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[ijirset@gmail.com](mailto:ijirset@gmail.com)

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# Applications of Real Vector Spaces: QR-Factorization

Parshuram A. Ahire

Head, Department of Mathematics, M.S.G. College Malegaon-Camp, Dist- Nasik, Maharashtra, India

**ABSTRACT:** We are familiar with the LU- factorization of a matrix and showed how it leads to a very efficient method for solving a linear system. We now discuss another factorization of a matrix  $A$ , called a QR-factorization (or QR-decomposition) of  $A$ , This type of factorization is widely used in computer codes to find the eigenvalues of the matrix, to solve linear system, and to find least squares approximation.

**KEYWORDS:** Orthonormal basis, QR-factorization, Gram-Schmidt process.

## I. INTRODUCTION

In this paper we introduce the QR-factorization of matrix is a decomposition of matrix into an orthonormal and a triangular matrix. A QR-decomposition of a real square matrix  $A$  is a decomposition of  $A$  as

$$A = QR$$

where  $Q$  is an orthogonal matrix (i.e.  $Q^T Q = I$ ) and  $R$  is an upper triangular matrix. If  $A$  is nonsingular, then this factorization is unique. There are several methods for actually computing the QR decomposition. One of such method is the Gram-Schmidt process.

## II. COMPUTING THE QR FACTORIZATION

Let  $A$  be an  $m \times n$  matrix with linearly independent columns. Applying the Gram-Schmidt orthogonalization process to the columns of  $A$  produces an  $m \times n$  matrix  $Q$  whose columns are orthonormal. In fact,  $A = QR$ , where  $R$  is an  $n \times n$  upper triangular matrix with positive entries on the diagonal.

**THEOREM (I) :** If  $A$  is an  $m \times n$  matrix with linearly independent columns, then  $A$  can be factored as  $A = QR$ , where  $Q$  is an  $m \times n$  matrix whose columns form an orthonormal basis for the column space of  $A$  and  $R$  is an  $n \times n$  non-singular upper triangular matrix.

**PROOF :** Let  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$  denotes the linearly independent columns of  $A$ , which form a basis for a column space of  $A$ . By using the Gram-Schmidt process, we can obtain an orthonormal basis  $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n$  for the column space of  $A$ . Recall how this orthonormal basis was obtained. We first constructed an orthogonal basis

$\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  as follows :

$\mathbf{v}_1 = \mathbf{u}_1$  and then for  $i = 1, 2, \dots, n$  we have

$$\mathbf{v}_i = \mathbf{u}_i - \frac{\mathbf{u}_i \cdot \mathbf{v}_1}{\mathbf{u}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 - \frac{\mathbf{u}_i \cdot \mathbf{v}_2}{\mathbf{u}_2 \cdot \mathbf{v}_2} \mathbf{v}_2 - \dots - \frac{\mathbf{u}_i \cdot \mathbf{v}_{i-1}}{\mathbf{u}_{i-1} \cdot \mathbf{v}_{i-1}} \mathbf{v}_{i-1} \quad \text{----- (1)}$$

Finally,  $\mathbf{w}_i = \frac{1}{\|\mathbf{v}_i\|} \cdot \mathbf{v}_i$ , for  $i = 1, 2, 3, \dots, n$ . Now each of the vectors  $\mathbf{u}_i$  can be

written as a linear combination of the  $\mathbf{w}$  – vectors as:

$$\begin{aligned} \mathbf{u}_1 &= r_{11} \mathbf{w}_1 + r_{22} \mathbf{w}_2 + \dots + r_{n1} \mathbf{w}_n \\ \mathbf{u}_2 &= r_{12} \mathbf{w}_1 + r_{22} \mathbf{w}_2 + \dots + r_{n2} \mathbf{w}_n \end{aligned} \quad \text{----- (2)}$$

⋮

$$\mathbf{u}_n = r_{1n} \mathbf{w}_1 + r_{2n} \mathbf{w}_2 + \dots + r_{nn} \mathbf{w}_n .$$

[But, if  $S = \{ \mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n \}$  is an orthogonal basis for  $\mathbf{R}^n$  and  $\mathbf{u}$  any vector in  $\mathbf{R}^n$  then  $\mathbf{u} = c_1 \mathbf{w}_1 + c_2 \mathbf{w}_2 + \dots + c_n \mathbf{w}_n$  where  $c_i = \mathbf{u} \cdot \mathbf{w}_i$  ]

Therefore, we have  $r_{ij} = \mathbf{u}_i \cdot \mathbf{w}_j$



Also, from equation (1), we see that  $u_i$  lies in

$$\text{Span} \{ v_1, v_2, \dots, v_i \} = \text{span} \{ w_1, w_2, \dots, w_i \}$$

Since,  $w_j$  is orthogonal to  $\text{span} \{ w_1, w_2, \dots, w_i \}$  for  $j > i$ , it is orthogonal to  $u_i$ .

Hence  $r_{ij} = 0$  for  $j > i$ . Let  $Q$  be the matrix whose columns are,  $w_1, w_2, \dots, w_i$

Let

$$r_j = \begin{pmatrix} r_{1j} \\ r_{2j} \\ \vdots \\ r_{nj} \end{pmatrix}$$

Then the equation in (2) can be written in the matrix form as :

$$A = [u_1, u_2, \dots, u_n] = [Qr_1, Qr_2, \dots, Qr_n] = QR$$

Where  $R$  is the matrix whose columns are  $r_1, r_2, \dots, r_n$ . Thus

$$R = \begin{pmatrix} r_{11} & r_{12} & \dots & r_{1n} \\ 0 & r_{22} & \dots & r_{2n} \\ 0 & 0 & \dots & r_{3n} \\ \dots & \dots & \ddots & \vdots \\ 0 & 0 & \dots & r_{nn} \end{pmatrix}$$

We now show that  $R$  is non singular. Let  $x$  be the solution to the linear system  $Rx = 0$ .

Multiplying this equation by  $Q$  on the left, we have

$$Q(Rx) = Q0 \Rightarrow (QR)x = 0 \Rightarrow Ax = 0$$

But, the homogeneous system can be written as

$$x_1 u_1 + x_2 u_2 + \dots + x_n u_n = 0$$

where  $x = (x_1, x_2, \dots, x_n)$ . since the column of  $A = [u_1 | u_2 | \dots | u_n]$  are linearly independent, Therefore

$$x_1 = x_2 = \dots = x_n = 0,$$

so  $x$  must be the zero vector. Also,  $A$  is non-singular implies that  $R$  is non singular. In Exercise we ask you to show that the diagonal entries  $r_{ii}$  of  $R$  are non zero by first expressing  $u_i$  as a linear combination of  $v_1, v_2, \dots, v_i$  and then computing  $r_{ij} = u_i \cdot w_j$ . This provides another proof of the non singularity of  $R$ .

The procedure for finding the QR-factorization on an  $m \times n$  matrix  $A$  with linearly independent columns is as follows:

**Step 1:** Let the column of  $A$  be denoted by  $u_1, u_2, \dots, u_n$  and let  $W$  be the subspace of  $R^n$  with these vectors as basis.

**Step 2:** Transform the basis  $\{u_1, u_2, \dots, u_n\}$  for  $W$  by using the Gram- Schmidt process to an orthonormal basis  $\{w_1, w_2, \dots, w_n\}$ .

Let  $Q = [w_1 | w_2 | \dots | w_n]$  be the matrix whose columns are  $w_1, w_2, \dots, w_n$ .

**Step 3:** Compute  $R = [r_{ij}]$ , where,  $r_{ij} = u_i \cdot w_j$ .

**Example(1):** Compute the QR-factorization of  $A = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 2 & 0 \\ -1 & -2 & 2 \end{pmatrix}$

**Soluton:** Let  $A = [u_1 | u_2 | u_3]$ , where  $u_1 = (1, -1, -1)^T$ ,  $u_2 = (0, 2, -2)^T$ , and  $u_3 = (2, 0, 2)^T$ .

Now we will drop  $T$  notation for simplicity, but we have to remember that all the vectors are column vectors of  $A$ . Using the Gram-Schmidt procedure, we obtain

$$v_1 = u_1 = (1, -1, -1)$$

$$w_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{3}}(1, -1, -1)$$

$$v_2 = u_2 - (u_2 \cdot w_1) w_1 = (0, 2, -2) - \frac{1}{\sqrt{3}}(0) \frac{1}{\sqrt{3}}(1, -1, -1) = (0, 2, -2)$$

$$w_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{2}}(0, 1, -1)$$

$$v_3 = u_3 - (u_3 \cdot w_1) w_1 - (u_3 \cdot w_2) w_2 = (2, 0, 2) - \frac{1}{\sqrt{3}}(2+0-2) \frac{1}{\sqrt{3}}(1, -1, -1) - \frac{1}{\sqrt{2}}(0+0-2) \frac{1}{\sqrt{2}}(0, 1, -1) = (2, 0, 2) - 0 + (0, 1, -1) = (2, 1, 1)$$

$$w_3 = \frac{v_3}{\|v_3\|} = \frac{1}{\sqrt{6}}(2, 1, 1).$$



$$\text{Thus } \mathbf{Q} = [\mathbf{w}_1 | \mathbf{w}_2 | \mathbf{w}_3] = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix} = \begin{pmatrix} 0.5774 & 0 & 0.8165 \\ -0.5774 & 0.7071 & 0.4082 \\ -0.5774 & -0.7071 & 0.4082 \end{pmatrix}$$

Here,  $\mathbf{Q}$  has orthonormal columns

Therefore,  $\mathbf{Q}$  has left inverse as  $\mathbf{Q}^T$ . i.e.  $\mathbf{Q} \cdot \mathbf{Q}^T = \mathbf{I}$

Now, we can find  $\mathbf{R}$  as  $\mathbf{A} = \mathbf{QR}$

Where  $\mathbf{Q}^T \mathbf{A} = \mathbf{Q}^T (\mathbf{QR}) = (\mathbf{Q}^T \mathbf{Q}) \mathbf{R} = \mathbf{I} \mathbf{R} = \mathbf{R}$

$$\text{Therefore } \mathbf{R} = \mathbf{Q}^T \mathbf{A} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ -1 & 2 & 0 \\ -1 & -2 & 2 \end{pmatrix} = \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{8} & -\sqrt{2} \\ 0 & 0 & \sqrt{6} \end{pmatrix}$$

$$= \begin{pmatrix} 1.7321 & 0 & 0 \\ 0 & 2.8284 & -1.4142 \\ 0 & 0 & 2.4495 \end{pmatrix}. \text{ As you can verify, } \mathbf{A} = \mathbf{QR}.$$

**Example(2)** : Find the QR-factorization of  $\mathbf{A} = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$

**Solution** : Letting the columns of  $\mathbf{A}$  be denoted by  $\mathbf{u}_1, \mathbf{u}_2$  and  $\mathbf{u}_3$ , we let  $W$  be the subspace of  $\mathbb{R}^4$  with  $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$  as a basis. Applying the Gram-Schmidt process to this basis, starting with  $\mathbf{u}_1$ , we find (verify) the following orthogonal basis

$$\mathbf{w}_1 = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 0 \end{pmatrix} \quad \mathbf{w}_2 = \begin{pmatrix} \frac{-1}{\sqrt{15}} \\ \frac{\sqrt{15}}{2} \\ \frac{\sqrt{15}}{3} \\ \frac{-1}{\sqrt{15}} \\ \frac{\sqrt{15}}{3} \\ \frac{3}{\sqrt{15}} \end{pmatrix} \quad \mathbf{w}_3 = \begin{pmatrix} \frac{-4}{\sqrt{35}} \\ \frac{\sqrt{35}}{3} \\ \frac{\sqrt{35}}{3} \\ \frac{1}{\sqrt{35}} \\ \frac{\sqrt{35}}{3} \\ \frac{-3}{\sqrt{35}} \end{pmatrix}$$

$$\text{Then } \mathbf{Q} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{15}} & \frac{-4}{\sqrt{35}} \\ \frac{1}{\sqrt{3}} & \frac{\sqrt{15}}{2} & \frac{\sqrt{35}}{3} \\ \frac{1}{\sqrt{3}} & \frac{\sqrt{15}}{3} & \frac{\sqrt{35}}{3} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{15}} & \frac{1}{\sqrt{35}} \\ \frac{1}{\sqrt{3}} & \frac{\sqrt{15}}{3} & \frac{\sqrt{35}}{3} \\ 0 & \frac{3}{\sqrt{15}} & \frac{-3}{\sqrt{35}} \end{pmatrix} = \begin{pmatrix} 0.5774 & -0.2582 & -0.6761 \\ 0.5774 & 0.5164 & 0.5071 \\ 0.5774 & -0.2582 & 0.1690 \\ 0 & 0.7746 & -0.5071 \end{pmatrix}$$

$$\text{And } \mathbf{R} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix}$$

Where,  $r_{ij} = \mathbf{u}_i \cdot \mathbf{w}_j$ .

$$\text{Thus, } \mathbf{R} = \begin{pmatrix} \frac{3}{\sqrt{3}} & \frac{-2}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \\ 0 & \frac{5}{\sqrt{15}} & \frac{-2}{\sqrt{15}} \\ 0 & \frac{7}{\sqrt{15}} & \frac{7}{\sqrt{35}} \end{pmatrix} \approx \begin{pmatrix} 1.7321 & -1.1547 & -0.5774 \\ 0 & 1.2910 & -0.5164 \\ 0 & 0 & 1.1832 \end{pmatrix}$$

As you can verify,  $\mathbf{A} = \mathbf{QR}$ .

**3.Note:**

- 1.In the proof of Theorem 1,  $r_{ii}$  is non zero by first expressing  $\mathbf{u}_i$  as a linear combination of  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  and then computing  $r_{ii} = \mathbf{u}_i \cdot \mathbf{w}_i$ .
- 2.Every non singular matrix has a QR- factorization.



**4.Remark :** State-of-the-art computer implementations (such as in MATLAB) yield an alternative QR-factorization of an  $m \times n$  matrix  $A$  as the product of an  $m \times m$  matrix  $Q$  and an  $m \times n$  matrix  $R = [r_{ij}]$ , where  $r_{ij} = 0$  if  $i > j$ . Thus, if  $A$  is  $5 \times 3$ ,

$$\text{then } R = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

### III. CONCLUSION

In general, This decomposition leads to an algorithm for solving a linear system  $\mathbf{AX} = \mathbf{b}$  that is widely used method on computers for solving a linear system. A main reason for the popularity of this method is that it provides the cheapest way of solving a linear system for which we repeatedly have to change the right side. This type of situation occurs often in applied problems. For example, an electric utility company must determine the inputs ( the unknowns) needed to produce some required outputs ( the right sides). The inputs and outputs might be related by a linear system, whose coefficient matrix is fixed, while the right side changes from day to day, or even hours to hours. The QR-factorization discussed is also useful in solving other problems in linear algebra.

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