



e-ISSN: 2319-8753 | p-ISSN: 2347-6710

IJIRSET

International Journal of Innovative Research in
SCIENCE | ENGINEERING | TECHNOLOGY

INTERNATIONAL JOURNAL OF INNOVATIVE RESEARCH

IN SCIENCE | ENGINEERING | TECHNOLOGY

Volume 12, Issue 12, December 2023

ISSN INTERNATIONAL
STANDARD
SERIAL
NUMBER
INDIA

Impact Factor: 8.423

9940 572 462

6381 907 438

ijirset@gmail.com

www.ijirset.com



On The Construction of Cantor - $\frac{3}{8}$ Set

Satish B. Khobragade, Kailash R. Mule

Department of Mathematics, Ghulam Nabi Azad Art's, Commerce and Science College, Barshitakli, Dist. Akola, Maharashtra, India

Department of Mathematics, Shri. D. M. Burungale College, Shegaon, Dist. Buldhana, Maharashtra, India

ABSTRACT : In this paper we construct a Cantor - $\frac{3}{8}$ set S from $[0,1]$ such that $S_n \supset S_{n+1}, m(S_n) = (\frac{6}{8})^n$ and $S = \cap S_n$. Further we find the measure of Cantor - $\frac{3}{8}$ set S i.e. we find that if S is Cantor - $\frac{3}{8}$ set then $m(S) = 0$. Here in the construction of Cantor - $\frac{3}{8}$ set the length of remaining closed intervals at each stage are equal to $(\frac{3}{8})^k, k = 1,2,3,.....$ so we call this Cantor like set as Cantor - $\frac{3}{8}$ set.

KEY WORDS : Cantor set, Cantor like sets, Cantor - $\frac{3}{8}$ set.

I. INTRODUCTION

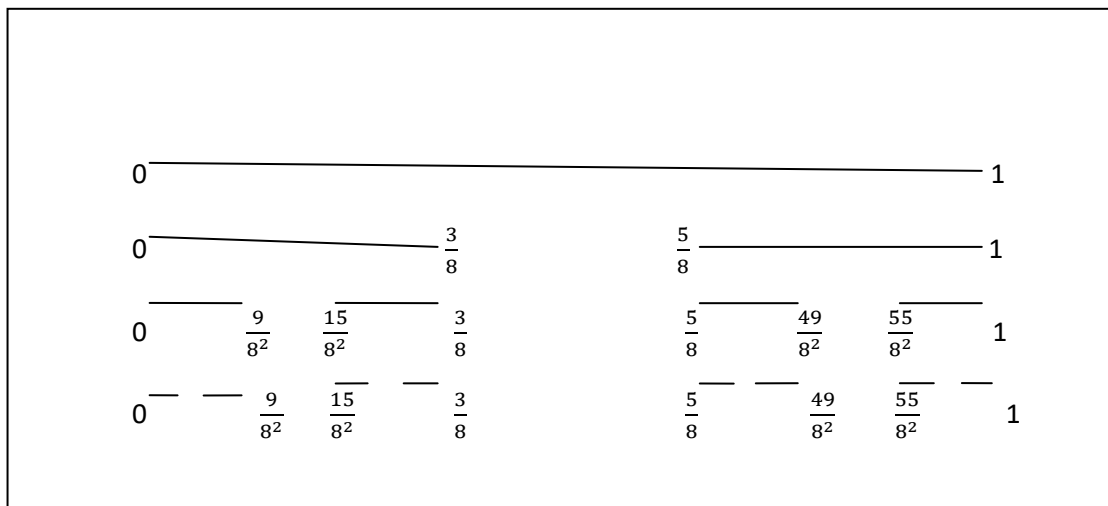
Originator of Cantor set was Georg Ferdinand Ludwig Philipp Cantor (1845 - 1918). He was born on March 3, 1845, in St. Petersburg. The Cantor set and Cantor like sets have been constructed to give examples of uncountable sets with measure zero. These sets are found to have astonishing properties.

e.g. Though they are uncountable they have empty interior, are closed, bounded, compact, perfect and nowhere dense sets.[3]

These sets are used to clear the concepts in Real Analysis and are found very useful in constructing counter examples. These sets are also used in variety of branches of mathematics such as Real Analysis, Number Theory, Measure Theory, General Topology, Functional Analysis, Dimension Theory, Theory of Fractals etc.

II. MAIN RESULT

1. Construction of Cantor- $\frac{3}{8}$ set



First stage:

At first stage we divide $I = [0,1]$ in to eight equal parts and remove middle open interval $I_{1,1} = (\frac{3}{8}, \frac{5}{8})$. The remaining two closed intervals are denoted by $J_{1,1} = [0, \frac{3}{8}]$ and $J_{1,2} = [\frac{5}{8}, 1]$. Let S_1 be union of remaining two closed intervals at first stage.

$$\therefore S_1 = J_{1,1} \cup J_{1,2}$$

Second stage:

Now we divide the remaining two closed intervals $J_{1,1}, J_{1,2}$ again into eight equal parts and remove the middle open intervals i.e. we remove $I_{2,2} = (\frac{9}{8^2}, \frac{15}{8^2})$ and $I_{2,2} = (\frac{49}{8^2}, \frac{55}{8^2})$. The remaining four closed intervals are denoted by $J_{2,1} = [0, \frac{9}{8^2}]$, $J_{2,2} = [\frac{15}{8^2}, \frac{3}{8}]$, $J_{2,3} = [\frac{5}{8}, \frac{49}{8^2}]$, $J_{2,4} = [\frac{55}{8^2}, 1]$. Let S_2 be union of remaining four closed intervals at second stage.

$$\therefore S_2 = J_{2,1} \cup J_{2,2} \cup J_{2,3} \cup J_{2,4}$$

Third stage:

Now we divide the remaining four closed intervals $J_{2,1}, J_{2,2}, J_{2,3}$ and $J_{2,4}$ again into eight equal parts and remove the middle open intervals i.e. we remove $I_{3,3} = (\frac{27}{8^3}, \frac{45}{8^3})$, $I_{3,3} = (\frac{147}{8^3}, \frac{165}{8^3})$, $I_{3,3} = (\frac{347}{8^3}, \frac{365}{8^3})$ and $I_{3,3} = (\frac{467}{8^3}, \frac{485}{8^3})$. The remaining eight closed intervals are denoted by $J_{3,1} = [0, \frac{27}{8^3}]$, $J_{3,2} = [\frac{45}{8^3}, \frac{9}{8^2}]$, $J_{3,3} = [\frac{15}{8^2}, \frac{147}{8^3}]$, $J_{3,4} = [\frac{165}{8^3}, \frac{3}{8}]$, $J_{3,5} = [\frac{5}{8}, \frac{347}{8^3}]$, $J_{3,6} = [\frac{365}{8^3}, \frac{49}{8^2}]$, $J_{3,7} = [\frac{55}{8^2}, \frac{467}{8^3}]$ and $J_{3,8} = [\frac{485}{8^3}, \frac{3}{8}]$. Let S_2 be union of remaining eight closed intervals at third stage.

$$\therefore S_3 = J_{3,1} \cup J_{3,2} \cup J_{3,3} \cup J_{3,4} \cup J_{3,5} \cup J_{3,6} \cup J_{3,7} \cup J_{3,8}$$

Continuing in this way at the n^{th} stage we remove 2^{n-1} open intervals $I_{j,n}, j = 1, 2, \dots, 2^{n-1}, n \geq 1$ obtaining 2^n closed intervals $J_{n,j}, j = 1, 2, \dots, 2^n, n \geq 1$ of length $(\frac{3}{8})^n$. Let S_n be union of remaining 2^n closed intervals at n^{th} stage.

$$\therefore S_n = \bigcup_{j=1}^{2^n} J_{n,j}, \quad n \geq 1$$

We define Cantor - $\frac{3}{8}$ set S by

$$S = \bigcap_{n=1}^{\infty} S_n$$

$$\text{Where } S_n = \bigcup_{j=1}^{2^n} J_{n,j}, \quad n \geq 1$$

Now we determine the measure of Cantor - $\frac{3}{8}$ set in two ways. In the first way we use the length of remaining closed intervals at each stage, use the following theorem and in second way we use the length of removed open intervals at each stage.

Theorem 2 Let $\{E_i\}$ be a sequence of measurable sets. We have

- i) If $E_1 \subseteq E_2 \subseteq \dots$, then $\mu(\bigcup_{n=1}^{\infty} E_n) = \lim \mu(E_n)$.
- ii) If $E_1 \supseteq E_2 \supseteq \dots$, then $\mu(E_i) < \infty$, $\mu(\bigcap_{n=1}^{\infty} E_n) = \lim \mu(E_n)$.

This is theorem 10 in section 5.5 on page 103 of chapter 5, from G. de Barra [1].

Theorem 3 If S is Cantor - $\frac{3}{8}$ set then $m(S) = 0$.

Proof 1

At first stage remaining two closed intervals are each of length $\frac{3}{8}$.

\therefore The sum of the lengths of remaining closed intervals at first stage = $m(S_1) = \frac{6}{8}$.

At second stage remaining four closed intervals are each of length $\frac{9}{8^2}$

\therefore The sum of the lengths of remaining closed intervals at second stage = $m(S_2) = \frac{6^2}{8^2} = (\frac{6}{8})^2$.

At third stage remaining eight closed intervals are each of length $\frac{27}{8^3}$.

\therefore The sum of the lengths of remaining closed intervals at third stage = $m(S_3) = \frac{6^3}{8^3} = (\frac{6}{8})^3$.

Continuing in this way at n^{th} stage remaining 2^n closed intervals are each of length $(\frac{3}{8})^n$.

\therefore The sum of the lengths of remaining closed intervals at n^{th} stage = $m(S_n) = 2^n (\frac{3}{8})^n = (\frac{6}{8})^n$.

Since $S = \bigcap S_n$ and by using theorem 2 we get,

$$\begin{aligned} \therefore m(S) &= m(\bigcap S_n) \\ &= \lim_{n \rightarrow \infty} m(S_n) \\ &= \lim_{n \rightarrow \infty} (\frac{6}{8})^n \end{aligned}$$

$$\therefore m(S) = 0$$

\therefore Lebesgue measure of Cantor - $\frac{3}{8}$ set is zero.

Proof 2

At first stage the length of removed interval is $\frac{2}{8}$ i.e. $m(I_{1,1}) = \frac{2}{8}$.

In the second stage we remove two open intervals each of length $\frac{6}{8^2}$.

\therefore The sum of the lengths of the removed disjoint intervals = $\frac{6}{8^2} + \frac{6}{8^2} = \frac{12}{8^2}$

In the third stage we remove four open intervals each of length $\frac{18}{8^3}$.



∴ The sum of the lengths of the removed disjoint intervals = $\frac{18}{8^3} + \frac{18}{8^3} + \frac{18}{8^3} + \frac{18}{8^3} = \frac{72}{8^3}$.

Similarly sum of the lengths of the removed disjoint intervals at fourth stage = $\frac{54}{8^4} + \frac{54}{8^4} + \frac{54}{8^4} + \frac{54}{8^4} + \frac{54}{8^4} + \frac{54}{8^4} + \frac{54}{8^4} + \frac{54}{8^4} + \frac{54}{8^4} = \frac{432}{8^4}$.

∴ The sum of the lengths of the removed disjoint intervals in the geometric construction of S

$$= \frac{2}{8} + \frac{12}{8^2} + \frac{72}{8^3} + \frac{432}{8^4} + \dots$$

$$= \frac{1}{4} + 3 \left(\frac{2^2}{8^2}\right) + 9 \left(\frac{2^3}{8^3}\right) + 27 \left(\frac{2^4}{8^4}\right) + \dots$$

$$= \frac{1}{4} + 3 \left(\frac{1}{4}\right)^2 + 3^2 \left(\frac{1}{4}\right)^3 + 3^3 \left(\frac{1}{4}\right)^4 + \dots$$

$$= \frac{1}{4} + \frac{1}{4} \left[\frac{3}{4} + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \dots \right]$$

$$= \frac{1}{4} + \frac{1}{4} \left[\frac{\frac{3}{4}}{1 - \frac{3}{4}} \right] = \frac{1}{4} + \frac{1}{4} \left[\frac{\frac{3}{4}}{\frac{1}{4}} \right] = \frac{1}{4} + \frac{3}{4} = 1 \dots\dots\dots(1)$$

$$\text{Since } S = [0, 1] \setminus \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{2^{k-1}} I_{j,k}$$

(The symbol “\” denote the complementation)

∴ m(S) = m(I) – m(Union of removed intervals)

$$= 1 - 1 = 0 \quad \text{(From equations (1) and } m(I) = 1 \text{)}$$

$$\therefore m(S) = 0$$

∴ Lebesgue measure of Cantor - $\frac{3}{8}$ set is zero.

III. CONCLUSION

we can construct different Cantor like sets from [0, 1].

REFERENCES

[1] G.de. Barra, “ Measure Theory And Integration”, New Age International (p) Limited, 2003.
 [2] Kenneth Falconer, “Fractal Geometry : Mathematical Foundations and Applications”, John Willey and Sons.1990
 [3] Kannan V. “Cantor set: From Classical To Modern”. The Mathematical Student, Vol 63, No.1-4(1994)P.243-257.
 [4] Dr. S.M. Padhye, Satish B. Khobragade “ On the construction of Cantor like sets”, The IJES, Volume 4 ,Issue 4, pages 49-54, 2015.
 [5] E. C. Titchmarsh , “The Theory Of Functions”, Oxford University Press, Oxford Second Edition -1987.



INNO SPACE
SJIF Scientific Journal Impact Factor
Impact Factor: 8.423



ISSN INTERNATIONAL
STANDARD
SERIAL
NUMBER
INDIA



INTERNATIONAL JOURNAL OF INNOVATIVE RESEARCH

IN SCIENCE | ENGINEERING | TECHNOLOGY

9940 572 462 **6381 907 438** **ijirset@gmail.com**



www.ijirset.com

Scan to save the contact details