function.



III. MAJOR RESULTS

**Theorem 3.1:** Let  $0 < \alpha \leq 1$ . Then

$$({}_0D_{\theta}^{\alpha})[arcsin_{\alpha}(\theta^{\alpha})] = \left[ 1 - \left( \frac{1}{\Gamma(\alpha+1)} \theta^{\alpha} \right)^{\otimes 2} \right]^{\otimes \left( -\frac{1}{2} \right)}, \tag{32}$$

$$({}_0D_{\theta}^{\alpha})[arccos_{\alpha}(\theta^{\alpha})] = - \left[ 1 - \left( \frac{1}{\Gamma(\alpha+1)} \theta^{\alpha} \right)^{\otimes 2} \right]^{\otimes \left( -\frac{1}{2} \right)}, \tag{33}$$

$$({}_0D_{\theta}^{\alpha})[arctan_{\alpha}(\theta^{\alpha})] = \left[ 1 + \left( \frac{1}{\Gamma(\alpha+1)} \theta^{\alpha} \right)^{\otimes 2} \right]^{\otimes (-1)}, \tag{34}$$

$$({}_0D_{\theta}^{\alpha})[arccot_{\alpha}(\theta^{\alpha})] = - \left[ 1 + \left( \frac{1}{\Gamma(\alpha+1)} \theta^{\alpha} \right)^{\otimes 2} \right]^{\otimes (-1)}. \tag{35}$$

**Proof** Using chain rule for fractional derivatives and Theorem 2.13 yields

$$\begin{aligned} &({}_0D_{\theta}^{\alpha}) \left[ \sin_{\alpha} (arcsin_{\alpha}(\theta^{\alpha})) \right] \\ &= ({}_0D_{\theta}^{\alpha}) \left[ \sin_{\alpha}(\theta^{\alpha}) \right] (arcsin_{\alpha}(\theta^{\alpha})) \otimes ({}_0D_{\theta}^{\alpha}) [arcsin_{\alpha}(\theta^{\alpha})] \\ &= \cos_{\alpha}(arcsin_{\alpha}(\theta^{\alpha})) \otimes ({}_0D_{\theta}^{\alpha}) [arcsin_{\alpha}(\theta^{\alpha})]. \end{aligned} \tag{36}$$

Since  $\sin_{\alpha}(arcsin_{\alpha}(\theta^{\alpha})) = \frac{1}{\Gamma(\alpha+1)} \theta^{\alpha}$ , it follows that

$$\cos_{\alpha}(arcsin_{\alpha}(\theta^{\alpha})) \otimes ({}_0D_{\theta}^{\alpha}) [arcsin_{\alpha}(\theta^{\alpha})] = 1. \tag{37}$$

By Eq. (23), we obtain

$$\begin{aligned} & \cos_{\alpha}(arcsin_{\alpha}(\theta^{\alpha})) \\ &= \left[ 1 - [\sin_{\alpha}(arcsin_{\alpha}(\theta^{\alpha}))]^{\otimes 2} \right]^{\otimes \left( \frac{1}{2} \right)} \\ &= \left[ 1 - \left[ \frac{1}{\Gamma(\alpha+1)} \theta^{\alpha} \right]^{\otimes 2} \right]^{\otimes \left( \frac{1}{2} \right)}. \end{aligned} \tag{38}$$

Therefore,

$$({}_0D_{\theta}^{\alpha}) [arcsin_{\alpha}(\theta^{\alpha})] = \left[ 1 - \left( \frac{1}{\Gamma(\alpha+1)} \theta^{\alpha} \right)^{\otimes 2} \right]^{\otimes \left( -\frac{1}{2} \right)}.$$

Similarly,

$$\begin{aligned} &({}_0D_{\theta}^{\alpha}) [cos_{\alpha}(arccos_{\alpha}(\theta^{\alpha}))] \\ &= ({}_0D_{\theta}^{\alpha}) [cos_{\alpha}(\theta^{\alpha})] (arccos_{\alpha}(\theta^{\alpha})) \otimes ({}_0D_{\theta}^{\alpha}) [arccos_{\alpha}(\theta^{\alpha})] \\ &= - \sin_{\alpha}(arccos_{\alpha}(\theta^{\alpha})) \otimes ({}_0D_{\theta}^{\alpha}) [arccos_{\alpha}(\theta^{\alpha})]. \end{aligned} \tag{39}$$

By  $\cos_{\alpha}(arccos_{\alpha}(\theta^{\alpha})) = \frac{1}{\Gamma(\alpha+1)} \theta^{\alpha}$ , we have

$$\sin_{\alpha}(arccos_{\alpha}(\theta^{\alpha})) \otimes ({}_0D_{\theta}^{\alpha}) [arccos_{\alpha}(\theta^{\alpha})] = - 1. \tag{40}$$

Since

$$\begin{aligned} &\sin_{\alpha}(arccos_{\alpha}(\theta^{\alpha})) \\ &= \left[ 1 - [cos_{\alpha}(arccos_{\alpha}(\theta^{\alpha}))]^{\otimes 2} \right]^{\otimes \left( \frac{1}{2} \right)} \\ &= \left[ 1 - \left[ \frac{1}{\Gamma(\alpha+1)} \theta^{\alpha} \right]^{\otimes 2} \right]^{\otimes \left( \frac{1}{2} \right)}. \end{aligned} \tag{41}$$

It follows that

$$({}_0D_{\theta}^{\alpha}) [arccos_{\alpha}(\theta^{\alpha})] = - \left[ 1 - \left( \frac{1}{\Gamma(\alpha+1)} \theta^{\alpha} \right)^{\otimes 2} \right]^{\otimes \left( -\frac{1}{2} \right)}.$$



On the other hand,

$$\begin{aligned} & ({}^0D_{\theta}^{\alpha})\left[\tan_{\alpha}\left(\arctan_{\alpha}(\theta^{\alpha})\right)\right] \\ &= ({}^0D_{\theta}^{\alpha})\left[\tan_{\alpha}(\theta^{\alpha})\right]\left(\arctan_{\alpha}(\theta^{\alpha})\right) \otimes ({}^0D_{\theta}^{\alpha})\left[\arctan_{\alpha}(\theta^{\alpha})\right] \\ &= \left[\sec_{\alpha}\left(\arctan_{\alpha}(\theta^{\alpha})\right)\right]^{\otimes 2} \otimes ({}^0D_{\theta}^{\alpha})\left[\arctan_{\alpha}(\theta^{\alpha})\right]. \end{aligned} \tag{42}$$

By Eq. (24), we have

$$\left[1 + \left[\tan_{\alpha}\left(\arctan_{\alpha}(\theta^{\alpha})\right)\right]^{\otimes 2}\right] \otimes ({}^0D_{\theta}^{\alpha})\left[\arctan_{\alpha}(\theta^{\alpha})\right] = 1. \tag{43}$$

Hence,

$$({}^0D_{\theta}^{\alpha})\left[\arctan_{\alpha}(\theta^{\alpha})\right] = \left[1 + \left(\frac{1}{\Gamma(\alpha+1)}\theta^{\alpha}\right)^{\otimes 2}\right]^{\otimes (-1)}$$

In addition,

$$\begin{aligned} & ({}^0D_{\theta}^{\alpha})\left[\cot_{\alpha}\left(\operatorname{arccot}_{\alpha}(\theta^{\alpha})\right)\right] \\ &= ({}^0D_{\theta}^{\alpha})\left[\cot_{\alpha}(\theta^{\alpha})\right]\left(\operatorname{arccot}_{\alpha}(\theta^{\alpha})\right) \otimes ({}^0D_{\theta}^{\alpha})\left[\operatorname{arccot}_{\alpha}(\theta^{\alpha})\right] \\ &= -\left[\csc_{\alpha}\left(\operatorname{arccot}_{\alpha}(\theta^{\alpha})\right)\right]^{\otimes 2} \otimes ({}^0D_{\theta}^{\alpha})\left[\operatorname{arccot}_{\alpha}(\theta^{\alpha})\right]. \end{aligned} \tag{44}$$

Using Eq. (25) yields

$$({}^0D_{\theta}^{\alpha})\left[\operatorname{arccot}_{\alpha}(\theta^{\alpha})\right] = -\left[1 + \left(\frac{1}{\Gamma(\alpha+1)}\theta^{\alpha}\right)^{\otimes 2}\right]^{\otimes (-1)} \quad \text{Q.e.d.}$$

**Theorem 3.2:** Suppose that  $0 < \alpha \leq 1$ , then

$$\arcsin_{\alpha}(\theta^{\alpha}) + \arccos_{\alpha}(\theta^{\alpha}) = \frac{T_{\alpha}}{4}, \tag{45}$$

$$\arctan_{\alpha}(\theta^{\alpha}) + \operatorname{arccot}_{\alpha}(\theta^{\alpha}) = \frac{T_{\alpha}}{4}. \tag{46}$$

**Proof** Since

$$\begin{aligned} & ({}^0D_{\theta}^{\alpha})\left[\arcsin_{\alpha}(\theta^{\alpha}) + \arccos_{\alpha}(\theta^{\alpha})\right] \\ &= \left[1 - \left(\frac{1}{\Gamma(\alpha+1)}\theta^{\alpha}\right)^{\otimes 2}\right]^{\otimes \left(-\frac{1}{2}\right)} - \left[1 - \left(\frac{1}{\Gamma(\alpha+1)}\theta^{\alpha}\right)^{\otimes 2}\right]^{\otimes \left(-\frac{1}{2}\right)} \\ &= 0. \end{aligned}$$

It follows that  $\arcsin_{\alpha}(\theta^{\alpha}) + \arccos_{\alpha}(\theta^{\alpha})$  is a constant. Moreover, using  $\arcsin_{\alpha}(0) = 0$  and  $\arccos_{\alpha}(0) = \frac{T_{\alpha}}{4}$  yields

$$\arcsin_{\alpha}(0) + \arccos_{\alpha}(0) = \frac{T_{\alpha}}{4}. \tag{47}$$

And hence,

$$\arcsin_{\alpha}(\theta^{\alpha}) + \arccos_{\alpha}(\theta^{\alpha}) = \frac{T_{\alpha}}{4}.$$

On the other hand,

$$\begin{aligned} & ({}^0D_{\theta}^{\alpha})\left[\arctan_{\alpha}(\theta^{\alpha}) + \operatorname{arccot}_{\alpha}(\theta^{\alpha})\right] \\ &= \left[1 + \left(\frac{1}{\Gamma(\alpha+1)}\theta^{\alpha}\right)^{\otimes 2}\right]^{\otimes (-1)} - \left[1 + \left(\frac{1}{\Gamma(\alpha+1)}\theta^{\alpha}\right)^{\otimes 2}\right]^{\otimes (-1)} \\ &= 0. \end{aligned}$$



Therefore,  $\arctan_{\alpha}(\theta^{\alpha}) + \operatorname{arccot}_{\alpha}(\theta^{\alpha})$  is also a constant. By  $\arctan_{\alpha}(1) = \frac{\tau_{\alpha}}{8}$  and  $\operatorname{arccot}_{\alpha}(1) = \frac{\tau_{\alpha}}{8}$ , we obtain

$$\arctan_{\alpha}(\theta^{\alpha}) + \operatorname{arccot}_{\alpha}(\theta^{\alpha}) = \frac{\tau_{\alpha}}{4}. \quad \text{Q.e.d.}$$

**Theorem 3.3:** Let  $0 < \alpha \leq 1$ . Then

$$(O I_{\theta}^{\alpha}) \left[ \left[ 1 - \left( \frac{1}{\Gamma(\alpha+1)} \theta^{\alpha} \right)^{\otimes 2} \right]^{\otimes \left( -\frac{1}{2} \right)} \right] = \arcsin_{\alpha}(\theta^{\alpha}), \quad (48)$$

$$\frac{\tau_{\alpha}}{4} - (O I_{\theta}^{\alpha}) \left[ \left[ 1 - \left( \frac{1}{\Gamma(\alpha+1)} \theta^{\alpha} \right)^{\otimes 2} \right]^{\otimes \left( -\frac{1}{2} \right)} \right] = \arccos_{\alpha}(\theta^{\alpha}), \quad (49)$$

$$(O I_{\theta}^{\alpha}) \left[ \left[ 1 + \left( \frac{1}{\Gamma(\alpha+1)} \theta^{\alpha} \right)^{\otimes 2} \right]^{\otimes (-1)} \right] = \arctan_{\alpha}(\theta^{\alpha}), \quad (50)$$

$$\frac{\tau_{\alpha}}{4} - (O I_{\theta}^{\alpha}) \left[ \left[ 1 + \left( \frac{1}{\Gamma(\alpha+1)} \theta^{\alpha} \right)^{\otimes 2} \right]^{\otimes (-1)} \right] = \operatorname{arccot}_{\alpha}(\theta^{\alpha}). \quad (51)$$

**Proof** By Theorem 3.1, we have

$$(O I_{\theta}^{\alpha}) \left[ (O D_{\theta}^{\alpha}) \left[ \arcsin_{\alpha}(\theta^{\alpha}) \right] \right] = (O I_{\theta}^{\alpha}) \left[ \left[ 1 - \left( \frac{1}{\Gamma(\alpha+1)} \theta^{\alpha} \right)^{\otimes 2} \right]^{\otimes \left( -\frac{1}{2} \right)} \right]. \quad (52)$$

And hence,

$$(O I_{\theta}^{\alpha}) \left[ \left[ 1 - \left( \frac{1}{\Gamma(\alpha+1)} \theta^{\alpha} \right)^{\otimes 2} \right]^{\otimes \left( -\frac{1}{2} \right)} \right] = \arcsin_{\alpha}(\theta^{\alpha}).$$

By Theorem 3.2, we immediately obtain

$$\frac{\tau_{\alpha}}{4} - (O I_{\theta}^{\alpha}) \left[ \left[ 1 - \left( \frac{1}{\Gamma(\alpha+1)} \theta^{\alpha} \right)^{\otimes 2} \right]^{\otimes \left( -\frac{1}{2} \right)} \right] = \arccos_{\alpha}(\theta^{\alpha}).$$

Moreover, using Theorem 3.1 yields

$$(O I_{\theta}^{\alpha}) \left[ (O D_{\theta}^{\alpha}) \left[ \arctan_{\alpha}(\theta^{\alpha}) \right] \right] = (O I_{\theta}^{\alpha}) \left[ \left[ 1 + \left( \frac{1}{\Gamma(\alpha+1)} \theta^{\alpha} \right)^{\otimes 2} \right]^{\otimes (-1)} \right]. \quad (53)$$

Thus,

$$(O I_{\theta}^{\alpha}) \left[ \left[ 1 + \left( \frac{1}{\Gamma(\alpha+1)} \theta^{\alpha} \right)^{\otimes 2} \right]^{\otimes (-1)} \right] = \arctan_{\alpha}(\theta^{\alpha}).$$

Again by Theorem 3.2, we can easily obtain

$$\frac{\tau_{\alpha}}{4} - (O I_{\theta}^{\alpha}) \left[ \left[ 1 + \left( \frac{1}{\Gamma(\alpha+1)} \theta^{\alpha} \right)^{\otimes 2} \right]^{\otimes (-1)} \right] = \operatorname{arccot}_{\alpha}(\theta^{\alpha}). \quad \text{Q.e.d.}$$

#### IV. CONCLUSION

In this paper, we obtain some fractional derivative and integral formulas of inverse fractional trigonometric functions based on Jumarie type of modified R-L fractional calculus and a new multiplication of fractional analytic functions. These results are the same as those in traditional calculus. The main method we used is the chain rule for fractional derivatives. Moreover, the new multiplication we defined is a natural operation of fractional analytic functions. In the





future, we will make use of the chain rule for fractional derivatives and the new multiplication of fractional analytic functions to study the problems in applied mathematics and fractional calculus.

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