

# Some Interesting Properties and Extended Binet Formula for the Generalized Lucas Sequence

Daksha Diwan<sup>1</sup>, Devbhadra V. Shah<sup>2</sup>

Assistant Professor, Department of Mathematics, Government Engineering College, Gandhinagar, Gujarat, India<sup>1</sup>

Associate Professor, Department of Mathematics, Sir P.T.Sarvajani College of Science, Surat, Gujarat, India<sup>2</sup>

**ABSTRACT:** Lucas sequence  $\{L_n\}_{n=0}^{\infty}$  is defined by the recurrence relation  $L_n = L_{n-1} + L_{n-2}$ ;  $n \geq 2$  with initial condition  $L_0 = 2, L_1 = 1$ . One of the generalizations of the Lucas sequence is the class of sequences  $\{L_n^{L(a,b)}\}_{n=0}^{\infty}$  generated by the recurrence relation

$$L_n^{L(a,b)} = \begin{cases} aL_{n-1}^{L(a,b)} + L_{n-2}^{L(a,b)} ; & \text{when } n \text{ is odd} \\ bL_{n-1}^{L(a,b)} + L_{n-2}^{L(a,b)} ; & \text{when } n \text{ is even} \end{cases} \quad (n \geq 2)$$

with initial condition  $L_0^{L(a,b)} = 2, L_1^{L(a,b)} = 1$  and  $a, b$  are positive integers. The Lucas sequence is a special case of these sequences with  $a = b = 1$ .

In this paper we obtain extended Binet's formula for  $\{L_n^{L(a,b)}\}_{n=0}^{\infty}$  and derive some interesting properties of this Generalized Lucas sequence.

**KEYWORDS:** Lucas sequence, Generalized Lucas sequence, Binet formula.

## I. INTRODUCTION

In recent years, many interesting properties of classic Fibonacci numbers, classic Lucas number and their generalizations have been shown by researchers and applied to almost every field of science and art. Cennet, Ahmet, Hasan [1], Kenan, Adem [7] and Shah, Shah [11] defined new generalizations of Lucas sequence and gave various identities along with extended Binet formula for the concerned new generalizations. For the rich and related applications of Lucas numbers, one can refer to the nature and different areas of the science [2, 3, 4, 5, 6, 8, 13, 12, 14]. The classic Lucas sequence  $\{L_n\}$  is defined as  $L_0 = 2, L_1 = 1$  and  $L_n = L_{n-1} + L_{n-2}$  for  $n \geq 2$ .

The first few terms of the Lucas sequence are : 2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, 1364, 2207, 3571... (Koshy [9])

We define a generalization of the Lucas sequence and call it the *generalized Lucas sequence*.

**Definition:** For any two positive numbers  $a$  and  $b$ , the *generalized Lucas sequence*  $\{L_n^G(a, b, 1, 1)\} = \{L_n^{L(a,b)}\} (= \{L_n^L\})$  is defined by the recurrence relation

$$L_n^L = a^{\chi(n)} b^{1-\chi(n)} L_{n-1}^L + L_{n-2}^L,$$

where  $L_0^L = 2, L_1^L = 1$  and  $a, b$  are any two positive integers.

This can be equivalently expressed as

$$L_n^L = \begin{cases} aL_{n-1}^L + L_{n-2}^L ; & \text{if } n \text{ is odd} \\ bL_{n-1}^L + L_{n-2}^L ; & \text{if } n \text{ is even} \end{cases} \quad (n \geq 2) \quad (1.1)$$

with  $L_0^L = 2, L_1^L = 1$ .

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The Lucas sequence is a special case of these sequences with  $a = b = 1$ . In this paper we derive extended Binet's formula for  $\{L_n^{L(a,b)}\}_{n=0}^{\infty}$  and some of its interesting properties.

## II. EXTENDED BINET'S FORMULA FOR $\{L_n^{L(a,b)}\}$

We first note down two results which will be useful for further manipulations.

**Lemma 2.1:**  $L_{2n+5}^L - (ab + 2)L_{2n+3}^L + L_{2n+1}^L = 0.$

*Proof:* By using definition (1.1), we get

$$\begin{aligned} L_{2n+5}^L - (ab + 2)L_{2n+3}^L + L_{2n+1}^L &= (ab + 1)L_{2n+3}^L + aL_{2n+2}^L - (ab + 2)L_{2n+3}^L + L_{2n+1}^L \\ &= (aL_{2n+2}^L + L_{2n+1}^L) - L_{2n+3}^L = 0. \end{aligned}$$

We now obtain the value of the following series related with  $L_n^L$ .

**Lemma 2.2:**  $\sum_{i=1}^{\infty} L_{2i-1}^L x^{2i-1} = \frac{x+(2a-1)x^3}{x^4-(ab+2)x^2+1}.$

*Proof:* Let  $l(x) = \sum_{i=1}^{\infty} L_{2i-1}^L x^{2i-1} = L_1^L x + L_3^L x^3 + L_5^L x^5 + \dots$

$\therefore x^4 l(x) = L_1^L x^5 + L_3^L x^7 + L_5^L x^9 + \dots$  and

$(ab + 2)x^2 l(x) = L_1^L x^5 + L_3^L x^7 + L_5^L x^9 + \dots$

Using lemma 2.1, we get

$$(x^4 - (ab + 2)x^2 + 1)l(x) = x + (2a - 1)x^3.$$

This gives  $l(x) = \sum_{i=1}^{\infty} L_{2i-1}^L x^{2i-1} = \frac{x+(2a-1)x^3}{x^4-(ab+2)x^2+1}.$

We next derive the generating function for  $\{L_n^L\}$ .

**Lemma 2.3:** The generating function for  $\{L_n^L\}$  is given by

$$L(x) = \frac{(2a-1)x^3 - (2ab+2-b)x^2 + x + 2}{x^4 - (ab+2)x^2 + 1}.$$

*Proof:* Define  $(x) = \sum_{i=0}^{\infty} L_i^L x^i = L_0^L + L_1^L x + L_2^L x^2 + L_3^L x^3 + \dots$ . Then after the rearrangement of terms, we get

$$(1 - ax - x^2)L(x) = 2 + (1 - 2a)x + (b - a)x \left( \sum_{i=1}^{\infty} L_{2i-1}^L x^{2i-1} \right).$$

Using Lemma 2.2 and on simplification we get the generating function for  $\{L_n^{L(a,b)}\}$  as

$$\begin{aligned} L(x) &= \frac{2 + (1 - 2a)x}{(1 - ax - x^2)} + \frac{(b - a)x}{(1 - ax - x^2)} \left\{ \frac{x + (2a - 1)x^3}{x^4 - (ab + 2)x^2 + 1} \right\} \\ &= \frac{((2a - 1)x^3 - (2ab + 2 - b)x^2 + x + 2)(1 - ax - x^2)}{(1 - ax - x^2)(x^4 - (ab + 2)x^2 + 1)} \end{aligned}$$

$$= \frac{((2a-1)x^3 - (2ab+2-b)x^2 + x + 2)}{(x^4 - (ab+2)x^2 + 1)}.$$

We are now all set to derive the extended Binet's formula for  $L_n^{L(a,b)}$ .

**Theorem 2.4:**  $L_n^L = \frac{1}{(ab)^{[n/2]} b^{\chi(n)}} \left( \frac{\gamma \alpha^n - \delta \beta^n}{\alpha - \beta} \right)$ , where  $\gamma = (b - 2ab + 2a)$ ,

$\delta = (b - 2ab + 2\beta)$  and  $\alpha = \frac{ab + \sqrt{a^2 b^2 + 4ab}}{2}, \beta = \frac{ab - \sqrt{a^2 b^2 + 4ab}}{2}$  be the roots of equation  $x^2 - abx - ab = 0$ .

*Proof:* By lemma 2.3, we have  $L(x) = \frac{((2a-1)x^3 - (2ab+2-b)x^2 + x + 2)}{(x^4 - (ab+2)x^2 + 1)}$ .

If we write  $L(x) = \frac{((2a-1)x^3 - (2ab+2-b)x^2 + x + 2)}{x^4 - (ab+2)x^2 + 1} = \frac{(Ax+B)}{x^2 - (\alpha+1)} + \frac{(Cx+D)}{x^2 - (\beta+1)}$ , then by using the method of partial fractions,

we get

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$$L(x) = \frac{1}{(\alpha - \beta)} \left[ \left\{ \frac{((2a-1)(\alpha+1)+1)x+2-((\alpha+1)(2ab+2-b))}{x^2-(\alpha+1)} \right\} - \left\{ \frac{1+(2a-1)(\beta+1)x+2-(2ab+2-b)(\beta+1)}{x^2-(\beta+1)} \right\} \right].$$

Then by using Maclaurin's expansion [10] of  $\frac{P-Qx}{x^2-R}$  as

$$\frac{P-Qx}{x^2-R} = \sum_{n=0}^{\infty} QR^{-n-1}x^{2n+1} - \sum_{n=0}^{\infty} PR^{-n-1}x^{2n},$$

we get

$$\frac{2 - (\alpha + 1)(2ab + 2 - b) - ((1 - 2a)(\alpha + 1) - 1)x}{x^2 - (\alpha + 1)} = \left[ \sum_{n=0}^{\infty} \frac{(1 - 2a)(\alpha + 1) - 1}{(\alpha + 1)^{n+1}} x^{2n+1} - \sum_{n=0}^{\infty} \frac{2 - (\alpha + 1)(2ab + 2 - b)}{(\alpha + 1)^{n+1}} x^{2n} \right]$$

and

$$\frac{-2 + (\beta + 1)(2ab + 2 - b) - ((2a - 1)(\beta + 1) + 1)x}{x^2 - (\beta + 1)} = \left[ \sum_{n=0}^{\infty} \frac{(2a-1)(\beta+1)+1}{(\beta+1)^{n+1}} x^{2n+1} - \sum_{n=0}^{\infty} \frac{-2-(\beta+1)(2ab+2-b)}{(\beta+1)^{n+1}} x^{2n} \right].$$

Thus,

$$L(x) = \frac{1}{(\alpha - \beta)} \left\{ \sum_{n=0}^{\infty} \left( \frac{(1 - 2a)(\alpha + 1) - 1}{(\alpha + 1)^{n+1}} - \frac{(1 - 2a)(\beta + 1) - 1}{(\beta + 1)^{n+1}} \right) x^{2n+1} \right\} + \frac{1}{(\alpha - \beta)} \sum_{n=0}^{\infty} \left\{ \left( \frac{2-(\beta+1)(2ab+2-b)}{(\beta+1)^{n+1}} - \frac{2-(\alpha+1)(2ab+2-b)}{(\alpha+1)^{n+1}} \right) \right\} x^{2n}.$$

Now using  $\frac{\alpha^2}{ab} = \alpha + 1$  and  $\frac{\beta^2}{ab} = \beta + 1$  we get

$$L(x) = \frac{1}{(\alpha - \beta)} \left\{ \sum_{n=0}^{\infty} \left[ (1 - 2a) \left( \frac{\alpha \left(\frac{\beta^2}{ab}\right)^{n+1} - \beta \left(\frac{\alpha^2}{ab}\right)^{n+1}}{\left(\frac{\alpha^2}{ab}\right)^{n+1} \left(\frac{\beta^2}{ab}\right)^{n+1}} \right) - 2a \left( \frac{\left(\frac{\beta^2}{ab}\right)^{n+1} - \left(\frac{\alpha^2}{ab}\right)^{n+1}}{\left(\frac{\alpha^2}{ab}\right)^{n+1} \left(\frac{\beta^2}{ab}\right)^{n+1}} \right) \right] x^{2n+1} \right\} + \frac{1}{(\alpha - \beta)} \left\{ \sum_{n=0}^{\infty} \left[ (b - 2ab - 2) \left( \frac{\left(\frac{\beta^2}{ab}\right)^{n+1} \left(\frac{\alpha^2}{ab}\right)^{n+1} - \left(\frac{\alpha^2}{ab}\right)^{n+1} \left(\frac{\beta^2}{ab}\right)^{n+1}}{\left(\frac{\beta^2}{ab}\right)^{n+1} \left(\frac{\alpha^2}{ab}\right)^{n+1}} \right) + 2 \left( \frac{\left(\frac{\alpha^2}{ab}\right)^{n+1} - \left(\frac{\beta^2}{ab}\right)^{n+1}}{\left(\frac{\beta^2}{ab}\right)^{n+1} \left(\frac{\alpha^2}{ab}\right)^{n+1}} \right) \right] x^{2n} \right\} = \frac{1}{(\alpha - \beta)} \left\{ \sum_{n=0}^{\infty} \left[ \frac{(1-2a)(ab)^{n+1}\alpha\beta(\beta^{2n+1}-\alpha^{2n+1})}{(\alpha\beta)\alpha^{2n+1}\beta^{2n+1}} + 2a \frac{(ab)^{n+1}(\alpha^{2n+2}-\beta^{2n+2})}{(\alpha\beta)^{2n+2}} \right] x^{2n+1} \right\} + \frac{1}{(\alpha - \beta)} \left\{ \sum_{n=0}^{\infty} \left[ (b - 2ab - 2) \frac{(ab)^n(\alpha\beta)^2(\alpha^{2n}-\beta^{2n})}{(\alpha\beta)^2\alpha^{2n}\beta^{2n}} + 2 \frac{(ab)^{n+1}(\alpha^{2n+2}-\beta^{2n+2})}{(\alpha\beta)^{2n+2}} \right] x^{2n} \right\}.$$

Using  $\alpha\beta = -ab$ , we get

$$L(x) = \frac{1}{(\alpha - \beta)} \left\{ \sum_{n=0}^{\infty} \left( \frac{(1 - 2a)(ab)^{n+1}(\beta^{2n+1} - \alpha^{2n+1})}{(-ab)^{2n+1}} + 2a \frac{(\alpha^{2n+2} - \beta^{2n+2})}{(ab)^{n+1}} \right) x^{2n+1} \right\} + \frac{1}{(\alpha - \beta)} \left\{ \sum_{n=0}^{\infty} \left( (b - 2ab - 2) \frac{(\alpha^{2n} - \beta^{2n})}{(ab)^n} + 2 \frac{(\alpha^{2n+2} - \beta^{2n+2})}{(ab)^{n+1}} \right) x^{2n} \right\}$$

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$$= \frac{1}{(\alpha - \beta)} \left\{ \sum_{n=0}^{\infty} \frac{x^{2n+1}}{b(ab)^n} \{ (b - 2ab + 2\alpha)\alpha^{2n+1} - (b - 2ab + 2\beta)\beta^{2n+1} \} \right\}$$

$$+ \frac{1}{(\alpha - \beta)} \left\{ \sum_{n=0}^{\infty} \frac{x^{2n}}{(ab)^n} \left[ \left( (b - 2ab - 2) + 2 \left( \frac{\alpha^2}{ab} \right) \right) \alpha^{2n} - \left( (b - 2ab - 2) + 2 \left( \frac{\beta^2}{ab} \right) \right) \beta^{2n} \right] \right\}$$

Using  $\frac{\alpha^2}{ab} = \alpha + 1$  and  $\frac{\beta^2}{ab} = \beta + 1$  we get

$$L(x) = \frac{1}{(\alpha - \beta)} \left\{ \sum_{n=0}^{\infty} \frac{x^{2n+1}}{b(ab)^n} [(b - 2ab + 2\alpha)\alpha^{2n+1} - (b - 2ab + 2\beta)\beta^{2n+1}] \right\}$$

$$+ \frac{1}{(\alpha - \beta)} \left\{ \sum_{n=0}^{\infty} \frac{x^{2n}}{(ab)^n} \left[ ((b - 2ab - 2) + 2(\alpha + 1))\alpha^{2n} \right] \right\}$$

$$- \frac{1}{(\alpha - \beta)} \left\{ \sum_{n=0}^{\infty} \frac{x^{2n}}{(ab)^n} \left[ -((b - 2ab - 2) + 2(\beta + 1))\beta^{2n} \right] \right\}$$

$$= \frac{1}{(\alpha - \beta)} \left\{ \sum_{n=0}^{\infty} \frac{x^{2n+1}}{b(ab)^n} [(b - 2ab + 2\alpha)\alpha^{2n+1} - (b - 2ab + 2\beta)\beta^{2n+1}] \right\}$$

$$+ \frac{1}{(\alpha - \beta)} \left\{ \sum_{n=0}^{\infty} \frac{x^{2n}}{(ab)^n} [(b - 2ab + 2\alpha)\alpha^{2n} - (b - 2ab + 2\beta)\beta^{2n}] \right\}.$$

Thus if we write  $\gamma = (b - 2ab + 2\alpha)$  and  $\delta = (b - 2ab + 2\beta)$ , then

$$L(x) = \sum_{n=0}^{\infty} \frac{1}{(ab)^{\lfloor n/2 \rfloor} b^{\chi(n)}} \left( \frac{\gamma \alpha^n - \delta \beta^n}{\alpha - \beta} \right) x^n.$$

Hence the required extended Binet formula is

$$L_n^L = \frac{1}{(ab)^{\lfloor n/2 \rfloor} b^{\chi(n)}} \left( \frac{\gamma \alpha^n - \delta \beta^n}{\alpha - \beta} \right)$$

where  $\gamma = (b - 2ab + 2\alpha)$ ,  $\delta = (b - 2ab + 2\beta)$ .

### III. SOME INTERESTING PROPERTIES OF $L_n^L$

Here we derive some interesting properties for  $L_n^L$  from its extended Binet formula. We first find expressions which characterizes  $L_n^L$  in terms of  $L_{n-1}^L$  and powers of  $\alpha$  or  $\beta$ .

**Theorem 3.1:**  $L_n^L = \frac{\alpha}{a^{1-\chi(n)} b^{\chi(n)}} L_{n-1}^L + \frac{\beta^{n-1}}{(ab)^{\lfloor n/2 \rfloor} a^{1-\chi(n)}} ; n \geq 2.$

*Proof:* By theorem 2.4 we have  $L_n^L = C(\gamma \alpha^n - \delta \beta^n)$ ,

where  $C = \frac{1}{b^{\chi(n)} (ab)^{\lfloor n/2 \rfloor} (\alpha - \beta)}$ ,  $\gamma = (b - 2ab + 2\alpha)$ ,  $\delta = (b - 2ab + 2\beta)$ .

Now consider the cases  $n = 2k$  and  $n = 2k + 1$ .

$$\therefore L_{2k}^L = \frac{1}{(ab)^{\lfloor 2k/2 \rfloor}} \left( \frac{\gamma \alpha^{2k} - \delta \beta^{2k}}{\alpha - \beta} \right) = \frac{1}{(ab)^k} \left( \frac{\gamma \alpha^{2k} - \delta \beta^{2k}}{\alpha - \beta} \right) \text{ and } L_{2k+1}^L = \frac{1}{b(ab)^k} \left( \frac{\gamma \alpha^{2k+1} - \delta \beta^{2k+1}}{\alpha - \beta} \right).$$

$$\therefore bL_{2k+1}^L - \alpha L_{2k}^L = \frac{1}{(ab)^k (\alpha - \beta)} (\gamma \alpha^{2k+1} - \delta \beta^{2k+1} - \gamma \alpha^{2k+1} + \delta \beta^{2k}) = \frac{\delta \beta^{2k} (\alpha - \beta)}{(ab)^k (\alpha - \beta)}.$$

$$\therefore L_{2k+1}^L - \frac{\alpha}{b} L_{2k}^L = \frac{\delta \beta^{2k}}{b(ab)^k}. \text{ Thus}$$

$$L_{2k+1}^L = \frac{\alpha}{b} L_{2k}^L + \frac{\delta \beta^{2k}}{b(ab)^k}. \quad (3.1)$$

Again considering  $n = 2k + 1$  and  $n = 2k + 2$  in theorem 2.4, we get

$$L_{2k+1}^L = \frac{1}{b(ab)^{\lfloor 2k+1/2 \rfloor}} \left( \frac{\gamma \alpha^{2k+1} - \delta \beta^{2k+1}}{\alpha - \beta} \right) = \frac{1}{b(ab)^k} \left( \frac{\gamma \alpha^{2k+1} - \delta \beta^{2k+1}}{\alpha - \beta} \right) \text{ and } L_{2k+2}^L = \frac{1}{(ab)^{k+1}} \left( \frac{\gamma \alpha^{2k+2} - \delta \beta^{2k+2}}{\alpha - \beta} \right).$$

Using these two results, we get

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$$abL_{2k+2}^L - b\alpha L_{2k+1}^L = \frac{1}{(ab)^{k+1}(\alpha-\beta)} (ab\gamma\alpha^{2k+2} - \delta ab\beta^{2k+2} - \gamma ab\alpha^{2k+2} + \delta ab\alpha\beta^{2k+1}) = \frac{\delta\beta^{2k+1}(\alpha-\beta)}{(ab)^k(\alpha-\beta)}.$$

$$\therefore L_{2k+2}^L - \frac{\alpha}{a} L_{2k+1}^L = \frac{\delta\beta^{2k+1}}{(ab)^{k+1}}. \text{ This gives}$$

$$L_{2k+2}^L = \frac{\alpha}{a} L_{2k+1}^L + \frac{\delta\beta^{2k+1}}{(ab)^{k+1}}. \quad (3.2)$$

Finally, combining (3.1) and (3.2), we get

$$L_n^L = \frac{\alpha}{a^{1-\chi(n)}b^{\chi(n)}} L_{n-1}^L + \frac{\delta\beta^{n-1}}{(ab)^{\lfloor n/2 \rfloor} b^{\chi(n)}}.$$

**Theorem 3.2:**  $L_n^L = \frac{\beta}{a^{1-\chi(n)}b^{\chi(n)}} L_{n-1}^L + \frac{\alpha^{n-1}}{(ab)^{\lfloor n/2 \rfloor} b^{\chi(n)}}; n \geq 2.$

*Proof:* From theorem 2.4 we have,  $L_n^L = C(\gamma\alpha^n - \delta\beta^n)$ , where  $C = \frac{1}{b^{\chi(n)}(ab)^{\lfloor n/2 \rfloor}(\alpha-\beta)}$ ,

$$\gamma = (b - 2ab + 2\alpha), \delta = (b - 2ab + 2\beta).$$

Now consider the cases when  $n = 2k$  or  $n = 2k + 1$ .

$$\therefore L_{2k}^L = \frac{1}{(ab)^{\lfloor 2k/2 \rfloor}} \left( \frac{\gamma\alpha^{2k} - \delta\beta^{2k}}{\alpha-\beta} \right) = \frac{1}{(ab)^k} \left( \frac{\gamma\alpha^{2k} - \delta\beta^{2k}}{\alpha-\beta} \right) \text{ and } L_{2k+1}^L = \frac{1}{(ab)^k} \left( \frac{\gamma\alpha^{2k+1} - \delta\beta^{2k+1}}{\alpha-\beta} \right).$$

Using these two results we get

$$bL_{2k+1}^L - \beta L_{2k}^L = \frac{1}{(ab)^k(\alpha-\beta)} (\gamma\alpha^{2k+1} - \delta\beta^{2k+1} - \gamma\alpha^{2k}\beta + \delta\beta^{2k+1}) = \frac{\gamma\alpha^{2k}(\alpha-\beta)}{(ab)^k(\alpha-\beta)}.$$

$$\therefore L_{2k+1}^L - \frac{\beta}{b} L_{2k}^L = \frac{\gamma\alpha^{2k}}{b(ab)^k}. \text{ Thus}$$

$$L_{2k+1}^L = \frac{\beta}{b} L_{2k}^L + \frac{\gamma\alpha^{2k}}{b(ab)^k}. \quad (3.3)$$

Again considering  $n = 2k + 1$  and  $n = 2k + 2$  in theorem 2.4, we get

$$L_{2k+1}^L = \frac{1}{b(ab)^{\lfloor 2k+1/2 \rfloor}} \left( \frac{\gamma\alpha^{2k+1} - \delta\beta^{2k+1}}{\alpha-\beta} \right) = \frac{1}{b(ab)^k} \left( \frac{\gamma\alpha^{2k+1} - \delta\beta^{2k+1}}{\alpha-\beta} \right) \text{ and } L_{2k+2}^L = \frac{1}{(ab)^{k+1}} \left( \frac{\gamma\alpha^{2k+2} - \delta\beta^{2k+2}}{\alpha-\beta} \right).$$

$$\therefore abL_{2k+2}^L - b\beta L_{2k+1}^L = \frac{1}{(ab)^{k+1}(\alpha-\beta)} (ab\gamma\alpha^{2k+2} - \delta ab\beta^{2k+2} - \gamma ab\alpha^{2k+1}\beta + \delta ab\alpha\beta^{2k+2}) = \frac{\gamma\alpha^{2k+1}(\alpha-\beta)}{(ab)^k(\alpha-\beta)}.$$

$$\therefore L_{2k+2}^L - \frac{\beta}{a} L_{2k+1}^L = \frac{\gamma\alpha^{2k+1}}{(ab)^k}. \text{ Hence}$$

$$L_{2k+2}^L = \frac{\beta}{a} L_{2k+1}^L + \frac{\gamma\alpha^{2k+1}}{(ab)^k}. \quad (3.4)$$

Combining (3.3) and (3.4) we get

$$L_n^L = \frac{\beta}{a^{1-\chi(n)}b^{\chi(n)}} L_{n-1}^L + \frac{\gamma\alpha^{n-1}}{(ab)^{\lfloor n/2 \rfloor} b^{\chi(n)}}.$$

**Corollary 3.3:**  $2L_n^L = \frac{\alpha+\beta}{a^{1-\chi(n)}b^{\chi(n)}} L_{n-1}^L + \frac{\gamma\alpha^{n-1} + \delta\beta^{n-1}}{(ab)^{\lfloor n/2 \rfloor} b^{\chi(n)}}.$

*Proof:* Adding the results of theorems 3.1 and 3.2, we get

$$2L_n^L = \frac{\alpha+\beta}{a^{1-\chi(n)}b^{\chi(n)}} L_{n-1}^L + \frac{\gamma\alpha^{n-1} + \delta\beta^{n-1}}{(ab)^{\lfloor n/2 \rfloor} b^{\chi(n)}}, \text{ as required.}$$

### IV. CONCLUSION

In this paper we derived extended Binet's formula for  $\{L_n^{L(a,b)}\}_{n=0}^{\infty}$  in theorem 2.4 and derive some interesting properties of this Generalized Lucas sequence in section 3.

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