Lowener, Star and “θ” Partial Ordering of S-Unitary Matrices

A. Govindarasu
Associate Professor, Dept. of Mathematics, A.V.C. College (Autonomous), Mannampandal, Mayiladuthurai, Tamilnadu, India

ABSTRACT: Some results relating to ‘θ’ partial ordering on s-unitary matrices are given. Theorems relating to Lowener, Star and ‘θ’ partial ordering on s-unitary matrices are obtained. Further ‘θ’ partial ordering preserved under s-unitary similarity has been proved.

MATHEMATICS SUBJECT CLASSIFICATION: 15A45,15B57,15A09

KEYWORDS: Lowener partial ordering, Partial ordering, Star partial ordering, s-unitary matrix s-unitary similarity, ‘θ’ partial ordering.

I. INTRODUCTION

Some characterizations of the star partial ordering and rank subactivity for matrices was discussed by Hartwig and Styan in [3]. Jurgen Grob observed some remarks on partial ordering of Hermitian Matrices in [5]. Xifu Liu and Hu yang discussed some results on the partial ordering of block matrices [8]. In [4], Jorma K. Merikoski and Xiaogi Liu have developed star partial ordering on Normal Matrices. Krishnamoorthy and Govindarasu introduced the concept of ‘θ’ partial ordering [7].

II. PRELIMINARIES AND NOTATIONS:

Let \( C_{nn} \) be the space of \( nn \) complex matrices of order \( n \). For \( A \in C_{nn} \) let
\[
A^T, A^*, A^S, A^\theta = A^\theta
\]
denote transpose, conjugate, conjugate transpose, secondary transpose, conjugate secondary transpose of a matrix \( A \) respectively. Anna Lee [1] has initiated the study of secondary symmetric matrices. Also she has shown that for a complex matrix \( A \), the usual transpose \( A^T \) and secondary transpose \( A^\bar{S} \) are related as
\[
A^S = VA^TV \quad \text{where} \quad V \quad \text{is the associated permutation matrix whose elements on the secondary diagonal are 1 and other elements are zero. Also} \quad A^\bar{S} = (c_{ij}) \quad \text{where} \quad c_{ij} = d_{n-j+1, n-i+1} \quad \text{[2]} \quad \text{and} \quad A^\bar{S} = VA^S = A^\theta . \quad \text{Also} \quad V \quad \text{satisfies the following properties.} \quad V^T = V^\theta = V = V^* = V \quad \text{and} \quad V^2 = I .
\]
A matrix \( A \in C_{nn} \) is said to be s-unitary if \( A^\theta A = AA^\theta = I \) [6].

Definition 2.1: [Lowener Partial Ordering]

For \( A, B \in C_{nn} \), we define \( A \geq_L B \) if \( A - B \geq 0 \).
Definition 2.2: [Star Partial Ordering]

For $A, B \in C_{nxn}$, we define $A \succ^* B$ if $B^* B = B^* A$ and $BB^* = AB^*$.

III. LOWENER, STAR AND “$\theta$” PARTIAL ORDERING OF S-UNITARY MATRICES

Definition 3.1 [“$\theta$” PARTIAL ORDERING] [7]

Let $A, B \in C_{nxn}$, $A \preceq_\theta B$ iff $A^\theta A = A^\theta B$ and $AA^\theta = BA^\theta$.

Theorem 3.2

Let $VA \preceq_\theta AV$ If $A$ is s-unitary then $A$ is unitary.

Proof:

Let $VA \preceq_\theta AV$

$\Rightarrow (VA)^\theta VA = (VA)^\theta AV$ [By Definition 3.1]

$\Rightarrow A^\theta VVA = A^\theta VAV$

$\Rightarrow A^\theta A = A^\theta VAV$

$\Rightarrow I = A^\theta VAV = VA^* VVAV$ [Since $A$ is s-unitary]

$\Rightarrow I = VA^* AV$

$\Rightarrow VIV = A^* A$

$\Rightarrow I = A^* A$ \quad \therefore (3.2.1)

$VA \preceq_\theta AV \Rightarrow VA(VA)^\theta = AV(VA)^\theta$ [By definition 4.3.1]

$\Rightarrow VAA^\theta V = AVA^\theta V$ [Since $A$ is s-unitary]

$\Rightarrow VIV = AVA^* VV$

$\Rightarrow I = AA^*$

Therefore $A^* A = I$ \quad \therefore (3.2.2)

From (3.2.1) and (3.2.2) we have $A^* A = AA^* = I$

Therefore $A$ is unitary.

Theorem 3.3

If $A$ and $B$ are s-unitary matrices and $AV = VA, VB = BV$ then $A \preceq_\theta B \Rightarrow AV \preceq_\theta BV$

Proof:

Given $A \preceq_\theta B$
\[ A \preceq B \Rightarrow A^\theta A = A^\theta B \quad \text{and} \quad AA^\theta = BA^\theta \]  

[By Definition.3.1]

By definition \( A^\theta A = A^\theta B \):

\[ \Rightarrow VA^*VA = VA^*VB \]  

[Since \( A \) is s-unitary]

\[ \Rightarrow (AV)^*VA = (AV)^*VB \]  

\[ \Rightarrow (AV)^*AV = (AV)^*BV \quad \text{---------}(3.3.1) \]  

[Since \( AV = VA, VB = BV \)]

By definition \( AA^\theta = BA^\theta \):

\[ \Rightarrow AVA^*V = BVA^*V \]  

[Since \( A \) is s-unitary]

Premultiplying and postmultiplying the above equation by \( V \) we get,

\[ \Rightarrow VAVA^*VV = VBVA^*VV \]  

\[ \Rightarrow VAVA^* = VBVA^* \]  

[Since \( V^2 = I \)]

\[ \Rightarrow VA(AV)^* = VB(AV)^* \]  

\[ \Rightarrow AV(AV)^* = BV(AV)^* \quad \text{---------}(3.3.2) \]  

[Since \( AV = VA, VB = BV \)]

From (3.3.1.) and (3.3.2) we have \( AV \preceq BV \).

Therefore \( A \preceq B \Rightarrow AV \preceq BV \).

**Theorem 3.4**

If \( A \) and \( B \) are s-unitary matrices and \( AV = VA, VB = BV \) then \( A \preceq B \Rightarrow AV \preceq BV \).

**Proof:** \( A \preceq B \Rightarrow A^*A = A^*B \) and \( AA^* = BA^* \)

Take \( A^*A = A^*B \):

\[ \Rightarrow (VA^*V)VA = (VA^*V)VB \]  

\[ \Rightarrow A^\theta VA = A^\theta VB \]  

[Since \( A \) is s-unitary]

\[ \Rightarrow VA^\theta VA = VA^\theta VB \]  

\[ \Rightarrow (AV)^\theta VA = (AV)^\theta VB \]  

\[ \Rightarrow (AV)^\theta AV = (AV)^\theta BV \quad \text{---------}(3.4.1) \]

Take \( AA^* = BA^* \):

\[ \Rightarrow AVA^* = BVVA^* \]  

\[ \Rightarrow AV(VA^*V) = BV(VA^*V) \]  

[Since \( A \) is s-unitary]

\[ \Rightarrow AV(A^\theta V) = BV(A^\theta V) \]  

[Since \( A \) is s-unitary]
\[
\Rightarrow AV(A^\theta V) = BV(A^\theta V)
\]
\[
\Rightarrow AV(VA)^\theta = BV(VA)^\theta
\]
\[
\Rightarrow AV(\theta) = BV(\theta) \quad \text{(3.4.2)} \quad \text{[Since } AV = VA, VB = BV]\]

From (3.4.1) and (3.4.2) we get \( AV \lesssim_{\theta} BV \). Therefore \( A \lesssim B \Rightarrow AV \lesssim_{\theta} BV \).

**Theorem 3.5**

If \( VA \lesssim_{\theta} VB \) and \( VA \lesssim_{\theta} VB \), then \( VA = VB \).

**Proof:**

Given \( VA \lesssim_{\theta} VB \),

\( VA \lesssim_{\theta} VB \Rightarrow (VA)^\theta VA = (VA)^\theta VB, \) and \( VA(\theta) = VB(\theta) \).

Given \( VA \lesssim_{\theta} VB \Rightarrow (VA)^\theta VA = (VA)^\theta VB \)

\( \Rightarrow (VA)^\theta (VB - VA) = 0 \quad \text{(3.5.1)} \)

Taking conjugate secondary transpose on both sides of (3.5.1) we get

\( (VB - VA)^\theta (VA) = 0 \)

Since \( VA \lesssim_{\theta} VB \) we have \( (VB - VA)^\theta VB = V(VB - VA)V \)

\( (3.5.1) \Rightarrow VV(VB - VA)^\theta VVA = 0 \)

\( \Rightarrow V(VB - VA)VVA = 0 \)

\( \Rightarrow I(VB - VA)VVA = 0 \) \quad \text{[Since } V^2 = I]\)

\( \Rightarrow VBVA - (VA)^2 = 0 \)

\( \Rightarrow (VB - VA)VVA = 0 \)

\( \Rightarrow VBVA = (VA)^2 \)

\( \Rightarrow VB = VA \) Hence the theorem is proved by this paper.

**Theorem 3.6**

For \( A, B \in C_{n\times n} \) and \( V \) is the permutation matrix with units in the secondary diagonal.

\( A \lesssim_{\theta} B \Leftrightarrow VA \lesssim_{\theta} VB \Leftrightarrow AV \lesssim_{\theta} BV \)

**Proof:**

\( A \lesssim_{\theta} B \Rightarrow A^\theta A = A^\theta B \) and \( AA^\theta = BA^\theta \) \quad \text{[By Definition 3.1]} \)

\( \Rightarrow A^\theta VVA = A^\theta VVB \) and \( VAA^\theta V = VBA^\theta V \)
\[ (VA)^{\theta}VA = (VA)^{\theta}VB \quad \text{and} \quad VA(VA)^{\theta} = VB(VA)^{\theta} \]

Therefore \[ A \prec^{\theta} B \Leftrightarrow VA \prec^{\theta} VB \]

Similarly we can prove \[ A \prec^{\theta} B \Leftrightarrow AV \prec^{\theta} BV \]

Hence \[ A \prec^{\theta} B \Leftrightarrow VA \prec^{\theta} VB \Leftrightarrow AV \prec^{\theta} BV \]

**Theorem 3.7**
Let \( A, B \in C_{n \times n} \), *θ* partial ordering is preserved under \( s \)-unitary similarity.

**Proof:** It is enough to prove \[ A \prec^{\theta} B \Leftrightarrow VP^{-1}VAP \prec^{\theta} VP^{-1}VBP \] for some \( s \)-unitary matrix \( P \)

\[
A \prec^{\theta} B \Leftrightarrow VA \prec^{\theta} VB \\
\Leftrightarrow P^{\theta}VAP \prec^{\theta} P^{\theta}VBP \\
\Leftrightarrow VP^{\theta}VAP \prec^{\theta} VP^{\theta}VBP \\
\Leftrightarrow (VP^{-1}V)AP \prec^{\theta} (VP^{-1}V)BP \\
\Leftrightarrow C \prec^{\theta} D \quad \text{where} \quad C = (VP^{-1}V)AP \quad \text{and} \quad D = (VP^{-1}V)BP
\]

Therefore *θ* partial ordering is preserved under \( s \)-unitary similarity

**IV. CONCLUSION**

Results relating to *θ* partial ordering on \( s \)-unitary matrices are discussed. Relation between Lowener, Star and *θ* partial ordering on \( s \)-unitary matrices are derived. Also shown *θ* partial ordering preserved under \( s \)-unitary similarity. This concept may be applied to any other matrices.

**REFERENCES**


BIOGRAPHY

I, Dr. A. Govindarasu presently working as Associate professor in Mathematics at A.V.C.College (Autonomous), Mannampandal, Mayiladuthurai. I have 28 years of teaching experience till date. I had specialized in Linear Algebra, Graph Theory, Differential Equations, Integral equations. I did my Ph.D on “CONTRIBUTIONS TO THE STUDY ON S-UNITARY MATRICES”. To my credit I have published five international papers. I also served as coordinator at A.V.C.College (Autonomous), for Earn-While-Learn Scheme of Bharathidasan University and presently holding additional post of Assistant Controller of Examinations at A.V.C.College (Autonomous), Mayiladuthurai.