Some Euler Sequences Spaces in Non-Archimedean Fields

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ABSTRACT: The aim of this paper is to introduce some Euler sequence spaces in a Complete, Non-trivially valued, Non-Archimedean fields and prove a few inclusion theorems on it.

KEYWORDS: Non-Archimedean fields, Euler sequence space, sequence space.

I. INTRODUCTION

Throughout this paper, \( K \) denotes complete, non-trivially valued, non-archimedean fields. Infinite matrices and sequences considered in the sequel have entries in \( K \). A general reference for the study of summability methods in classical one is [9]. For analysis, in non-archimedean fields, see [3].

If \( A = (a_{n,k}) \), \( a_{n,k} \in K \), \( n,k = 0,1,2,... \) is an infinite matrix, then the \( A \)-transform of a sequence \( x = \{x_k\} \), \( x_k \in K \), \( k = 0,1,2,... \), is the sequence, where [16]

\[
(Ax)_n = \sum_{k=0}^{n} a_{nk}x_k , \quad n = 0,1,2,... \]

it being assumed that the series on the right converge.

In 1969, Maddox [4] studied Continuous and Kothe-Toeplitz duals of certain sequence spaces and later he worked with Lascarides [5] to determine the matrices of the classes \( (\ell(p),\ell(\infty)),(\ell(p),c) \) and \( (\ell(\infty),\ell(\infty)) \). In classical analysis, Nanda[8] studied some sequence spaces over valued fields [10]. Wang [11] studied sequence spaces for Nörlund (1978). In 1981, Ahmad and Saraswat [10] obtained the necessary and sufficient conditions to characterize the matrices of the classes \( c_0 \), \( c_0 \), \( c_\infty \) and \( \ell_p \). Altay et.al. [1] also introduced an Euler sequence spaces \( e_0 \) and \( e_\infty \) of non-absolute type that are \( BK \)-spaces and constructed the bases for the duals of the spaces \( e_0 \) and \( e_\infty \) [13]. Finally, they established the necessary and sufficient conditions of an infinite matrix belonging to the classes \( (e_0^r,\ell_p) \) and \( (e_\infty^r,\ell_p) \), where \( 1 \leq p \leq \infty \).

We introduce some Euler sequence spaces and prove a few inclusion theorems over Non-Archimedean Fields [13].

If \( x = \{x_k\} \) is a sequence defined over \( K \), then the sequence spaces are:

\[
c = \left\{ x = \{x_k\} \in K : \lim_{k \to \infty} x_k \text{ exists} \right\} , \quad \ell_p = \left\{ x = \{x_k\} \in K : \sum_{k=0}^{n} |x_k|^p < \infty , \quad p \geq 1 \right\}
\]

We introduce some Euler sequence spaces over \( K \) as [14]
The Euler matrix is defined by

\[ e^r_c = \left\{ x = \{x_k\} \in K : \lim_{n \to \infty} \sum_{k=0}^{n} nC_k r^k (1-r)^{n-k} x_k \text{ exists} \right\} \]

\[ e^r_p = \left\{ x = \{x_k\} \in K : \sum_{k=0}^{n} nC_k r^k (1-r)^{n-k} x_k \right\} < \infty \}

\[ e^r_{\infty} = \left\{ x = \{x_k\} \in K : \sup_{n,k} nC_k r^k (1-r)^{n-k} x_k \right\} < \infty \}

The Euler matrix \( e^r_{n,k} \) is defined by

\[ e^r_{n,k} = \begin{cases} nC_k r^k (1-r)^{n-k}, & k \leq n \\ 0, & k > n \end{cases} \]

for all \( k, n = 0, 1, 2, \ldots \), and the method \( E^r \) is regular for \( |1-r| < 1 \). Atlay et al [11] redefined the spaces \( e^r_p \), \( e^r_c \) and \( e^r_{\infty} \) as [11]

\[ e^r_p = (\ell^r_p) E^r e^r_c = (c) E^r \text{ and } e^r_{\infty} = (\ell^r_{\infty}) E^r \]

**Theorem 2.** The sets \( e^r_p \), \( e^r_c \) and \( e^r_{\infty} \) are linear spaces with co-ordinatewise addition and scalar multiplication that are BK- spaces with norm \( \|y\|_{e^r} = \|x\|_{e^r} = \|E^r x\|_{e^r} \), where \( 1 \leq p \leq \infty \)[12].

**Proof** Since (1) holds and \( c, \ell^r_{\infty}, \ell^r_p \) are the BK-spaces with respect to their norms in \( K \) and \( e^r_{n,k} \neq 0 \), \( e^r_{n,k} = 0 \) (\( k > n \)), for \( k, n = 0, 1, 2, \ldots \), Wilansky [10] gives the fact that the spaces \( e^r_p \), \( e^r_c \) and \( e^r_{\infty} \) are the BK- spaces, \( 1 \leq p \leq \infty \).

**Definition 1.** Let \( y_n = \sum_{k=0}^{n} e^r_{n,k} x_k \) be Euler transform \( [E, r] \) of sequence \( \{x_k\} \), where \( [E, r] = (e^r_{n,k}) \)

which we define by[09]

\[ y_n = [E, r] x_k = \sum_{k=0}^{n} nC_k r^k (1-r)^{n-k} x_k \]
II. SOME INCLUSION THEOREMS ON EULER SEQUENCE SPACES

Theorem 3. The inclusion $\ell_p \subseteq e_p^r$ strictly holds for $1 \leq p \leq \infty$

Proof. Let us take any $x \in \ell_p$. Now

$$y_k = \sum_{j=0}^{k} kC_j r^j (1-r)^{k-j} x_j$$

$$|y_k| = \left| \sum_{j=0}^{k} kC_j r^j (1-r)^{k-j} x_j \right|$$

$$\leq \max \left\{ |kC_0| |r|^0 |1-r|^k |kC_1| |r| |1-r|^{k-1}, \ldots, |kC_k| |r|^k |1-r|^0 \right\}$$

$$= |r|^k |x_k|$$

since $|r| = 1$

$$< |x_k|$$

since $|r| = 1$

$$|y_k|^p < |x_k|^p$$

$$\sum_{k=0}^{n} |y_k|^p < \sum_{k=0}^{n} |x_k|^p$$

This shows that $\ell_p \subseteq e_p^r$, $1 \leq p \leq \infty$

Theorem 4. The inclusion $c \subseteq e_p^r$ strictly holds.

Proof. Let us take $x \in c$. Now

$$y_n = \sum_{k=0}^{n} nC_k r^k (1-r)^{n-k} x_k$$

$$\lim_{n \to \infty} y_n = \lim_{n \to \infty} \sum_{k=0}^{n} nC_k r^k (1-r)^{n-k} x_k$$

$$\lim_{n \to \infty} |y_n| = \lim_{n \to \infty} \left| \sum_{k=0}^{n} nC_k r^k (1-r)^{n-k} x_k \right|$$

$$\leq \lim_{n \to \infty} \left( |r|^0 (1-r)^n |x_0|, |r|^1 (1-r)^{n-1} |x_1|, \ldots, |r|^n (1-r)^0 |x_n| \right)$$

$$= \lim_{n \to \infty} |r|^n |x_n|$$

$$= \lim_{n \to \infty} |x_n|, \quad (|r| = 1)$$
Theorem 5. The inclusion \( \ell_\infty \subset e^r_\infty \) hold strictly.

Proof. Let us take \( x \in \ell_\infty \). Now,

\[
 y_n = \sum_{k=0}^{n} nC_k r^k (1-r)^{n-k} x_k
\]

\[
 |y_n| \leq \max \left\{ \left| nC_0 r^0 (1-r)^{n} x_0 \right|, \left| nC_1 r^1 (1-r)^{n-1} x_1 \right|, \ldots, \left| nC_n r^n (1-r)^{0} x_n \right| \right\}
\]

\[
 = |r|^n |x_n| < |x_n|, (|r| = 1)
\]

\[
 \sup_n |y_n| < \sup_n |x_n|
\]

\[
 \Rightarrow x \in e^r_\infty \text{, i.e., } \ell_\infty \subset e^r_\infty .
\]

REFERENCES

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