

Generalized Monotonicities and Its Applications to the System of General Variational Inequalities

Khushbu¹, Zubair Khan²

Research Scholar, Department of Mathematics, Integral University, Lucknow, Uttar Pradesh, India¹

Associate Professor, Department of Mathematics, Integral University, Lucknow, Uttar Pradesh, India²

ABSTRACT: In this paper, we introduce and study the system of general variational inequalities which is equivalent to the general variational inequality problem over the product of sets. The usual concept of monotonicity has been extended here. We establish existence results for the solution of general variational inequality problem over the product of sets in the setting of real Hausdorff topological vector space.

KEYWORDS: General variational inequalities, Monotonicity, Multivalued mapping, Convexity, Reflexive Banach space.

I. INTRODUCTION

Variational inequality theory has emerged as a powerful tool for a wide class of unrelated problems arising in various branches of physical, engineering, pure and applied sciences in unified and general framework. Variational inequalities have been extended and generalized in different directions by using novel and innovative techniques and ideas, both for their own sake and for their application.

In the last three decades, the Nash equilibrium problem [1] has been studied by many authors either by using Ky Fan minimax inequality [2] or by using fixed point technique. In 2000, Ansari and Yao [3] gave a new direction to solve the Nash equilibrium problem for non-differentiable functions. They introduced a system of optimization problems which includes the Nash equilibrium problem as a special case. They proved that every solution of system of generalized variational inequalities is also a solution of system of optimization problems for non-differentiable functions and also for non-differentiable and non-convex functions.

It is mentioned by J. P. Aubin in his book [4] that the Nash equilibrium problem [1, 5] for differentiable functions can be formulated in the form of a variational inequality problem over product of sets (for short, VIPPS). Not only the Nash equilibrium problem but also various equilibrium-type problems, like, traffic equilibrium, spatial equilibrium, and general equilibrium programming problems, from operations research, economics, game theory, mathematical physics, and other areas, can also be uniformly modeled as a VIPPS, see for example [6,7,8] and the references therein. Pang [8] decomposed the original variational inequality problem defined on the product of sets into a system of variational inequalities (for short, SVI), which is easy to solve, to establish some solution methods for VIPPS. Later, it was found that these two problems VIPPS and SVI are equivalent. In the recent past VIPPS or SVI has been considered and studied by many authors, see for example [3, 6, 7, 9, 10, 11] and references therein. Konnov [10] extended the concept of (pseudo) monotonicity and established the existence results for a solution of VIPPS. Recently, Ansari and Yao [12] introduced the system of generalized implicit variational inequalities and proved the existence of its solution. They derived the existence results for a solution of system of generalized variational inequalities and used their results as tools to establish the existence of a solution of system of optimization problems, which include the Nash equilibrium problem as a special case, for non-differentiable functions. Later Ansari and Khan [13, 14] also generalized these existing results.

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Motivated by the works given in [13,14], in this paper we formulate the problem of system of general variational inequalities and general variational inequality problem over the product of sets. It is noticed that every solution of general variational inequality problem over the product of sets (GVIPPS) is a solution of system of general variational inequalities (SGVI) and vice-versa. We extend various kinds of monotonicities defined by Ansari and Yao [12], Ansari and Zubair [13,14]. We adopt the technique of Yang and Yao [15] to establish the existence results for a solution of general variational inequality problem over the product of sets (GVIPPS) and hence a solution of system of general variational inequality problem (SGVIP).

II. PRELIMINARIES AND FORMULATION OF PROBLEM

Let I be a finite index set, $I = \{1, 2, \dots, n\}$. For each $i \in I$, let X_i be a topological vector space with its dual X_i^* , K_i a nonempty and convex subset of X_i , $K = \prod_{i \in I} K_i$, $X = \prod_{i \in I} X_i$, and $X^* = \prod_{i \in I} X_i^*$. We denote by $\langle \cdot, \cdot \rangle$ the pairing between X_i^* and X_i . For each $i \in I$, when X_i is a normed space, its norm is denoted by $\|\cdot\|_i$ and the product norm on X will be denoted by $\|\cdot\|$. For each $x \in X$, we write $x = (x_i)_{i \in I}$, where $x_i \in X_i$, that is, for each $x \in X$, $x_i \in X_i$ denotes the i th component of x . For each $i \in I$, let $f_i : K \rightarrow X_i^*$ be a nonlinear operator such that

$$f(x) = (f_i(x))_{i \in I} \tag{1}$$

The following system of variational inequality problem (SVIP) and variational inequality problem over product of sets (VIPPS) was introduced and studied by Ansari and Khan [13]:

$$\begin{aligned} \text{(SVIP)} \quad & \text{Find } \bar{x} \in K \text{ such that for each } i \in I \langle f_i(\bar{x}), y_i - \bar{x}_i \rangle \geq 0 \text{ for all } y_i \in K_i \\ \text{(VIPPS)} \quad & \text{Find } \bar{x} \in K \text{ such that } \sum_{i \in I} \langle f_i(\bar{x}), y_i - \bar{x}_i \rangle \geq 0 \text{ for all } y_i \in K_i, i \in I \end{aligned} \tag{2}$$

Now let $F_i : K \rightarrow 2^{X_i^*}$ be a multivalued map with nonempty values so that if we set

$$F = (F_i : i \in I) \tag{3}$$

then $F : K \rightarrow 2^{X^*}$ is a multivalued map with nonempty values. Let $A_i : X_i^* \rightarrow X_i^*$ be another mapping such that $A = (A_i : i \in I)$ and $A : X^* \rightarrow X^*$.

We consider the following more general problem of system of general variational inequalities:

$$\text{(SGVIP)} \quad \begin{cases} \text{Find } \bar{x} \in K \text{ and } \bar{u} \in F(\bar{x}) \text{ such that} \\ \langle A_i(\bar{u}_i), y_i - \bar{x}_i \rangle \geq 0 \text{ for all } y_i \in K_i, i \in I \end{cases}$$

where u_i is the i th component of u .

Next we introduce the following general variational inequality problem over the product of sets:

$$\text{(GVIPPS)} \quad \begin{cases} \text{Find } \bar{x} \in K \text{ and } \bar{u} \in F(\bar{x}) \text{ such that} \\ \sum_{i \in I} \langle A_i(\bar{u}_i), y_i - \bar{x}_i \rangle \geq 0, \text{ for all } y_i \in K_i, i \in I \end{cases}$$

where u_i is the i th component of u .

It is easy to see that every solution of (GVIPPS) is a solution of (SGVIP) and vice-versa.

Now we shall use the following existence results of variational inequality problem over product of sets (VIPPS) and system of variational inequality problem (SVIP) by Ansari and Zubair [13]:

Theorem 2.1 [13] For each $i \in I$, let K_i be a nonempty, compact and convex subset of X_i , and f defined by (1) be hemicontinuous and densely relatively pseudomonotone on K . Then (VIPPS) has a solution and hence (SVIP) has a solution.

Corollary 2.1 [13] For each $i \in I$, let K_i be a nonempty, compact and convex subset of X_i , and f , defined by (1) be hemicontinuous and relatively quasimonotone on K such that K_f is segment-dense in K . Then (VIPPS) has a solution and hence (SVIP) has a solution.

Theorem 2.2 [13] For each $i \in I$, let K_i be a nonempty and convex subset of a real topological vector space (not necessarily Hausdorff) X_i . Let f , defined by (1) be relatively B-pseudomonotone such that for each $A \in F(K)$, $x \mapsto \sum_{i \in I} \langle f_i(x), y_i - x_i \rangle$ is upper semicontinuous on coA . Assume that there exists a nonempty, closed and compact subset D of K and an element $\tilde{y} \in D$ such that for all $x \in K \setminus D$, $\sum_{i \in I} \langle f_i(x), \tilde{y}_i - x_i \rangle < 0$. Then (VIPPS) has a solution and hence (SVI) has a solution.

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Corollary 2.2 [13] For each $i \in I$, let K_i be a nonempty, closed and convex subset of a real reflexive Banach space X_i . Let f , defined by (1) be relatively demimonotone such that for each $A \in F(K)$, $x \mapsto \sum_{i \in I} \langle f_i(x), y_i - x_i \rangle$ is upper semicontinuous on coA . Assume that there exists $\tilde{y} \in K$ such that

$$\lim_{\|x\| \rightarrow \infty, x \in K} \sum_{i \in I} \langle f_i(x), \tilde{y}_i - x_i \rangle < 0$$

Then (VIPPS) has a solution and hence (SVI) has a solution.

Now we define different kinds of relatively monotonicities for the multivalued map. The definitions here are the generalizations of the definitions given by Ansari and Zubair [13].

Definition 2.1 [16] A subset K^0 of K is said to be *segment-dense* in K if for all $x \in K$, there can be found $x^0 \in K^0$ such that x is a cluster point of the set $[x, x^0] \cap K^0$, where $[x, x^0]$ denotes the line segment joining x and x^0 including end points .

Definition 2.2 Let $A : X^* \rightarrow X^*$ be the mapping. The multivalued map $F : K \rightarrow 2^{X^*}$ defined by (3) is said to be:

(a) *generalized relatively pseudomonotone with respect to A* at $y \in K$ if for all $x \in K$ and for all $u \in F(x)$, $v \in F(y)$, we have

$$\sum_{i \in I} \langle A_i(u_i), y_i - x_i \rangle \geq 0 \Rightarrow \sum_{i \in I} \langle A_i(v_i), y_i - x_i \rangle \geq 0$$

and *relatively strictly pseudomonotone with respect to A* at $y \in K$ if the second inequality is strict for all $x \neq y$, where u_i and v_i are the i th components of u and v respectively.

(b) *generalized relatively quasimonotone with respect to A* at $y \in K$ if for all $x \in K$ and for all $u \in F(x)$, $v \in F(y)$, we have

$$\sum_{i \in I} \langle A_i(u_i), y_i - x_i \rangle > 0 \Rightarrow \sum_{i \in I} \langle A_i(v_i), y_i - x_i \rangle \geq 0$$

where u_i and v_i are the i th components of u and v respectively ;

(c) *generalized densely relatively pseudomonotone with respect to A* (respectively, *generalized densely relatively strictly pseudomonotone with respect to A*) on K if there exists a segment dense subset $K^0 \subseteq K$ such that F is generalized relatively pseudomonotone with respect to A (respectively, generalized relatively strictly pseudomonotone with respect to A) on K^0 .

If F is generalized relatively pseudomonotone with respect to A (respectively, generalized relatively strictly pseudomonotone with respect to A and generalized relatively quasimonotone with respect to A) at each $y \in K$, then we say that it is *generalized relatively pseudomonotone with respect to A* (respectively, *generalized relatively strictly pseudomonotone with respect to A* and *generalized relatively quasimonotone with respect to A*) on K .

Definition 2.3 The multivalued map $F : K \rightarrow 2^{X^*}$ defined by (3) is said to be *hemicontinuous* if for all $x, y \in K$ and $\lambda \in [0, 1]$, the mapping $\lambda \mapsto \langle F(x + \lambda z), z \rangle$ with $z = y - x$ is upper semicontinuous at 0.

Now we define the generalized relatively B-pseudomonotonicity with respect to A and generalized relatively demimonotonicity with respect to A which reduce to the definition of a pseudomonotone map, introduced by Brezis [17].

Definition 2.4 The multivalued map $F : K \rightarrow 2^{X^*}$ defined by (3) is said to be *generalized relatively B-pseudomonotone with respect to A* (respectively, *generalized relatively demimonotone with respect to A*) if for each $x \in K$ and every net $\{x^\alpha\}_{\alpha \in I}$ in K converging to x (respectively, weakly to x) with

$$\lim \inf_\alpha [\sum_{i \in I} \langle A_i(u_i^\alpha), x_i - x_i^\alpha \rangle] \geq 0, \quad \text{for all } u_i^\alpha \in F_i(x^\alpha)$$

we have, for all $u_i \in F_i(x)$

$$\sum_{i \in I} \langle A_i(u_i), y_i - x_i \rangle \geq \lim \sup_\alpha [\sum_{i \in I} \langle A_i(u_i^\alpha), y_i - x_i^\alpha \rangle] \quad \text{for all } u_i^\alpha \in F_i(x^\alpha) \text{ and } y \in K.$$

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Definition 2.5 Let Z be topological vector space with its topological dual Z^* and U a nonempty subset of Z . Let $G : U \rightarrow 2^{Z^*}$ be a multivalued map and $g : U \rightarrow Z^*$ a single valued map. The map g is called a *selection* of G on U if $g(x) \in G(x)$ for all $x \in U$. Furthermore, the map g is called a *continuous selection* of G on U if it is continuous on U and a selection of G on U .

For further detail on continuous selections of multivalued maps, we refer to [18].

It follows that if $f : K \rightarrow X^*$ defined by (1) is a selection of $F : K \rightarrow 2^{X^*}$ defined by (3), then for each $i \in I$, $f_i : K \rightarrow X_i^*$ is a selection of $F_i : K \rightarrow 2^{X_i^*}$ on K .

Lemma 2.1 Let $f : K \rightarrow X^*$, defined by (1), be a selection of multivalued map $F : K \rightarrow 2^{X^*}$ defined by (3) on K . Then:

- (a) if F is generalized relatively pseudomonotone with respect to A , then f is relatively pseudomonotone;
- (b) if F is generalized densely relatively pseudomonotone with respect to A , then f is densely relatively pseudomonotone ;
- (c) if F is generalized relatively B-pseudomonotone with respect to A (respectively, generalized relatively demimonotone with respect to A) then f is relatively B-pseudomonotone (respectively, relatively demimonotone).

III. EXISTENCE RESULTS

We adopt the technique of Yang and Yao [15] and also use the existence results for the solution of system of variational inequality problem (SVIP) and variational inequality problem over product of sets (VIPPS) by Ansari and Zubair [13] to derive the existence results for solution of general variational inequality problem over product of sets (GVIPPS) and system of general variational inequality problem (SGVIP).

First we will prove the following lemma:

Lemma 3.1 If $f : K \rightarrow X^*$, defined by (1) is a selection of $F : K \rightarrow 2^{X^*}$, defined by (3) on K and $\bar{x} \in K$ is a solution of (VIPPS), then (\bar{x}, \bar{u}) is a solution of (GVIPPS) with $\bar{u}_i = f_i(\bar{x})$ for all $i \in I$.

Proof. Assume that $\bar{x} \in K$ is a solution of (VIPPS). Then

$$\sum_{i \in I} \langle f_i(\bar{x}), y_i - \bar{x}_i \rangle \geq 0, \text{ for all } y_i \in K_i, i \in I$$

Let $\bar{u}_i = f_i(\bar{x})$, so that $\bar{u} \in f(\bar{x})$. Since f is a selection of F , we have $\bar{u} \in F(\bar{x})$ such that

$$\sum_{i \in I} \langle A_i(\bar{u}_i), y_i - \bar{x}_i \rangle \geq 0, \text{ for all } y_i \in K_i, i \in I$$

Hence (\bar{x}, \bar{u}) is a solution of (GVIPPS).

Rest of the section, we assume that the pairing $\langle \cdot, \cdot \rangle$ is continuous.

First we prove some existence results for a solution of (GVIPPS) under generalized densely relatively pseudomonotonicity with respect to A and generalized relatively quasimonotonicity with respect to A .

Theorem 3.1 For each $i \in I$, let K_i be a nonempty, compact and convex subset of a real Hausdorff topological vector space X_i . Assume that

- (i) $F : K \rightarrow 2^{X^*}$, defined by (3) is a generalized densely relatively pseudomonotone multivalued map with respect to A on K ;
- (ii) there exists a continuous selection $f : K \rightarrow X^*$, defined by (1), of F on K .

Then (GVIPPS) has a solution.

Proof. It follows from condition (ii) that there exists a continuous function f such that $f(x) \in F(x)$ for all $x \in K$, that is, for each $i \in I$, $f_i(x) \in F_i(x)$ for all $x \in K$. By lemma 2.1, f is relatively densely pseudomonotone and hemicontinuous. Then all the conditions of Theorem 2.1 are satisfied and hence there exists a solution $\bar{x} \in K$ of (VIPPS). For each $i \in I$, let $\bar{u}_i = f_i(\bar{x}) \in F_i(\bar{x})$. Then Lemma 3.1 implies that (\bar{x}, \bar{u}) is a solution of (GVIPPS).

Theorem 3.2 For each $i \in I$, let K_i be a nonempty, compact and convex subset of a real Hausdorff topological vector space X_i . Assume that:

- (i) the multivalued map $F : K \rightarrow 2^{X^*}$, defined by (3) is generalized relatively quasimonotone with respect to A on K ;

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- (ii) there exists a continuous selection $f : K \rightarrow X^*$, defined by (1), of F on K such that K_f is segment-dense in K

Then (GVIPPS) has a solution.

Proof. It follows from condition (ii) that there exists a continuous function f such that $f(x) \in F(x)$ for all $x \in K$, that is, for each $i \in I$, $f_i(x) \in F_i(x)$ for all $x \in K$. By lemma 2.1, f is relatively quasimonotone and hemicontinuous. Then all the conditions of Corollary 2.1 are satisfied and hence there exists a solution $\bar{x} \in K$ of (VIPPS). For each $i \in I$, let $\bar{u}_i = f_i(\bar{x}) \in F_i(\bar{x})$. Then Lemma 3.1 implies that (\bar{x}, \bar{u}) is a solution of (GVIPPS).

Now we prove the existence of a solution of (GVIPPS) under generalized relatively B-pseudomonotonicity with respect to A .

Theorem 3.3 For each $i \in I$, let K_i be a nonempty and convex subset of a real topological vector space (not necessarily, Hausdorff) X_i . Assume that:

- (i) the multivalued map $F : K \rightarrow 2^{X^*}$, defined by (3), is a generalized relatively B-pseudomonotone with respect to A on K
- (ii) there exists a continuous selection $f : K \rightarrow X^*$, defined by (1), of F on K ;
- (iii) there exists a nonempty, closed and compact subset D of K and an element $\tilde{y} \in D$ such that for all $x \in K \setminus D$, $\sum_{i \in I} \langle f_i(x), \tilde{y}_i - x_i \rangle < 0$.

Then (GVIPPS) has a solution.

Proof. From condition (ii), there exists a continuous functions $f : K \rightarrow X^*$ such that $f(x) \in F(x)$, for all $x \in K$, that is, for each $i \in I$, $f_i(x) \in F_i(x)$ for all $x \in K$. Since the pairing $\langle \cdot, \cdot \rangle$ is continuous, we have the map $x \mapsto \sum_{i \in I} \langle f_i(x), y_i - x_i \rangle$ is continuous on K . By lemma 2.1, f is relatively B-pseudomonotone. Therefore from Theorem 2.2, there exists a solution $\bar{x} \in K$ of (VIPPS). For each $i \in I$, let $\bar{u}_i = f_i(\bar{x}) \in F_i(\bar{x})$. Then by Lemma 3.1, (\bar{x}, \bar{u}) is a solution of (GVIPPS).

Corollary 3.1 For each $i \in I$, let K_i be a nonempty and convex subset of a real reflexive Banach space X_i . Assume that:

- (i) the multivalued map $F : K \rightarrow 2^{X^*}$, defined by (3) is generalized relatively demimonotone with respect to A on K ;
- (ii) there exists a continuous selection $f : K \rightarrow X^*$, defined by (1), of F on K ;
- (iii) there exists $\tilde{y} \in K$ such that

$$\lim_{\|x\| \rightarrow \infty, x \in K} \sum_{i \in I} \langle f_i(x), \tilde{y}_i - x_i \rangle < 0$$

Then (GVIPPS) has a solution.

Proof. From condition (ii), there exists a continuous functions $f : K \rightarrow X^*$ such that $f(x) \in F(x)$, for all $x \in K$, that is, for each $i \in I$, $f_i(x) \in F_i(x)$ for all $x \in K$. Since the pairing $\langle \cdot, \cdot \rangle$ is continuous, we have the map $x \mapsto \sum_{i \in I} \langle f_i(x), y_i - x_i \rangle$ is continuous on K . By lemma 2.1, f is relatively demimonotone. Therefore from Corollary 2.2, there exists a solution $\bar{x} \in K$ of (VIPPS). For each $i \in I$, let $\bar{u}_i = f_i(\bar{x}) \in F_i(\bar{x})$. Then by Lemma 3.1, (\bar{x}, \bar{u}) is a solution of (GVIPPS).

IV. CONCLUSION

We have successfully established the existence results for the solution of general variational inequality problem over the product of sets (GVIPPS) and hence a solution of system of general variational inequality problem (SGVIP) by using the existence results for the solution of variational inequality problem over product of sets (VIPPS) and system of variational inequality problem (SVIP). Our results can be used to study the optimization problem for nondifferentiable functions.

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