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On Some Properties of Cantor Set

Satish B. Khobragade¹, Kailash R. Kale²

Department of Mathematics, G.N.A. Arts, Commerce and Science College Barshitakli, Dist. Akola,

Maharashtra, India¹

Department of Statistics, G.N.A. Arts, Commerce and Science College Barshitakli, Dist. Akola, Maharashtra, India²

ABSTRACT :- In this paper we study some properties of Cantor set K such as $x \in K$ iff x has ternary expansion containing 0 and 2 only, Cantor set is uncountable, closed, bounded, compact, perfect, nowhere dense in \mathbb{R} and Lebesgue measure is zero.

KEY WORDS:- Cantor set, Cantor like set

I. INTRODUCTION

In the construction of Cantor set at first stage we divide $[0, 1]$ into three equal parts and remove middle third open interval $I_{1,1} = (\frac{1}{3}, \frac{2}{3})$. The remaining two closed intervals are denoted by $J_{1,1} = [0, \frac{1}{3}]$ and $J_{1,2} = [\frac{2}{3}, 1]$.

At second stage we divide the remaining two closed intervals i.e. $J_{1,1}$ and $J_{1,2}$ again into three equal parts and remove middle open intervals $I_{1,2} = (\frac{1}{9}, \frac{2}{9})$ and $I_{2,2} = (\frac{7}{9}, \frac{8}{9})$. The remaining four closed intervals are denoted by $J_{2,1} = [0, \frac{1}{9}]$, $J_{2,2} = [\frac{2}{9}, \frac{3}{9}]$, $J_{2,3} = [\frac{6}{9}, \frac{7}{9}]$ and $J_{2,4} = [\frac{8}{9}, 1]$. Continuing in this way, after $(n - 1)$ stages we have deleted $1 + 2 + \dots + 2^{n-2} = 2^{n-1} - 1$ disjoint open intervals $I_{j,k}$ ($1 \leq k \leq n - 1, 1 \leq j \leq 2^{k-1}$) and we have left 2^{n-1} closed intervals each of length $\frac{1}{3^{n-1}}$.

The n^{th} stage consists of leaving out the middle interval of each of these open intervals $I_{1,n}, \dots, I_{2^{n-1},n}$. Thus obtaining 2^n closed intervals of length $\frac{1}{3^n}$.

We define Cantor set K by

$$K = [0, 1] \setminus \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{2^{k-1}} I_{j,k}$$

(The symbol ' \setminus ' denote the complementation)

This also means that $K = \bigcap_{n=1}^{\infty} E_n$, where $E_n = \bigcup_{k=1}^{2^n} J_{n,k}$, $n \geq 1$.

i.e. E_n is the union of all remaining closed intervals in the construction of Cantor set.

II. REVIEW OF LITERATURE

Originator of Cantor set was Georg Ferdinand Ludwig Philipp Cantor (1845 - 1918). He was born on March 3, 1845, in St. Petersburg. The Cantor set and Cantor like sets have been constructed to give examples of uncountable sets with measure zero. These sets are found to have astonishing properties.



e.g. Though they are uncountable they have empty interior, are closed, bounded, compact, perfect and nowhere dense sets.[3]

These sets are used to clear the concepts in Real Analysis and are found very useful in constructing counter examples. These sets are also used in variety of branches of mathematics such as Real Analysis, Number Theory, Measure Theory, General Topology, Functional Analysis, Dimension Theory, Theory of Fractals etc.

III. METHODOLOGY

The construction of Cantor set is described. The measure theoretic, topological properties will be proved. Several possibilities of the proofs will be discussed. The word fractal is roughly translated as “broken” and is used to describe shapes that have a very ragged, broken appearance to them, that do not lend themselves to the usual notions of Euclidean Geometry.

IV. MAIN RESULT

Proposition 1:- $x \in K$ iff x has ternary expansion containing 0 and 2 only.

Proof :- Let $I = [0, 1]$

The numbers in the interval I can be written in decimal expansion as follows.

First we divide the interval I into ten equal parts; the number in the first part have 0 as their first decimal, similarly the number in the second part have 1 as their first decimal and so on.

Let us consider the number 0.391. It has three as the first decimal and so it is in the fourth of ten equal parts. When this fourth part is divided again into ten equal parts, this number is in the last of them, because its second decimal is 9 and so on.

Moreover, 0.391 is nothing but $\frac{3}{10} + \frac{9}{10^2} + \frac{1}{10^3}$. Let us apply this method for ternary expansion instead of decimal expansion. In this method every time we divide the interval into three equal parts instead of 10.

When $I = [0, 1]$ is first divided into three equal parts, the number in first part has 0 as their first ternary digit, the number in second part has 1 as their first ternary digit and the number in third part has 2 as their first ternary digit.

Now, consider the number whose ternary expansion is 0.201 which is in third part. When this third part is divided again into three equal parts, it is in first of them because its second ternary digit is 0 and so on.

Moreover, 0.201 is nothing but $\frac{2}{3} + \frac{0}{3^2} + \frac{1}{3^3}$.

In general, the number whose ternary expansion is $0.a_1a_2a_3 \dots a_n \dots$ is $\sum_{n=1}^{\infty} \frac{a_n}{3^n}$ and each digit $a_n \in \{0, 1, 2\}$.

It is observed that in ternary expansion some numbers have two different expansions.

e.g. $\frac{1}{3} = 0.100\dots = 0.022\dots$

It so happens that such numbers are precisely the end points of deleted intervals in the geometric construction of K . But it is observed that for $x \in K$, there will be only one ternary expansion containing 0 and 2.



∴ In terms of ternary expansion K may be described as the set of all those numbers in I that admits a ternary expansion where no digit is 1. Equivalently those numbers in I for which in a suitable ternary expansion every digit is either 0 or 2.

Consider the set of all possible finite sums

$$\left\{ \frac{2}{3^{n_1}} + \frac{2}{3^{n_2}} + \dots + \frac{2}{3^{n_k}} : n_1 < n_2 < \dots < n_k \text{ are in } \mathbb{N} \right\}.$$

Taking topological closure of this set we get Cantor set K.

Proposition 2 :- Cantor set K is uncountable.

Proof :- We can show that K is uncountable by many ways. Here we prove it by two ways. One by constructing a function $f : I \rightarrow K$ which is one-one and second by constructing a function

$g : K \rightarrow I$ which is onto. By both ways we conclude that the cardinality of $K \geq$ cardinality of $I =$ cardinality of \mathbb{R} .

a) Every $x \in [0, 1]$ has a unique binary expansion say $x = (0. x_1 x_2 x_3 \dots)_2$ where we

prefer to take nonterminating expansion. Then $f : I \rightarrow K$ be defined by

$$f(x) = (0. (2x_1)(2x_2)(2x_3) \dots)_3$$

Where expansion on right hand side is ternary expansion.

Now, we show that f is one-one.

Let $x, y \in I$ and $f(x) = f(y)$.

Suppose $x = (0. x_1 x_2 x_3 \dots)_2$ is binary expansion of x and $y = (0. y_1 y_2 y_3 \dots)_2$ is binary expansion of y then $f(y) = (0. (2y_1)(2y_2)(2y_3) \dots)_3 \in K$.

Since $f(x) = f(y)$

$$\therefore 2x_i = 2y_i \text{ for all } i = 1, 2, 3, \dots$$

$$\implies x = y$$

$$\therefore f(x) = f(y) \implies x = y$$

∴ f is one-one.

But $I = [0, 1]$ is uncountable set.

∴ We can say that cardinality of K is larger than cardinality of I.

∴ K is uncountable

b) We can also prove above result by constructing $g : K \rightarrow I$ which is onto.

Every $y \in I = [0, 1]$ has ternary expansion given by $y = (0. y_1 y_2 y_3 \dots)_3$, where each $y_i = 0$ or 1 or 2.



Also $y \in K$ if and only if $y = 0.y_1 y_2 y_3 \dots$ is ternary expansion of y , where each $y_i = 0$ or 2 . (By Proposition 1)

\therefore If $y \in K$ and $y = 0.y_1 y_2 y_3 \dots$, where each $y_i = 0$ or 2 . (By Proposition 1) then

$0.\left(\frac{y_1}{2}\right)\left(\frac{y_2}{2}\right)\left(\frac{y_3}{2}\right)\dots$ is a binary expansion of some real number belonging to $[0, 1]$.

Thus we can define $g : K \rightarrow I$ by $g(x) = \left(0.\left(\frac{x_1}{2}\right)\left(\frac{x_2}{2}\right)\left(\frac{x_3}{2}\right)\dots\right)_2$ a binary expansion of $g(x)$, where $x \in K$ and $x = 0.x_1 x_2 x_3 \dots$.

To prove that g is onto i.e. we have to show that $g(K) = I$.

Suppose $y \in I$ and $y = (0.y_1 y_2 y_3 \dots)_2$ is the binary expansion of y then take

$x = (0.(2y_1)(2y_2)(2y_3) \dots)_3$ which is ternary expansion of x . So that

$g(x) = (0.y_1 y_2 y_3 \dots)_2 = y$.

Also ternary expansion of x namely $0.(2y_1)(2y_2)(2y_3) \dots$ contains only 0 or 2 .

$\therefore x \in K$

$\therefore g(x) = y$ for some $x \in K$

$\therefore g$ is onto.

$\therefore I$ is the range of g .

\therefore Cardinality of K is not less than that of I .

But I is uncountable.

$\therefore K$ is uncountable.

\therefore By both ways we have shown that Cantor set K is uncountable.

Proposition 3:- Cantor set K is closed.

Proof:- We know that Cantor set K is constructed by removing disjoint open intervals $I_{j,k}$ i.e.

$$K = [0, 1] \setminus \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{2^{k-1}} I_{j,k}.$$

Since each $I_{j,k}$ is open interval.

$\therefore \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{2^{k-1}} I_{j,k}$ is open set and hence its complement is closed.

$\therefore K$ is closed.

Proposition 4:- Cantor set K is bounded.

Proof:- Since the Cantor set being subset of $[0, 1]$, it has a lower bound 0 and upper bound 1 .

∴ We can say that Cantor set is bounded above and below.

∴ K is bounded.

Proposition 5:- Cantor set K is compact.

Proof:- Since the Cantor set is a closed subset of reals and it is also bounded set. (By Proposition 4)

∴ By Heine Borel Theorem

∴ K is compact.

Definition 6 Perfect set

- E is perfect set if E is closed and if every point of E is a limit point of E.

Proposition 7:- Cantor set K is perfect.

Proof:- To show that K is perfect set, since K is closed, it is sufficient to show that every x in K is limit of K.

Let $x \in K$ and S be any segment containing x.

Suppose $S = (x - \delta, x + \delta)$.

We choose n such that $\frac{1}{3^n} < \delta$.

Further, $x \in E_m$ for every $m \geq 1$.

∴ $x \in E_n$ for this n.

∴ $x \in J_{n,k}$ for some $k \leq 2^n$,

Where $J_{n,k}$ is the interval of E_n which contains x.

Now, we show that $J_{n,k} \subset S$.

If $y \in J_{n,k}$

∴ $|x - y| < \frac{1}{3^n} < \delta$

∴ $|x - y| < \delta$

∴ $y \in (x - \delta, x + \delta) = S$

∴ $y \in S$

∴ $y \in J_{n,k} \Rightarrow y \in S$

∴ $J_{n,k} \subset S$

Let x_n be an end point of $J_{n,k}$ such that $x_n \neq x$.

It follows from the construction of K that $x_n \in K \cap S - \{x\}$.



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$$\therefore K \cap S - \{x\} \neq \emptyset$$

Hence x is a limit point of K.

\therefore K is perfect.

Definition 8 Nowhere dense set

\therefore A set is said to be nowhere dense if its closure has empty interior.

Theorem 9 :- Cantor set K is nowhere dense in \mathbb{R} .

Proof:- We first prove that, no E_n contains an interval of length $> \frac{1}{3^n}$.

$$\text{Suppose } (a, b) \subset E_n = \bigcup_{k=1}^{2^n} J_{n,k}$$

Where $J_{n,k} = k^{th}$ remaining closed intervals in the construction of Cantor set.

Since, $J_{n,k}$ are disjoint closed intervals, $(a, b) \subset J_{n,k}$ for some $k \leq 2^n$.

$$\therefore \ell(a, b) = b - a \leq \ell(J_{n,k}) = \frac{1}{3^n}$$

$$\Rightarrow b - a \leq \frac{1}{3^n}$$

Thus if $(a, b) \subset K$ then $(a, b) \subset E_n$ for each $n \geq 1$.

This implies that $b - a \leq \frac{1}{3^n}$ for each $n \geq 1$.

$$\text{i.e. } b = a$$

$$\text{i.e. } (a, b) = \emptyset$$

Thus K does not contain any open interval i.e. K is nowhere dense.

Theorem 10 :- Lebesgue measure of Cantor set is zero i.e. $m(K) = 0$.

Proof:- We know that

$$m(I) = 1 \quad \text{-----(1)}$$

In the construction of Cantor set at first stage we divide $[0, 1]$ into three equal parts and removed middle third open interval $I_{1,1} = (\frac{1}{3}, \frac{2}{3})$.

The length of deleted interval $I_{1,1}$ is $\frac{1}{3}$ i.e. $\ell(I_{1,1}) = \frac{1}{3}$.

In second stage we again removed the two middle third open intervals i.e. $I_{1,2} = (\frac{1}{9}, \frac{2}{9})$ and $I_{2,2} = (\frac{7}{9}, \frac{8}{9})$.

\therefore The lengths of deleted intervals are $\ell(I_{1,2}) = \frac{1}{9}$ and $\ell(I_{2,2}) = \frac{1}{9}$.

\therefore The sum of length of deleted disjoint intervals = $\frac{1}{9} + \frac{1}{9}$.



In the next stage, we have deleted four open intervals i.e. $I_{1,3} = (\frac{1}{27}, \frac{2}{27})$, $I_{2,3} = (\frac{7}{27}, \frac{8}{27})$.

$I_{3,3} = (\frac{19}{27}, \frac{20}{27})$ and $I_{4,3} = (\frac{25}{27}, \frac{26}{27})$.

The length of these intervals are $\ell(I_{1,3}) = \frac{1}{27}$, $\ell(I_{2,3}) = \frac{1}{27}$, $\ell(I_{3,3}) = \frac{1}{27}$ and $\ell(I_{4,3}) = \frac{1}{27}$.

∴ The sum of length of these deleted intervals = $\frac{1}{27} + \frac{1}{27} + \frac{1}{27} + \frac{1}{27}$ and so on.

∴ The sum of lengths of deleted intervals in the geometric construction of K =

$$\begin{aligned} &= \frac{1}{3} + (\frac{1}{9} + \frac{1}{9}) + (\frac{1}{27} + \frac{1}{27} + \frac{1}{27} + \frac{1}{27}) + \dots\dots \\ &= \frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \dots\dots\dots \\ &= \frac{1}{3} \sum_{n=0}^{\infty} \frac{2^n}{3^n} \\ &= \frac{1}{3} (\frac{1}{1-\frac{2}{3}}) = \frac{1}{3} (\frac{3}{1}) = 1 \end{aligned}$$

∴ The sum of lengths of deleted intervals in the geometric construction of K = 1

∴ Measure of Cantor set K i.e. $m(K) = (\text{Measure of } I) - (\text{Measure of union of removed intervals})$

$$\therefore m(K) = 1 - 1$$

$$\therefore m(K) = 0$$

∴ Lebesgue measure of Cantor set is zero.

V. CONCLUSION

Cantor like sets have been constructed to give examples of uncountable sets with measure zero. These sets are found to have astonishing properties.

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