

# Hausdorff and Box Dimensions of Cantor Like Sets

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**ABSTRACT:** The Cantor like sets  $S$  have been constructed in [3], for any sequence  $\{\epsilon_n\}$  with  $0 < \epsilon_n < 1$  with the help of sequence of sets  $\{S_n\}$  of subsets of  $[0, 1]$  such that  $S_n \supset S_{n+1}$ ,  $S = \bigcap S_n$  and  $m(S) = \prod_{n=1}^{\infty} (1 - \epsilon_n)$ . Further  $\sum \epsilon_n = \infty$  iff  $m(S) = 0$ .

In this paper we determine the estimates for the Hausdorff and upper / lower Box dimension of such Cantor like sets  $S$ . In particular it is proved that the Hausdorff dimension of the set  $S$  is given by  $\frac{-\log 2}{\gamma}$ , where  $\gamma =$

$$\lim_{k \rightarrow \infty} \log \left[ \prod_{n=1}^k \frac{(1 - \epsilon_n)}{2} \right]^{1/k} \text{ and } \overline{\lim}_{k \rightarrow \infty} \frac{\log 2}{-\log \left[ 2 \prod_{n=1}^k \frac{(1 - \epsilon_n)}{2} \right]^{1/k}} \geq \overline{\dim}_B(S) \geq \underline{\dim}_B(S) \geq$$

$$\underline{\lim}_{k \rightarrow \infty} \frac{\log 2}{-\log \left\{ \left[ \prod_{n=1}^k \frac{(1 - \epsilon_n)}{2} \right]^{1/k} \left[ \epsilon_{k+1} \right]^{1/k} \right\}}. \text{ If the above upper and lower limits are equal then } \underline{\dim}_B(S) = \overline{\dim}_B(S).$$

**KEYWORDS :** Cantor set, Cantor like set, Hausdorff dimension, upper Box dimension and lower Box dimension.

## I. INTRODUCTION

### Definition 1.1 Hausdorff measure

If  $U$  is any non-empty subset of  $n$ -dimensional Euclidean space,  $\mathbb{R}^n$ , then diameter of  $U$  is defined as  $|U| = \sup\{|x - y| : x, y \in U\}$ . If  $\{U_i\}$  is a countable (or finite) collection of sets of diameter at most  $\delta$  that cover  $F$ , i.e.  $F \subset \bigcup_{i=1}^{\infty} U_i$  with  $0 < |U_i| \leq \delta$  for each  $i$ , we say that  $\{U_i\}$  is a  $\delta$ -cover of  $F$ .

Suppose that  $F$  is a subset of  $\mathbb{R}^n$  and  $s$  is a non-negative number. For any  $\delta > 0$  we define

$$H^s_{\delta}(F) = \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } F \right\}$$

$$\text{We define } H^s(F) = \lim_{\delta \rightarrow 0} H^s_{\delta}(F)$$

This limit exists for any subset  $F$  of  $\mathbb{R}^n$ , though the limiting value can be 0 or  $\infty$ .

We call  $H^s(F)$  the  $s$ -dimensional Hausdorff measure of  $F$ .

$$\text{OR } H^s(F) = \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a countable cover of } F \right\}$$

### Definition 1.2 Hausdorff dimension

The Hausdorff dimension of  $F$  is defined as

$$\dim_H(F) = \inf \{ s : H^s(F) = 0 \}.$$

It is proved that  $\inf \{ s : H^s(F) = 0 \} = \sup \{ s : H^s(F) = \infty \}$ .

Above definitions and properties are taken from [2].

**Lemma 1.3 :-**  $(a + b)^{\alpha} \leq a^{\alpha} + b^{\alpha}$ ,  $a, b \geq 0$  and  $0 < \alpha < 1$ .

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Proof :-

Let  $f(c) = 1 + c^\alpha - (1 + c)^\alpha, c \geq 0$ .

We show that  $f$  is monotonic increasing. For this we show that its derivative is positive.

$$\begin{aligned} f'(c) &= \alpha c^{\alpha-1} - \alpha(1+c)^{\alpha-1} \\ &= \alpha[c^{\alpha-1} - (1+c)^{\alpha-1}] \\ &= \alpha \left[ \frac{1}{c^{1-\alpha}} - \frac{1}{(c+1)^{1-\alpha}} \right] \end{aligned}$$

Since  $c < c+1, c^{1-\alpha} < (c+1)^{1-\alpha}$ . Thus  $f'(c) > 0, c \geq 0$

This shows that  $f$  is monotonically increasing for  $c \geq 0$ . But  $f(0) = 0$  and hence  $f(c) \geq 0$  for all  $c \geq 0$ .

Thus for  $c = \frac{a}{b}, b > 0$  we get  $1 + (\frac{a}{b})^\alpha \geq (1 + \frac{a}{b})^\alpha$  i.e.  $(a+b)^\alpha \leq a^\alpha + b^\alpha$ .

This proves the result.

**Lemma 1.4 :** Let  $S$  be compact subset of  $\mathbb{R}$  and  $S \subset \bigcup_{n=1}^{\infty} C_n$  where  $C_n$ 's are intervals. Then for given  $\epsilon > 0$  and  $0 < \alpha < 1$ , there exists finitely many open intervals  $D_1, D_2, D_3, \dots, \dots, D_m$

such that  $\bigcup_{n=1}^{\infty} C_n \subseteq \bigcup_{j=1}^m D_j$  and  $\sum_{j=1}^m |D_j|^\alpha \leq \sum_{n=1}^{\infty} |C_n|^\alpha + \epsilon$

Proof : Suppose  $\epsilon > 0$  is given and  $0 < \alpha < 1$ . For  $S \subset \bigcup_{n=1}^{\infty} C_n$  where  $C_n$ 's are interval, define  $F_n$  to be a closed interval with end points same as that of  $C_n$ . Then  $S \subset \bigcup C_n \subset \bigcup F_n$  where each  $F_n$  is a closed interval.

Let  $F_n = [a_n, b_n]$ . For  $\epsilon > 0$  and  $\epsilon' = \epsilon \cdot 2^{1-\alpha}$ , consider  $E_n = (a_n - (\frac{\epsilon'}{2^{n+1}})^{1/\alpha}, b_n + (\frac{\epsilon'}{2^{n+1}})^{1/\alpha})$

Then  $S \subset \bigcup_{n=1}^{\infty} F_n \subset \bigcup_{n=1}^{\infty} E_n$ .

Since  $S$  is compact. There is a finite subfamily  $\{E_{n_1}, E_{n_2}, \dots, \dots, E_{n_m}\}$  of  $\{E_n\}$  such that  $S \subset \bigcup_{j=1}^m E_{n_j}$ . Put  $D_j = E_{n_j}, j = 1, 2, 3, \dots, m$

$$\therefore \sum_{j=1}^m |D_j|^\alpha \leq \sum_{n=1}^{\infty} |E_n|^\alpha = \sum_{n=1}^{\infty} \left\{ (b_n - a_n) + 2 \left( \frac{\epsilon'}{2^{n+1}} \right)^{1/\alpha} \right\}^\alpha$$

$$\therefore \sum_{j=1}^m |D_j|^\alpha \leq \sum_{n=1}^{\infty} \left[ (b_n - a_n)^\alpha + 2^\alpha \frac{\epsilon'}{2^{n+1}} \right] \quad (\because \text{By Lemma 1.3})$$

$$= \sum_{n=1}^{\infty} (b_n - a_n)^\alpha + 2^\alpha \sum_{n=1}^{\infty} \frac{\epsilon'}{2^{n+1}}$$

$$= \sum_{n=1}^{\infty} |C_n|^\alpha + 2^\alpha \frac{\epsilon'}{2}$$

$$= \sum_{n=1}^{\infty} |C_n|^\alpha + \epsilon \quad (\because \epsilon = 2^\alpha \frac{\epsilon'}{2})$$

$$\therefore \sum_{j=1}^m |D_j|^\alpha \leq \sum_{n=1}^{\infty} |C_n|^\alpha + \epsilon$$

## II. MAIN RESULTS

### 2.1 Construction of Cantor like sets

In the construction of Cantor like set  $S$  at first stage from  $I = [0, 1]$  we remove middle open interval  $I_{1,1} = (\frac{1-\epsilon_1}{2}, \frac{1+\epsilon_1}{2})$  of length  $\epsilon_1$ . We get two closed intervals denoted by  $J_{1,1} = [0, \frac{1-\epsilon_1}{2}]$  and  $J_{1,2} = [\frac{1+\epsilon_1}{2}, 1]$ . We get the set  $S_1 = J_{1,1} \cup J_{1,2}$  with measure  $(1 - \epsilon_1)$  i.e.  $m(S_1) = (1 - \epsilon_1)$ . In second stage from  $J_{1,1}$  and  $J_{1,2}$  we remove two

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middle open intervals  $I_{1,2} = (\frac{(1-\epsilon_1)(1-\epsilon_2)}{4}, \frac{(1-\epsilon_1)(1+\epsilon_2)}{4})$  and  $I_{2,2} = (\frac{4-(1-\epsilon_1)(1+\epsilon_2)}{4}, \frac{4-(1-\epsilon_1)(1-\epsilon_2)}{4})$ . The remaining four closed intervals are denoted by

$$J_{2,1} = [0, \frac{(1-\epsilon_1)(1-\epsilon_2)}{4}], J_{2,2} = [\frac{(1-\epsilon_1)(1+\epsilon_2)}{4}, \frac{(1-\epsilon_1)}{2}], J_{2,3} = [\frac{(1+\epsilon_1)}{2}, \frac{4-(1-\epsilon_1)(1+\epsilon_2)}{4}],$$

$$J_{2,4} = [\frac{4-(1-\epsilon_1)(1-\epsilon_2)}{4}, 1].$$

$\therefore$  Length of removed intervals =  $m(I_{1,2}) + m(I_{2,2}) = (1 - \epsilon_1)\epsilon_2 < (1 - \epsilon_1)$

We get the set  $S_2$  union of remaining four closed intervals i.e.  $S_2 = J_{2,1} \cup J_{2,2} \cup J_{2,3} \cup J_{2,4}$  with measure  $(1 - \epsilon_1)(1 - \epsilon_2)$  i.e.  $m(S_2) = (1 - \epsilon_1)(1 - \epsilon_2)$

Continuing in this way we can construct Cantor like set  $S = \bigcap_{n=1}^{\infty} S_n$  with  $m(S) = \lim_{n \rightarrow \infty} m(S_n) = \prod_{n=1}^{\infty} (1 - \epsilon_n)$ . This is from [3].

**Theorem 2.2 :** Hausdorff Dimension of Cantor like set  $S$  defined above is  $\frac{-\log 2}{\lim_{k \rightarrow \infty} \log [\prod_{n=1}^k \frac{(1-\epsilon_n)}{2}]^{1/k}}$  i.e.  $\frac{-\log 2}{\gamma}$  if the

required limit exists and  $\inf\{2^j [\prod_{n=1}^j \frac{(1-\epsilon_n)}{2}]^\alpha [\epsilon_{j+1}]^\alpha : j \geq 1\} > 0$ .

Proof : Let  $\alpha = \frac{-\log 2}{\gamma}$ , where  $\gamma = \lim_{k \rightarrow \infty} \log [\prod_{n=1}^k \frac{(1-\epsilon_n)}{2}]^{1/k}$

We first show that if  $\beta > \alpha$  then  $H^\beta(S) = 0$ .

Take  $k \geq 1$  and let  $J_{1,k}, J_{2,k}, J_{3,k}, \dots, J_{2^k,k}$  be the family of  $2^k$  intervals remaining at  $k^{th}$  step whose union is  $S_k$ .

$$|J_{j,k}| = \text{Length of each } J_{j,k} = \prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \therefore |J_{j,k}|^\beta = [\prod_{n=1}^k \frac{(1-\epsilon_n)}{2}]^\beta$$

$$\sum_{j=1}^{2^k} |J_{j,k}|^\beta = \sum_{j=1}^{2^k} [\prod_{n=1}^k \frac{(1-\epsilon_n)}{2}]^\beta = 2^k [\prod_{n=1}^k \frac{(1-\epsilon_n)}{2}]^\beta \dots \dots \dots (1)$$

Now  $\alpha = \frac{-\log 2}{\gamma}$ , where  $\gamma = \lim_{k \rightarrow \infty} \log [\prod_{n=1}^k \frac{(1-\epsilon_n)}{2}]^{1/k}$

$$\gamma \leq \lim_{k \rightarrow \infty} \log [(\frac{1}{2})^k]^{1/k} \quad (\because (1 - \epsilon_n) < 1 \forall n = 1, 2, 3, \dots, k)$$

$$\gamma \leq \lim_{k \rightarrow \infty} \log(\frac{1}{2}) = \log(\frac{1}{2}) = -\log 2$$

$$\therefore \gamma \leq -\log 2. \text{ Thus } \gamma < 0$$

Now, since  $\beta > \alpha = \frac{-\log 2}{\gamma}$

$$\therefore \beta\gamma < -\log 2 \quad (\because \gamma < 0)$$

$$\Rightarrow \log 2 + \beta\gamma < 0$$

$$\Rightarrow \log 2 + \beta \lim_{k \rightarrow \infty} \log [\prod_{n=1}^k \frac{(1-\epsilon_n)}{2}]^{1/k} < 0 \dots \dots \dots (2)$$

Now,

$$2^k [\prod_{n=1}^k \frac{(1-\epsilon_n)}{2}]^\beta = e^{\log \{2^k [\prod_{n=1}^k \frac{(1-\epsilon_n)}{2}]^\beta\}}$$

$$= e^{\{\log 2^k + \log [\prod_{n=1}^k \frac{(1-\epsilon_n)}{2}]^\beta\}}$$

$$= e^{\{k \log 2 + \beta \log [\prod_{n=1}^k \frac{(1-\epsilon_n)}{2}]\}}$$

$$= e^{k \{\log 2 + \frac{\beta}{k} \log [\prod_{n=1}^k \frac{(1-\epsilon_n)}{2}]\}}$$

$$= e^{k \{\log 2 + \beta \log [\prod_{n=1}^k \frac{(1-\epsilon_n)}{2}]\}^{1/k}}$$

Putting in equation (1)

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$$\sum_{j=1}^{2^k} |J_{j,k}|^\beta = e^{k \{ \log 2 + \beta \log \left[ \prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]^{1/k} \}} \dots\dots\dots(3)$$

We claim that  $\lim_{k \rightarrow \infty} e^{k \{ \log 2 + \beta \log \left[ \prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]^{1/k} \}} = 0$

Since  $\log 2 + \beta \gamma$  is negative by equation (2), we can put  $\log 2 + \beta \gamma = -\delta_1, \delta_1 > 0$ .

If  $\gamma_k = \log \left[ \prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]^{1/k}$ , then  $\lim \gamma_k = \gamma$  and hence  $\exists K_1$  such that  $k \geq K_1$

$$\Rightarrow |(\log 2 + \beta \gamma_k) + \delta_1| < \frac{\delta_1}{2} \quad \Rightarrow |(\log 2 + \beta \gamma_k) - (-\delta_1)| < \frac{\delta_1}{2}$$

$$\Rightarrow -\delta_1 - \frac{\delta_1}{2} < (\log 2 + \beta \gamma_k) < -\delta_1 + \frac{\delta_1}{2}$$

$$-\frac{3\delta_1}{2} < (\log 2 + \beta \gamma_k) < -\frac{\delta_1}{2}$$

$$\Rightarrow k(\log 2 + \beta \gamma_k) < -\frac{k\delta_1}{2} \therefore e^{k(\log 2 + \beta \gamma_k)} < e^{-\frac{k\delta_1}{2}}, k \geq 1.$$

$$0 < e^{k(\log 2 + \beta \gamma_k)} < e^{-\frac{k\delta_1}{2}} \therefore \lim_{k \rightarrow \infty} e^{k(\log 2 + \beta \gamma_k)} = 0$$

This proves that  $\lim_{k \rightarrow \infty} e^{k \{ \log 2 + \beta \log \left[ \prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]^{1/k} \}} = 0 \dots\dots\dots(4)$

Now, for  $\delta > 0$ , choose  $k \geq 1$  such that  $|J_{j,k}| = \prod_{n=1}^k \frac{(1-\epsilon_n)}{2} < \delta, j = 1, 2, \dots, 2^k$ .

This is possible if  $\lim_{k \rightarrow \infty} \prod_{n=1}^k (1 - \epsilon_n)$  exists and then  $\prod_{n=1}^\infty \frac{(1-\epsilon_n)}{2} = \lim_{k \rightarrow \infty} \frac{\prod_{n=1}^k (1-\epsilon_n)}{2^k} = \lim_{k \rightarrow \infty} \prod_{n=1}^k (1 - \epsilon_n) \lim_{k \rightarrow \infty} \frac{1}{2^k} = 0$ . Then  $\{J_{j,k} : j = 1, 2, \dots, 2^k\}$  is a  $\delta$ -cover of  $S_k$  and therefore of  $S$ .

Now  $H^\beta_\delta(S) = \inf \{ \sum_{i=1}^\infty |U_i|^\beta : \{U_i\} \text{ is a } \delta\text{-cover of } S \}$ . Since  $S = \bigcap_{k=1}^\infty S_k, S \subset S_k$  and  $S_k = \bigcup_{j=1}^{2^k} J_{j,k}$ ,

$$\therefore H^\beta_\delta(S) \leq \sum_{j=1}^{2^k} |J_{j,k}|^\beta = e^{k \{ \log 2 + \beta \log \left[ \prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]^{1/k} \}}$$

As  $\delta \rightarrow 0, k \rightarrow \infty$

Now  $H^\beta(S) = \lim_{\delta \rightarrow 0} H^\beta_\delta(S) \leq \lim_{k \rightarrow \infty} e^{k \{ \log 2 + \beta \log \left[ \prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]^{1/k} \}} = 0$  ( By equation (4))

$$\therefore H^\beta(S) = 0$$

Thus  $(\alpha, \infty) \subset \{ \beta > 0 : H^\beta(S) = 0 \}$

$$\therefore \inf \{ \beta > 0 : H^\beta(S) = 0 \} \leq \alpha$$

$$\therefore \dim_H(S) \leq \alpha \dots\dots\dots(5)$$

Now to show that  $\dim_H(S) \geq \alpha$

suppose further that  $\eta = \inf \{ 2^j \left[ \prod_{n=1}^j \frac{(1-\epsilon_n)}{2} \right]^\alpha [\epsilon_{j+1}]^\alpha : j \geq 1 \} > 0 \dots\dots\dots(6)$

We prove that  $H^\alpha(S) > 0$ .

This will imply that  $\dim_H(S) \geq \alpha$  as by property of Hausdorff Dimension

$$\dim_H(S) = \sup \{ \gamma \geq 0 : H^\gamma(S) > 0 \}$$

Now we claim that  $H^\alpha(S) \geq \eta$

Suppose  $\{C_i\}$  is any countable cover of  $S$  by nonempty intervals.

We show that  $\sum |C_i|^\alpha \geq \eta$ .

For any  $\epsilon > 0$  we choose  $\{D_i\}$  a finite collection of open or closed intervals covering  $S$  such that

$$\sum_{i=1}^m |D_i|^\alpha < \sum_{i=1}^\infty |C_i|^\alpha + \epsilon \quad \text{(By Lemma 1.4)} \dots\dots\dots(7)$$

By taking intersection of  $D_i$  with  $[0,1]$  we may assume that  $D_i \subset [0,1], \forall i = 1,2,3, \dots, m$

For each  $D_i$  choose  $k_i, i = 1,2,3, \dots, m$  such that

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$$\left[ \prod_{n=1}^{k_i} \frac{(1-\epsilon_n)}{2} \right] \epsilon_{k_i+1} \leq |D_i| < \left[ \prod_{n=1}^{k_i-1} \frac{(1-\epsilon_n)}{2} \right] \epsilon_{k_i} \dots\dots\dots(8)$$

Then  $D_i$  can intersect at most one basic interval of  $S_{k_i}$ . If  $D_i$  intersects two intervals each of length  $\prod_{n=1}^{k_i} \frac{(1-\epsilon_n)}{2}$  of  $S_{k_i}$  then  $|D_i| \geq \left[ \prod_{n=1}^{k_i-1} \frac{(1-\epsilon_n)}{2} \right] \epsilon_{k_i}$  which is distance between consecutive intervals of  $S_{k_i}$  which is false.

If  $j \geq k_i$  then the basic intervals of  $S_j$  are  $[0, \prod_{n=1}^j \frac{(1-\epsilon_n)}{2}] \cup \dots \dots \dots \cup \left[ \frac{2^j - \prod_{n=1}^j (1-\epsilon_n)}{2^j}, 1 \right]$

These are  $2^j$  basic intervals.

Choose  $j \geq k_1, k_2, \dots, k_m$  so that  $|D_i| \geq \left[ \prod_{n=1}^j \frac{(1-\epsilon_n)}{2} \right] \epsilon_{j+1}$  for all  $i = 1, 2, 3, \dots, m$ .

For  $i = 1, 2, 3, \dots, m$ , if  $N_{k_i}$  is the number of sets  $D_i$  which intersect the basic intervals of  $S_j$ . Then  $\{D_i\}$  intersects all  $2^j$  basic intervals of length  $\prod_{n=1}^j \frac{(1-\epsilon_n)}{2}$  in  $S_j$  and  $\sum_{i=1}^m N_{k_i} = 2^j$ .

$$\begin{aligned} \text{From equation (8) } |D_i| &\geq \left[ \prod_{n=1}^j \frac{(1-\epsilon_n)}{2} \right] \epsilon_{j+1} \\ \therefore |D_i|^\alpha &\geq \left\{ \left[ \prod_{n=1}^j \frac{(1-\epsilon_n)}{2} \right] \epsilon_{j+1} \right\}^\alpha \\ |D_i|^\alpha &\geq \left[ \prod_{n=1}^j \frac{(1-\epsilon_n)}{2} \right]^\alpha [\epsilon_{j+1}]^\alpha \\ \sum_{N_{k_i}} |D_i|^\alpha &\geq \left[ \prod_{n=1}^j \frac{(1-\epsilon_n)}{2} \right]^\alpha [\epsilon_{j+1}]^\alpha N_{k_i} \\ \therefore \sum_{i=1}^m |D_i|^\alpha &\geq \left[ \prod_{n=1}^j \frac{(1-\epsilon_n)}{2} \right]^\alpha [\epsilon_{j+1}]^\alpha \sum_{i=1}^m N_{k_i} \\ \sum_{i=1}^m |D_i|^\alpha &\geq \left[ \prod_{n=1}^j \frac{(1-\epsilon_n)}{2} \right]^\alpha [\epsilon_{j+1}]^\alpha 2^j \quad (\because \sum_{i=1}^m N_{k_i} = 2^j) \\ \therefore \sum_{i=1}^m |D_i|^\alpha &\geq 2^j \left[ \prod_{n=1}^j \frac{(1-\epsilon_n)}{2} \right]^\alpha [\epsilon_{j+1}]^\alpha \end{aligned}$$

$$\begin{aligned} \text{But } \sum_{i=1}^\infty |C_i|^\alpha + \epsilon &> \sum_{i=1}^m |D_i|^\alpha \\ \sum_{i=1}^\infty |C_i|^\alpha + \epsilon &> 2^j \left[ \prod_{n=1}^j \frac{(1-\epsilon_n)}{2} \right]^\alpha [\epsilon_{j+1}]^\alpha \geq \eta \quad (\text{From equation(6)}) \end{aligned}$$

$$\begin{aligned} \therefore \sum_{i=1}^\infty |C_i|^\alpha + \epsilon &\geq \eta. \text{ Since } \epsilon > 0 \text{ is arbitrary, } \sum_{i=1}^\infty |C_i|^\alpha + \epsilon \geq \eta \\ \text{i.e. } \inf \{ \sum_{i=1}^\infty |C_i|^\alpha + \epsilon : \{C_i\} \text{ is countable cover of } S \} &\geq \eta \end{aligned}$$

$$\therefore H^\alpha(S) \geq \eta > 0 \dots\dots\dots(9)$$

By property of Hausdorff Dimension

$$\dim_H(S) = \sup \{ \gamma \geq 0 : H^\gamma(S) > 0 \}$$

Since  $H^\alpha(S) > 0$

( By equation (9) )

$$\therefore \alpha \in \{ \gamma \geq 0 : H^\gamma(S) > 0 \}$$

$$\therefore \sup \{ \gamma \geq 0 : H^\gamma(S) > 0 \} \geq \alpha$$

$$\therefore \dim_H(S) \geq \alpha \dots\dots\dots(10)$$

From equations (5) and (10), we get,

$$\dim_H(S) = \alpha$$

We consider now several particular cases of  $\{ \epsilon_n \}$  where the condition in equation (6) is satisfied so that the result in Theorem 2.2 holds.

**Case I :** Cantor ternary set. It is obtained by taking  $\epsilon_n = \frac{1}{3}$  for all  $n \geq 1$ .

$$\text{Thus } \prod_{n=1}^k \frac{(1-\epsilon_n)}{2} = \prod_{n=1}^k \frac{(1-\frac{1}{3})}{2} = \left(\frac{1}{3}\right)^k$$

$$\therefore \left[ \prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]^{1/k} = \frac{1}{3}$$

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$$\therefore \log \left[ \prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]^{1/k} = \log\left(\frac{1}{3}\right)$$

$$\alpha = \frac{-\log 2}{\lim_{k \rightarrow \infty} \log \left[ \prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]^{1/k}}$$

$$= \frac{-\log 2}{\log\left(\frac{1}{3}\right)}$$

$$\alpha = \frac{\log 2}{\log 3}$$

$$\therefore 3^\alpha = 2$$

$$\begin{aligned} \text{Now } 2^j \left[ \prod_{n=1}^j \frac{(1-\epsilon_n)}{2} \right]^\alpha [\epsilon_{j+1}]^\alpha &= 2^j \left(\frac{1}{3}\right)^{j\alpha} \left(\frac{1}{3}\right)^\alpha \\ &= 2^j 2^{-j \frac{1}{2}} \\ &= \frac{1}{2} \end{aligned} \quad (\because 3^\alpha = 2)$$

$$\therefore \inf\{2^j \left[ \prod_{n=1}^j \frac{(1-\epsilon_n)}{2} \right]^\alpha [\epsilon_{j+1}]^\alpha : j \geq 1\} = \frac{1}{2}$$

**Case II :** Consider the case  $\epsilon_n = \epsilon$  for all  $n \geq 1$ ,  $\epsilon < 1$ .

$$\text{Thus } \prod_{n=1}^k \frac{(1-\epsilon_n)}{2} = \prod_{n=1}^k \frac{(1-\epsilon)}{2} = \left(\frac{1-\epsilon}{2}\right)^k$$

$$\therefore \left[ \prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]^{1/k} = \frac{1-\epsilon}{2}$$

$$\therefore \log \left[ \prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]^{1/k} = \log\left(\frac{1-\epsilon}{2}\right)$$

$$\therefore \lim_{k \rightarrow \infty} \log \left[ \prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]^{1/k} = \log\left(\frac{1-\epsilon}{2}\right)$$

$$\text{Also } \alpha = \frac{-\log 2}{\lim_{k \rightarrow \infty} \log \left[ \prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]^{1/k}}$$

$$= \frac{-\log 2}{\log\left(\frac{1-\epsilon}{2}\right)}$$

$$\alpha \log\left(\frac{1-\epsilon}{2}\right) = -\log 2$$

$$\therefore \log\left(\frac{1-\epsilon}{2}\right)^\alpha = \log 2^{-1}$$

$$\therefore \left(\frac{1-\epsilon}{2}\right)^\alpha = 2^{-1}$$

$$\begin{aligned} \text{Now } 2^j \left[ \prod_{n=1}^j \frac{(1-\epsilon_n)}{2} \right]^\alpha [\epsilon_{j+1}]^\alpha &= 2^j \left(\frac{1-\epsilon}{2}\right)^{j\alpha} \epsilon^\alpha \\ &= 2^j 2^{-j} \epsilon^\alpha \\ &= \epsilon^\alpha \end{aligned} \quad (\because \left(\frac{1-\epsilon}{2}\right)^\alpha = 2^{-1})$$

$$\therefore \inf\{2^j \left[ \prod_{n=1}^j \frac{(1-\epsilon_n)}{2} \right]^\alpha [\epsilon_{j+1}]^\alpha : j \geq 1\} = \epsilon^\alpha$$

**Case III :** Consider the case  $\epsilon_n = \frac{1}{n+1}$  for all  $n \geq 1$

$$\begin{aligned} \text{Thus } \prod_{n=1}^k \frac{(1-\epsilon_n)}{2} &= \prod_{n=1}^k \frac{\left[1 - \frac{1}{(n+1)}\right]}{2} \\ &= \prod_{n=1}^k \frac{[(n+1)-1]}{2(n+1)} \\ &= \prod_{n=1}^k \frac{n}{2(n+1)} \\ &= \frac{1}{2^k} \prod_{n=1}^k \frac{n}{n+1} \end{aligned}$$

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$$\begin{aligned}
 &= \frac{1}{2^k} \frac{1}{2} \frac{2}{3} \frac{3}{4} \frac{4}{5} \cdots \frac{k}{(k+1)} \\
 \prod_{n=1}^k \frac{(1-\epsilon_n)}{2} &= \frac{1}{2^k} \frac{1}{k+1} \\
 \therefore \left[ \prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]^{1/k} &= \left[ \frac{1}{2^{k(k+1)}} \right]^{1/k} = \frac{1}{2^{(k+1)1/k}} = \frac{1}{2} \left( \frac{1}{k+1} \right)^{1/k} \\
 \therefore \log \left[ \prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]^{1/k} &= \log \left[ \frac{1}{2} \left( \frac{1}{k+1} \right)^{1/k} \right] \\
 \alpha &= \frac{-\log 2}{\lim_{k \rightarrow \infty} \log \left[ \prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]^{1/k}} \\
 &= \frac{-\log 2}{\lim_{k \rightarrow \infty} \log \left[ \frac{1}{2} \left( \frac{1}{k+1} \right)^{1/k} \right]} \\
 &= \frac{-\log 2}{\log \left( \frac{1}{2} \right)} \\
 \alpha &= \frac{-\log 2}{-\log 2} = 1 \\
 \therefore \alpha &= 1
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } 2^j \left[ \prod_{n=1}^j \frac{(1-\epsilon_n)}{2} \right]^\alpha [\epsilon_{j+1}]^\alpha &= 2^j \left[ \frac{1}{2^{j(j+1)}} \right] \frac{1}{(j+2)} = \frac{1}{(j+1)(j+2)} \\
 \therefore \inf \{ 2^j \left[ \prod_{n=1}^j \frac{(1-\epsilon_n)}{2} \right]^\alpha [\epsilon_{j+1}]^\alpha : j \geq 1 \} &= 0
 \end{aligned}$$

$\alpha = 1$  but  $\inf \{ 2^j \left[ \prod_{n=1}^j \frac{(1-\epsilon_n)}{2} \right]^\alpha [\epsilon_{j+1}]^\alpha : j \geq 1 \} = 0$  i.e. condition is not satisfied.

**Case IV :** Consider the case  $\epsilon_n = \frac{n}{2(n+1)}$  for all  $n \geq 1$ .

$$\begin{aligned}
 \text{Thus } \prod_{n=1}^k \frac{(1-\epsilon_n)}{2} &= \prod_{n=1}^k \frac{[1 - \frac{n}{2(n+1)}]}{2} \\
 &= \prod_{n=1}^k \frac{[2(n+1)-n]}{4(n+1)} \\
 &= \prod_{n=1}^k \frac{n+2}{4(n+1)} \\
 &= \frac{1}{4^k} \prod_{n=1}^k \frac{n+2}{n+1} \\
 &= \frac{1}{4^k} \frac{3}{2} \frac{4}{3} \frac{5}{4} \frac{6}{5} \cdots \frac{(k+2)}{(k+1)} \\
 \prod_{n=1}^k \frac{(1-\epsilon_n)}{2} &= \frac{1}{4^k} \frac{(k+2)}{2} \\
 \therefore \left[ \prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]^{1/k} &= \left[ \frac{(k+2)}{2 \cdot 4^k} \right]^{1/k} = \frac{(k+2)^{1/k}}{4 \cdot 2^{1/k}} = \frac{1}{4} \left( 1 + \frac{k}{2} \right)^{1/k} \\
 \therefore \log \left[ \prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]^{1/k} &= \log \left[ \frac{1}{4} \left( 1 + \frac{k}{2} \right)^{1/k} \right] \\
 \alpha &= \frac{-\log 2}{\lim_{k \rightarrow \infty} \log \left[ \prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]^{1/k}} \\
 &= \frac{-\log 2}{\lim_{k \rightarrow \infty} \log \left[ \frac{1}{4} \left( 1 + \frac{k}{2} \right)^{1/k} \right]} \\
 &= \frac{-\log 2}{\log \left( \frac{1}{4} \right)} \\
 \alpha &= \frac{\log 2}{\log 4} = \frac{1}{2}
 \end{aligned}$$

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$$\begin{aligned} \therefore \alpha &= \frac{1}{2} \\ \text{Now } 2^j \left[ \prod_{n=1}^j \frac{(1-\epsilon_n)}{2} \right]^\alpha [\epsilon_{j+1}]^\alpha &= 2^j \left[ \frac{1}{4^j} \left( \frac{j+2}{2} \right) \right]^{1/2} \left[ \frac{j+1}{2(j+2)} \right]^{1/2} \\ &= \frac{2^j}{2^j} \left( \frac{j+2}{2} \right)^{1/2} \left[ \frac{j+1}{2(j+2)} \right]^{1/2} \\ &= \left( \frac{j+1}{4} \right)^{1/2} \\ &= \frac{1}{2} (j+1)^{1/2} \\ \therefore \inf \{ 2^j \left[ \prod_{n=1}^j \frac{(1-\epsilon_n)}{2} \right]^\alpha [\epsilon_{j+1}]^\alpha : j \geq 1 \} &= \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} \end{aligned}$$

### Definition 2.3 Box counting dimension

Let  $F$  be any nonempty bounded subset of  $\mathbb{R}^n$  and let  $N_\delta(F)$  be the smallest number of sets of diameter at most  $\delta$  which can cover  $F$ . The lower and upper box dimensions of  $F$  are defined respectively as  $\underline{\dim}_B(F) = \lim_{\delta \rightarrow 0} \frac{\log N_\delta(F)}{-\log \delta}$  and  $\overline{\dim}_B(F) = \overline{\lim}_{\delta \rightarrow 0} \frac{\log N_\delta(F)}{-\log \delta}$ .

If the lower and upper box-counting dimensions of  $F$  are equal then the common value is called box-counting dimension of  $F$  or box- dimension of  $F$  and  $\dim_B(F) = \lim_{\delta \rightarrow 0} \frac{\log N_\delta(F)}{-\log \delta}$ . Above definitions are taken from [2].

**Theorem 2.4:** Lower and upper box-dimensions of Cantor like set  $S$  associated with  $\{\epsilon_n\}$  of positive numbers less than 1 with  $\sum \epsilon_j = \infty$  satisfy  $\underline{\dim}_B(S) \geq \lim_{k \rightarrow \infty} \frac{\log 2}{-\log \left[ \prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]^{1/k} [\epsilon_{k+1}]^{1/k}}$  and  $\overline{\dim}_B(S) \leq$

$$\overline{\lim}_{k \rightarrow \infty} \frac{\log 2}{-\log \left[ 2 \prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]^{1/k}}.$$

Proof : We know that in the construction of Cantor like set  $S$ , at  $k^{th}$  stage the set  $S_k$  consists of union of  $2^k$  disjoint intervals each of length  $\prod_{n=1}^k \frac{(1-\epsilon_n)}{2}$ . For  $0 < \delta < 1$ , let  $U$  be an interval with  $|U| < \delta$ . Choose  $k \geq 1$  such that  $2 \prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \geq \delta > 2 \prod_{n=1}^{k+1} \frac{(1-\epsilon_n)}{2}$ . This is possible since  $\lim_{k \rightarrow \infty} \prod_{n=1}^k (1 - \epsilon_n)$  exists. Then  $U$  can cover at most one interval among  $2^k$  intervals of length  $\prod_{n=1}^k \frac{(1-\epsilon_n)}{2}$ .

For if  $U$  contains more than one interval from  $J_{j,k}$  then  $\delta > |U| \geq 2 \prod_{n=1}^k \frac{(1-\epsilon_n)}{2}$ .

$\therefore \delta > 2 \prod_{n=1}^k \frac{(1-\epsilon_n)}{2}$  which is not possible because  $\delta \leq 2 \prod_{n=1}^k \frac{(1-\epsilon_n)}{2}$ .

$\therefore$  At most  $2^k$  intervals of diameter less than  $\delta$  are required to cover  $S$ .

$\therefore N_\delta(S) \leq 2^k$

Now  $\delta \leq 2 \prod_{n=1}^k \frac{(1-\epsilon_n)}{2}$

$$\log \delta \leq \log \left[ 2 \prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]$$

$$-\log \delta \geq -\log \left[ 2 \prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]$$

$$\therefore \frac{1}{-\log \delta} \leq \frac{1}{-\log \left[ 2 \prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]} \quad (\because -\log \delta > 0)$$



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By definition  $\overline{dim}_B(S) = \overline{lim}_{\delta \rightarrow 0} \frac{\log N_\delta(S)}{-\log \delta}$ . Since  $\delta \rightarrow 0$  if and only if  $k \rightarrow \infty$ , we have

$$\begin{aligned} \overline{dim}_B(S) &\leq \overline{lim}_{k \rightarrow \infty} \frac{\log 2^k}{-\log [2 \prod_{n=1}^k \frac{(1-\epsilon_n)}{2}]} \\ &\leq \overline{lim}_{k \rightarrow \infty} \frac{\log 2^k}{-\log [2 \prod_{n=1}^k \frac{(1-\epsilon_n)}{2}]^{1/k}} \\ &\leq \overline{lim}_{k \rightarrow \infty} \frac{k \log 2}{-k \log [2 \prod_{n=1}^k \frac{(1-\epsilon_n)}{2}]^{1/k}} \\ \therefore \overline{dim}_B(S) &\leq \overline{lim}_{k \rightarrow \infty} \frac{\log 2}{-\log [2 \prod_{n=1}^k \frac{(1-\epsilon_n)}{2}]^{1/k}} \end{aligned}$$

Now to show that  $\underline{dim}_B(S) \geq \underline{lim}_{k \rightarrow \infty} \frac{\log 2}{-\log \{ [\prod_{n=1}^k \frac{(1-\epsilon_n)}{2}]^{1/k} [\epsilon_{k+1}]^{1/k} \}}$ , consider an interval  $U$  of length less than  $\delta$ ,  $\delta > 0$  and choose  $k$  such that

$$[\prod_{n=1}^k \frac{(1-\epsilon_n)}{2}] \epsilon_{k+1} \leq \delta < [\prod_{n=1}^{k-1} \frac{(1-\epsilon_n)}{2}] \epsilon_k, \text{ then } U \text{ intersects at most one of the basic intervals of length } \prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \text{ used in the construction of } S.$$

For if  $U$  intersects more than one interval of  $S_k$  then  $|U| \geq$  distance between two disjoint intervals of  $S_k$

i.e.  $|U| \geq [\prod_{n=1}^{k-1} \frac{(1-\epsilon_n)}{2}] \epsilon_k$  i.e.  $\delta > [\prod_{n=1}^{k-1} \frac{(1-\epsilon_n)}{2}] \epsilon_k$  which is false because

$\delta < [\prod_{n=1}^{k-1} \frac{(1-\epsilon_n)}{2}] \epsilon_k$ . These are  $2^k$  such intervals.

$\therefore$  At least  $2^k$  intervals of length  $\delta$  are required to cover  $S_k \supset S$ .

$\therefore N_\delta(S) \geq 2^k$

Now  $\delta \geq [\prod_{n=1}^k \frac{(1-\epsilon_n)}{2}] \epsilon_{k+1}$

$\log \delta \geq \log \{ [\prod_{n=1}^k \frac{(1-\epsilon_n)}{2}] \epsilon_{k+1} \}$

$-\log \delta \leq -\log \{ [\prod_{n=1}^k \frac{(1-\epsilon_n)}{2}] \epsilon_{k+1} \}$

$\frac{1}{-\log \{ [\prod_{n=1}^k \frac{(1-\epsilon_n)}{2}] \epsilon_{k+1} \}} \leq \frac{1}{-\log \delta}$

$\frac{1}{-\log \delta} \geq \frac{1}{-\log \{ [\prod_{n=1}^k \frac{(1-\epsilon_n)}{2}] \epsilon_{k+1} \}}$

By definition  $\underline{dim}_B(S) = \underline{lim}_{\delta \rightarrow 0} \frac{\log N_\delta(S)}{-\log \delta}$

$$\begin{aligned} &\geq \underline{lim}_{k \rightarrow \infty} \frac{\log 2^k}{-\log \{ [\prod_{n=1}^k \frac{(1-\epsilon_n)}{2}] \epsilon_{k+1} \}} \\ &\geq \underline{lim}_{k \rightarrow \infty} \frac{\log 2^k}{-\log \{ [\prod_{n=1}^k \frac{(1-\epsilon_n)}{2}] \epsilon_{k+1} \}^{1/k}} \\ &\geq \underline{lim}_{k \rightarrow \infty} \frac{k \log 2}{-k \log \{ [\prod_{n=1}^k \frac{(1-\epsilon_n)}{2}] \epsilon_{k+1} \}^{1/k}} \\ &\geq \underline{lim}_{k \rightarrow \infty} \frac{\log 2}{-\log \{ [\prod_{n=1}^k \frac{(1-\epsilon_n)}{2}] \epsilon_{k+1} \}^{1/k}} \end{aligned}$$

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$$\therefore \underline{\dim}_B(S) \geq \lim_{k \rightarrow \infty} \frac{\log 2}{-\log \left\{ \left[ \prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]^{1/k} [\epsilon_{k+1}]^{1/k} \right\}}$$

**Weierstrass theorem :**

If  $\text{Re } z > 0$  and  $\gamma$  denotes Euler's constant then  $\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}$ . This is theorem, §

4.41, p. 148 of [6].

**Lemma 2.5 :**

If  $a_i$  and  $b_i$  are complex numbers such that  $\sum a_i = \sum b_i$  and  $\text{Re}(1 - a_i) > 0$ ,  $\text{Re}(1 - b_i) > 0$  then

$$\prod_{n=1}^{\infty} \frac{(n-a_1)(n-a_2)\dots(n-a_k)}{(n-b_1)(n-b_2)\dots(n-b_k)} = \prod_{i=1}^k \frac{\Gamma(1-b_i)}{\Gamma(1-a_i)}$$

Proof : Consider

$$\frac{(n-a_1)(n-a_2)\dots(n-a_k)}{(n-b_1)(n-b_2)\dots(n-b_k)} = \frac{n^k \left(1-\frac{a_1}{n}\right)\left(1-\frac{a_2}{n}\right)\dots\left(1-\frac{a_k}{n}\right)}{n^k \left(1-\frac{b_1}{n}\right)\left(1-\frac{b_2}{n}\right)\dots\left(1-\frac{b_k}{n}\right)} = \frac{\left(1-\frac{a_1}{n}\right)\left(1-\frac{a_2}{n}\right)\dots\left(1-\frac{a_k}{n}\right)}{\left(1-\frac{b_1}{n}\right)\left(1-\frac{b_2}{n}\right)\dots\left(1-\frac{b_k}{n}\right)} \dots\dots\dots(1)$$

Since  $\sum a_i = \sum b_i$ ,  $i = 1, 2, 3, \dots, k$

$\therefore \sum a_i - \sum b_i = 0$ ,  $i = 1, 2, 3, \dots, k$

$\therefore a_1 + a_2 + \dots + a_k - b_1 - b_2 - \dots - b_k = 0$

$\therefore e^{\left(\frac{a_1+a_2+\dots+a_k-b_1-b_2-\dots-b_k}{n}\right)} = e^0 = 1$

$\therefore$  Equation (1) can be written as

$$\begin{aligned} \frac{(n-a_1)(n-a_2)\dots(n-a_k)}{(n-b_1)(n-b_2)\dots(n-b_k)} &= \frac{\left(1-\frac{a_1}{n}\right)\left(1-\frac{a_2}{n}\right)\dots\left(1-\frac{a_k}{n}\right)}{\left(1-\frac{b_1}{n}\right)\left(1-\frac{b_2}{n}\right)\dots\left(1-\frac{b_k}{n}\right)} e^{\left(\frac{a_1+a_2+\dots+a_k-b_1-b_2-\dots-b_k}{n}\right)} \\ &= \frac{\left(1-\frac{a_1}{n}\right)\left(1-\frac{a_2}{n}\right)\dots\left(1-\frac{a_k}{n}\right)}{\left(1-\frac{b_1}{n}\right)\left(1-\frac{b_2}{n}\right)\dots\left(1-\frac{b_k}{n}\right)} e^{\left(\frac{a_1+a_2+\dots+a_k}{n}\right)} \\ &= \frac{\left(1-\frac{a_1}{n}\right)\left(1-\frac{a_2}{n}\right)\dots\left(1-\frac{a_k}{n}\right)}{\left(1-\frac{b_1}{n}\right)\left(1-\frac{b_2}{n}\right)\dots\left(1-\frac{b_k}{n}\right)} e^{\left(\frac{a_1}{n} + \frac{a_2}{n} + \dots + \frac{a_k}{n}\right)} \\ &= \frac{\left(1-\frac{a_1}{n}\right)\left(1-\frac{a_2}{n}\right)\dots\left(1-\frac{a_k}{n}\right)}{\left(1-\frac{b_1}{n}\right)\left(1-\frac{b_2}{n}\right)\dots\left(1-\frac{b_k}{n}\right)} e^{\frac{a_1}{n}} e^{\frac{a_2}{n}} \dots e^{\frac{a_k}{n}} \\ &= \frac{\left(1-\frac{a_1}{n}\right)e^{\frac{a_1}{n}} \left(1-\frac{a_2}{n}\right)e^{\frac{a_2}{n}} \dots \left(1-\frac{a_k}{n}\right)e^{\frac{a_k}{n}}}{\left(1-\frac{b_1}{n}\right)e^{\frac{b_1}{n}} \left(1-\frac{b_2}{n}\right)e^{\frac{b_2}{n}} \dots \left(1-\frac{b_k}{n}\right)e^{\frac{b_k}{n}}} \dots\dots\dots(2) \end{aligned}$$

By Weierstrass theorem

$$\begin{aligned} \frac{1}{\Gamma(z)} &= ze^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n} \\ &= ze^{\gamma z} \prod_{n=1}^{\infty} \left(\frac{n+z}{n}\right) e^{-z/n} \\ \therefore \Gamma(z) &= \frac{1}{ze^{\gamma z}} \prod_{n=1}^{\infty} \left(\frac{n}{n+z}\right) e^{z/n} \\ \Gamma(z) &= \frac{1}{ze^{\gamma z}} \prod_{n=1}^{\infty} \left(\frac{n}{n+z}\right) e^{z/n} \dots\dots\dots(3) \end{aligned}$$

For  $z = 1 - b_i$

$$\Gamma(1 - b_i) = \frac{1}{(1 - b_i)e^{\gamma(1 - b_i)}} \prod_{n=1}^{\infty} \left(\frac{n}{n+1 - b_i}\right) e^{\frac{1 - b_i}{n}} \dots\dots\dots(4)$$

For  $z = 1 - a_i$  Equation (3)  $\Rightarrow$

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$$\Gamma(1 - a_i) = \frac{1}{(1 - a_i)e^{\gamma(1 - a_i)}} \prod_{n=1}^{\infty} \left( \frac{n}{n+1 - a_i} \right) e^{\frac{1 - a_i}{n}} \dots\dots\dots(5)$$

Equation (4) ÷ Equation (5)

$$\begin{aligned} \frac{\Gamma(1 - b_i)}{\Gamma(1 - a_i)} &= \frac{(1 - a_i)e^{\gamma(1 - a_i)} \prod_{n=1}^{\infty} \left( \frac{n}{n+1 - b_i} \right) e^{\frac{1 - b_i}{n}}}{(1 - b_i)e^{\gamma(1 - b_i)} \prod_{n=1}^{\infty} \left( \frac{n}{n+1 - a_i} \right) e^{\frac{1 - a_i}{n}}} \\ &= \frac{(1 - a_i)e^{-\gamma a_i}}{(1 - b_i)e^{-\gamma b_i}} \prod_{n=1}^{\infty} \left[ \left( \frac{n/n+1 - b_i}{n/n+1 - a_i} \right) e^{\frac{1 - b_i}{n} - \frac{1 - a_i}{n}} \right] \\ &= \frac{(1 - a_i)}{(1 - b_i)} e^{-\gamma(b_i - a_i)} \prod_{n=1}^{\infty} \left( \frac{n+1 - a_i}{n+1 - b_i} \right) e^{-\frac{1}{n}(a_i - b_i)} \\ \therefore \prod_{i=1}^k \frac{\Gamma(1 - b_i)}{\Gamma(1 - a_i)} &= \left[ \prod_{i=1}^k \frac{(1 - a_i)}{(1 - b_i)} e^{-\gamma(b_i - a_i)} \right] \prod_{i=1}^k \left[ \prod_{n=1}^{\infty} \left( \frac{n+1 - a_i}{n+1 - b_i} \right) e^{-\frac{1}{n}(a_i - b_i)} \right] \\ &= \left[ \prod_{i=1}^k \frac{(1 - a_i)}{(1 - b_i)} e^{-\gamma(b_i - a_i)} \right] \prod_{n=1}^{\infty} \left[ \prod_{i=1}^k \left( \frac{n+1 - a_i}{n+1 - b_i} \right) e^{-\frac{1}{n}(a_i - b_i)} \right] \\ &= \frac{(1 - a_1)(1 - a_2) \dots (1 - a_k)}{(1 - b_1)(1 - b_2) \dots (1 - b_k)} e^{-\gamma \sum_{i=1}^k (a_i - b_i)} \prod_{n=1}^{\infty} \left[ \frac{(n+1 - a_1)(n+1 - a_2) \dots (n+1 - a_k)}{(n+1 - b_1)(n+1 - b_2) \dots (n+1 - b_k)} \right] e^{-\frac{1}{n} \sum_{i=1}^k (a_i - b_i)} \\ &= \frac{(1 - a_1)(1 - a_2) \dots (1 - a_k)}{(1 - b_1)(1 - b_2) \dots (1 - b_k)} \prod_{n=1}^{\infty} \left[ \frac{(n+1 - a_1)(n+1 - a_2) \dots (n+1 - a_k)}{(n+1 - b_1)(n+1 - b_2) \dots (n+1 - b_k)} \right] \end{aligned}$$

( $\because a_i = b_i, i = 1, 2, 3, \dots, k \implies e^{\sum_{i=1}^k (a_i - b_i)} = e^0 = 1$ )

Put  $n+1 = n'$  in second product

$$\begin{aligned} &= \frac{(1 - a_1)(1 - a_2) \dots (1 - a_k)}{(1 - b_1)(1 - b_2) \dots (1 - b_k)} \prod_{n=2}^{\infty} \left[ \frac{(n' - a_1)(n' - a_2) \dots (n' - a_k)}{(n' - b_1)(n' - b_2) \dots (n' - b_k)} \right] \\ &= \prod_{n'=1}^{\infty} \left[ \frac{(n' - a_1)(n' - a_2) \dots (n' - a_k)}{(n' - b_1)(n' - b_2) \dots (n' - b_k)} \right] \\ \therefore \prod_{i=1}^k \frac{\Gamma(1 - b_i)}{\Gamma(1 - a_i)} &= \prod_{n=1}^{\infty} \frac{(n - a_1)(n - a_2) \dots (n - a_k)}{(n - b_1)(n - b_2) \dots (n - b_k)} \\ \therefore \prod_{n=1}^{\infty} \frac{(n - a_1)(n - a_2) \dots (n - a_k)}{(n - b_1)(n - b_2) \dots (n - b_k)} &= \prod_{i=1}^k \frac{\Gamma(1 - b_i)}{\Gamma(1 - a_i)} \end{aligned}$$

**Corollary 2.6 :**  $\prod_{n=1}^{\infty} \frac{(n+1)^p - 1}{(n+1)^p} = \frac{1}{\Gamma(2 - \beta_2)\Gamma(2 - \beta_3) \dots \Gamma(2 - \beta_p)}$ , where  $\beta_1, \beta_2, \dots, \beta_p$  are  $p^{th}$  root of unity with  $\beta_1 = 1$ .

Proof : Here  $(x + 1)^p - 1$  is a polynomial in  $x$  of degree  $p$ . This polynomial has  $p$  solutions say  $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_p$ .  $\alpha_j$  is a solution of  $(x + 1)^p - 1 = 0$  iff  $\alpha_j + 1 = (1)^{1/p}$

i.e.  $(\alpha_j + 1)^p = 1$  iff  $\alpha_j + 1 = (1)^{1/p}$   
iff  $\alpha_j + 1$  is  $p^{th}$  root of unity  
iff  $\alpha_j = -1 + \beta_j$ , where  $\beta_j$  is the  $p^{th}$  root of unity

$$\begin{aligned} \therefore (x + 1)^p - 1 &= (x - \alpha_1)(x - \alpha_2) \dots (x - \alpha_p) \\ &= (x + 1 - \beta_1)(x + 1 - \beta_2) \dots (x + 1 - \beta_p) \\ \therefore \frac{(n+1)^p - 1}{(n+1)^p} &= \frac{(n+1 - \beta_1)(n+1 - \beta_2) \dots (n+1 - \beta_p)}{(n+1)(n+1) \dots (n+1)} \\ &= \frac{[n - (\beta_1 - 1)][n - (\beta_2 - 1)] \dots [n - (\beta_p - 1)]}{[n - (-1)][n - (-1)] \dots [n - (-1)]} \dots\dots\dots(1) \end{aligned}$$

Here  $a_i = \beta_i - 1$  and  $b_i = -1, i = 1, 2, 3, \dots, p$   
Since  $\beta_1, \beta_2, \dots, \beta_p$  are  $p^{th}$  root of unity  $\implies \beta_1 + \beta_2 + \dots + \beta_p = 0$  .....

Now  $1 - a_i = 1 - (\beta_i - 1) = 1 - \beta_i + 1 = 2 - \beta_i$   
 $\therefore \text{Re}(1 - a_i) > 0, i = 1, 2, 3, \dots, p$  .....

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Now  $1 - b_i = 1 - (-1) = 2, i = 1, 2, 3, \dots, p$   
 $\therefore \text{Re}(1 - b_i) > 0, i = 1, 2, 3, \dots, p$  .....(4)

Now  $\sum_{i=1}^p a_i = \sum_{i=1}^p (\beta_i - 1) = \sum_{i=1}^p \beta_i - \sum_{i=1}^p 1 = 0 - p = -p$  (From equation (2))

Also  $\sum_{i=1}^p b_i = \sum_{i=1}^p (-1) = -p$   
 $\therefore \sum_{i=1}^p a_i = \sum_{i=1}^p b_i$  .....(5)

From equations (3), (4), (5) and by Corollary 2.5 we get,

$$\begin{aligned} \therefore \prod_{n=1}^{\infty} \frac{(n+1)^p - 1}{(n+1)^p} &= \prod_{i=1}^p \frac{\Gamma(1 - (-1))}{\Gamma(1 - (\beta_i - 1))} = \prod_{i=1}^p \frac{\Gamma(2)}{\Gamma(2 - \beta_i)} = \prod_{i=1}^p \frac{1}{\Gamma(2 - \beta_i)} = \frac{1}{\Gamma(2 - \beta_1)\Gamma(2 - \beta_2)\dots\Gamma(2 - \beta_p)} \\ &= \frac{1}{\Gamma(2 - \beta_2)\Gamma(2 - \beta_3)\dots\Gamma(2 - \beta_p)} \quad (\because \beta_1 = 1) \\ \therefore \prod_{n=1}^{\infty} \frac{(n+1)^p - 1}{(n+1)^p} &= \frac{1}{\Gamma(2 - \beta_2)\Gamma(2 - \beta_3)\dots\Gamma(2 - \beta_p)} \end{aligned}$$

**Corollary 2.7:** If  $\epsilon_n$  is constant sequence or  $\epsilon_n = \frac{1}{(n+1)^p}$  where  $p$  is a positive integer then

$$\lim_{k \rightarrow \infty} \frac{\log 2}{-\log \left\{ \left[ \prod_{n=1}^k \frac{(1 - \epsilon_n)}{2} \right]^{1/k} [\epsilon_{k+1}]^{1/k} \right\}} = \overline{\lim}_{k \rightarrow \infty} \frac{\log 2}{-\log \left[ 2 \prod_{n=1}^k \frac{(1 - \epsilon_n)}{2} \right]^{1/k}}$$

Proof : Let  $\epsilon_n$  be constant. Let  $\epsilon_n = \epsilon \forall n$

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{\log 2}{-\log \left\{ \left[ \prod_{n=1}^k \frac{(1 - \epsilon_n)}{2} \right]^{1/k} [\epsilon_{k+1}]^{1/k} \right\}} &= \lim_{k \rightarrow \infty} \frac{\log 2}{-\log \left\{ \left[ \prod_{n=1}^k \frac{(1 - \epsilon)}{2} \right]^{1/k} [\epsilon]^{1/k} \right\}} \\ &= \lim_{k \rightarrow \infty} \frac{\log 2}{-\log \left\{ \left[ \left( \frac{1 - \epsilon}{2} \right)^k \right]^{1/k} [\epsilon]^{1/k} \right\}} \\ &= \lim_{k \rightarrow \infty} \frac{\log 2}{-\log \left\{ \left( \frac{1 - \epsilon}{2} \right) [\epsilon]^{1/k} \right\}} \\ &= \frac{\log 2}{-\log \left( \frac{1 - \epsilon}{2} \right)} \quad \dots\dots\dots(1) \end{aligned}$$

Now,

$$\begin{aligned} \overline{\lim}_{k \rightarrow \infty} \frac{\log 2}{-\log \left[ 2 \prod_{n=1}^k \frac{(1 - \epsilon_n)}{2} \right]^{1/k}} &= \overline{\lim}_{k \rightarrow \infty} \frac{\log 2}{-\log \left[ 2 \prod_{n=1}^k \frac{(1 - \epsilon)}{2} \right]^{1/k}} \\ &= \overline{\lim}_{k \rightarrow \infty} \frac{\log 2}{-\log \left[ 2 \left( \frac{1 - \epsilon}{2} \right)^k \right]^{1/k}} \\ &= \overline{\lim}_{k \rightarrow \infty} \frac{\log 2}{-\log \left[ \left( \frac{1 - \epsilon}{2} \right) (2) \right]^{1/k}} \\ &= \frac{\log 2}{-\log \left( \frac{1 - \epsilon}{2} \right)} \quad \dots\dots\dots(2) \end{aligned}$$

From equations (1) and (2) we get,

$$\therefore \lim_{k \rightarrow \infty} \frac{\log 2}{-\log \left\{ \left[ \prod_{n=1}^k \frac{(1 - \epsilon_n)}{2} \right]^{1/k} [\epsilon_{k+1}]^{1/k} \right\}} = \overline{\lim}_{k \rightarrow \infty} \frac{\log 2}{-\log \left[ 2 \prod_{n=1}^k \frac{(1 - \epsilon_n)}{2} \right]^{1/k}}$$

Let  $\epsilon_n = \frac{1}{(n+1)^p} \forall n$  and  $p$  is a positive integer

$$\begin{aligned} \therefore 1 - \epsilon_n &= 1 - \frac{1}{(n+1)^p} = \frac{(n+1)^p - 1}{(n+1)^p} \\ \prod_{n=1}^k \frac{(1 - \epsilon_n)}{2} &= \prod_{n=1}^k \frac{(n+1)^p - 1}{2(n+1)^p} = \frac{1}{2^k} \prod_{n=1}^k \frac{(n+1)^p - 1}{(n+1)^p} \\ \left[ \prod_{n=1}^k \frac{(1 - \epsilon_n)}{2} \right]^{1/k} &= \left[ \frac{1}{2^k} \prod_{n=1}^k \frac{(n+1)^p - 1}{(n+1)^p} \right]^{1/k} \end{aligned}$$

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$$\begin{aligned} \therefore \left[ \prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]^{1/k} &= \frac{1}{2} \left[ \prod_{n=1}^k \frac{(n+1)^p - 1}{(n+1)^p} \right]^{1/k} \\ \log \left[ \prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]^{1/k} &= \log \left\{ \frac{1}{2} \left[ \prod_{n=1}^k \frac{(n+1)^p - 1}{(n+1)^p} \right]^{1/k} \right\} \\ \therefore \lim_{k \rightarrow \infty} \log \left[ \prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]^{1/k} &= \lim_{k \rightarrow \infty} \log \left\{ \frac{1}{2} \left[ \prod_{n=1}^k \frac{(n+1)^p - 1}{(n+1)^p} \right]^{1/k} \right\} \\ &= -\log 2 + \lim_{k \rightarrow \infty} \log \left\{ \left[ \prod_{n=1}^k \frac{(n+1)^p - 1}{(n+1)^p} \right]^{1/k} \right\} \\ &= -\log 2 + \lim_{k \rightarrow \infty} \frac{\log \left[ \prod_{n=1}^k \frac{(n+1)^p - 1}{(n+1)^p} \right]}{k} \\ &= -\log 2 + \frac{\log \left[ \frac{1}{\Gamma(2-\beta_2)\Gamma(2-\beta_3)\dots\Gamma(2-\beta_p)} \right]}{\infty} \end{aligned} \tag{By Corollary 2.6}$$

$$= -\log 2 + 0$$

$$\therefore \lim_{k \rightarrow \infty} \log \left[ \prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]^{1/k} = \lim_{k \rightarrow \infty} \log \left\{ \frac{1}{2} \left[ \prod_{n=1}^k \frac{(n+1)^p - 1}{(n+1)^p} \right]^{1/k} \right\} = -\log 2 \tag{3}$$

$$\begin{aligned} \text{Now } \underline{\lim}_{k \rightarrow \infty} \frac{\log 2}{-\log \left\{ \left[ \prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]^{1/k} \left[ \epsilon_{k+1} \right]^{1/k} \right\}} &= \underline{\lim}_{k \rightarrow \infty} \frac{\log 2}{-\log \left\{ \left[ \prod_{n=1}^k \frac{\left(1-\frac{1}{(n+1)^p}\right)}{2} \right]^{1/k} \left[ \frac{1}{(k+2)^p} \right]^{1/k} \right\}} \\ &= \underline{\lim}_{k \rightarrow \infty} \frac{\log 2}{-\log \left\{ \left[ \prod_{n=1}^k \frac{(n+1)^p - 1}{2(n+1)^p} \right]^{1/k} \left[ \frac{1}{(k+2)^p} \right]^{1/k} \right\}} \\ &= \underline{\lim}_{k \rightarrow \infty} \frac{\log 2}{-\log \left\{ \frac{1}{2} \left[ \prod_{n=1}^k \frac{(n+1)^p - 1}{(n+1)^p} \right]^{1/k} \left[ \frac{1}{(k+2)^p} \right]^{1/k} \right\}} \\ &= \underline{\lim}_{k \rightarrow \infty} \frac{\log 2}{-\log \left\{ \frac{1}{2} \left[ \prod_{n=1}^k \frac{(n+1)^p - 1}{(n+1)^p} \right]^{1/k} \right\} - \log \left[ \frac{1}{(k+2)^p} \right]^{1/k}} \\ &= \frac{\log 2}{-(-\log 2) - 0} \end{aligned} \tag{From equation (3)}$$

$$= \frac{\log 2}{\log 2} = 1 \tag{4}$$

$$\begin{aligned} \text{Now, } \overline{\lim}_{k \rightarrow \infty} \frac{\log 2}{-\log \left[ 2 \prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]^{1/k}} &= \overline{\lim}_{k \rightarrow \infty} \frac{\log 2}{-\log \left[ 2 \prod_{n=1}^k \frac{\left(1-\frac{1}{(n+1)^p}\right)}{2} \right]^{1/k}} \\ &= \overline{\lim}_{k \rightarrow \infty} \frac{\log 2}{-\log \left[ 2 \prod_{n=1}^k \frac{(n+1)^p - 1}{(n+1)^p} \right]^{1/k}} \\ &= \overline{\lim}_{k \rightarrow \infty} \frac{\log 2}{-\log (2)^{1/k} - \log \left\{ \frac{1}{2} \left[ \prod_{n=1}^k \frac{(n+1)^p - 1}{(n+1)^p} \right]^{1/k} \right\}} \\ &= \frac{\log 2}{0 - (-\log 2)} \end{aligned} \tag{From equation(3)}$$

$$\therefore \overline{\lim}_{k \rightarrow \infty} \frac{\log 2}{-\log \left[ 2 \prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]^{1/k}} = 1 \tag{5}$$

From equations (4) and (5) we get,

$$\underline{\lim}_{k \rightarrow \infty} \frac{\log 2}{-\log \left\{ \left[ \prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]^{1/k} \left[ \epsilon_{k+1} \right]^{1/k} \right\}} = \overline{\lim}_{k \rightarrow \infty} \frac{\log 2}{-\log \left[ 2 \prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]^{1/k}}$$

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**Remark 2.8 :** According to Corollary 2.7 we observe that  $\lim_{k \rightarrow \infty} \frac{\log 2}{-\log \{ [\prod_{n=1}^k \frac{(1-\epsilon_n)}{2}]^{1/k} [\epsilon_{k+1}]^{1/k} \}} = \overline{\dim}_B(S)$  for  $\epsilon_n = \text{constant sequence}$  or  $\epsilon_n = \frac{1}{(n+1)^p}$  for p is positive integer.  $\overline{\dim}_B(S) = \underline{\dim}_B(S)$ .

### III CONCLUSION

We have obtained Hausdorff dimension of Cantor like set S constructed in [3] as  $\dim_H(S) = \frac{-\log 2}{\gamma}$ , where  $\gamma = \lim_{k \rightarrow \infty} \log [ \prod_{n=1}^k \frac{(1-\epsilon_n)}{2} ]^{1/k}$ . Also we have estimated the upper / lower Box dimensions of such Cantor like set S. It is proved that  $\lim_{k \rightarrow \infty} \frac{\log 2}{-\log [ 2 \prod_{n=1}^k \frac{(1-\epsilon_n)}{2} ]^{1/k}} \geq \overline{\dim}_B(S) \geq \underline{\dim}_B(S) \geq \lim_{k \rightarrow \infty} \frac{\log 2}{-\log \{ [\prod_{n=1}^k \frac{(1-\epsilon_n)}{2}]^{1/k} [\epsilon_{k+1}]^{1/k} \}}$ . Also we conclude that  $\underline{\dim}_B(S) = \overline{\dim}_B(S)$  for  $\epsilon_n = \epsilon, n \geq 1$  or  $\epsilon_n = \frac{1}{(n+1)^p}, n \geq 1$ , positive integer p.

### REFERENCES

- [1] G.de Barra, " Measure Theory And Integration", New Age International (p) Limited, 2003.
- [2] Kenneth Falconer, "Fractal Geometry : Mathematical Foundations and Applications" ,John Willey and Sons.1990.
- [3] Dr. S.M. Padhye, Satish B. Khobragade , " On the construction of Cantor like sets", The IJES, Volume 4 ,Issue 4, pages 49-54, 2015.
- [4] J. Shah. Hausdorff dimension and its application. Manuscript.  
<http://math.uchicago.edu/~may/VIGRE/VIGRE2009/REUPapers/Shah.pdf>
- [5] Kannan V. , "Cantor set: From Classical To Modern". The Mathematical Student, Vol 63, No.1-4(1994)P.243-257.
- [6] E. C. Titchmarsh , "The Theory Of Functions", Oxford University Press, Oxford Second .Edition -1987.