

Boundedness Properties for Some Integral Transform

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ABSTRACT: There are various integral transforms which have widely used in physics, astronomy as well as in engineering. In order to solve the differential equations, the integral transform were extensively used and thus there are several works on the theory and application of integral transform such as the Laplace, Fourier, Mellin and Hankel etc. and these transforms have various properties which are very applicable which is given in our previous papers. In this paper, we have discussed the boundedness properties of Fourier-Laplace transform and Fourier-Finite Mellin transform.

KEYWORDS: Fourier transform; Laplace transform; Fourier- Laplace transform; Finite Mellin Transform; Fourier-Finite Mellin Transform; Integral Transform.

I.INTRODUCTION

Various integral transforms have been extensively used in the formulation of electromagnetic scattering, radiation, antennas, and electromagnetic interference related problems. The integral transform technique is an indispensable tool for representing the fields in the unbounded (open) region [1]. Integral transforms play an important role in Analytic number theory [2]. An integral transform maps some function onto another one and as a consequence a function space into or onto another one. Operations in the original space are converted in general into operation in the image space. Integral transforms are therefore used in the first place if handling with the operations in the image space is easier to do or is better known as in the original space. As an example: the Laplace transforms converts differentiation in the space of definition into a simple algebraic operation in the image space [3].

The most common integral transforms that are used are: Fourier transforms (Joseph Fourier, 1768-1830), Laplace transforms (Pierre-Simon, marquis de Laplace, 1749-1827) and Mellin transforms (Robert Hjalmar Mellin, 1854-1933). The Fourier transform technique has long been used for electromagnetic scattering, diffraction, and antenna applications [1]. The Laplace transform is also very useful in the solution of many partial differential equations and it can be used as a very effective tool in simplifying the calculations in many fields of Engineering and Mathematics [4]. The Mellin transform is a basic tool for analyzing the behavior of many important functions in Mathematics and mathematical physics. Mellin transform has many applications such as quantum calculus, radar classification of ships, electromagnetic stress distribution, agriculture, medical stream, statistics, probability, signal processing, optics, pattern recognition, algorithms, correlators, navigation, vowel recognition and cryptographic scheme [5]. It is also used in solution of fractional differential equations [6].

Many authors studied various properties of integral transforms. Dhunde R. R. et.al. [4] discussed some remarks on the properties of double Laplace transform, V. D. Sharma and P. B. Deshmukh [5] have also discussed operation transform formulae for two dimensional fractional Mellin transform. V. D. Sharma and A. N. Rangari have already described operation transform formulae of Fourier-Laplace transform [7] and operational calculus on Generalized Fourier-Laplace transform [8]. Motivated by the work here we discussed and proved the boundedness properties for Fourier-Laplace transform and Fourier-Finite Mellin transform. So these transforms are as follows:

$$FL\{f(t, x)\} = F(s, p) = \int_{-\infty}^{\infty} \int_0^{\infty} K(t, x) f(t, x) dt dx$$

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where, $K(t, x) = e^{-i(st-ipx)}$. Also

$$FM_f \{f(t, x)\} = F(s, p) = \int_{-\infty}^{\infty} \int_0^a K(t, x) f(t, x) dt dx$$

where, $K(t, x) = e^{-ist} \left(\frac{a^{2p}}{x^{p+1}} - x^{p-1} \right)$.

And the development of these transforms requires some testing function spaces and Distributional Fourier-Laplace transform as well as Distributional Fourier-Finite Mellin transform as follows:

1.1. The space $FL_{a,b,\alpha}$

This space is given by

$$FL_{a,b,\alpha} = \left\{ \phi : \phi \in E_+ / \xi_{a,b,k,q,l} \phi(t, x) = 0 < t < \infty \left| t^k e^{ax} D_t^l D_x^q \phi(t, x) \right| \leq C_{lq} A^k k^{k\alpha} \right. \\ \left. 0 < x < \infty \right\} \quad (1.1.1)$$

Where the constants A and C_{lq} depend on the testing function ϕ .

1.2. The space $FM_{f,b,c,\alpha}$

This space is given by

$$FM_{f,b,c,\alpha} = \left\{ \phi : \phi \in E_+ / \xi_{b,c,k,q,l} \phi(t, x) = 0 < t < \infty \left| t^k \lambda_{b,c}(x) x^{q+1} D_t^l D_x^q \phi(t, x) \right| \leq C_{lq} A^k k^{k\alpha} \right. \\ \left. 0 < x < a \right\} \quad (1.2.1)$$

for each $k, l, q = 0, 1, 2, 3, \dots$ where, $\lambda_{b,c}(x) = \begin{cases} x^{+b} & 0 < x < 1 \\ x^{+c} & 1 < x < a \end{cases}$

where the constants A and C_{lq} depend on the testing function ϕ .

1.3. Distributional Generalized Fourier-Laplace Transforms (FLT)

For $f(t, x) \in FL_{a,\alpha}^{*\beta}$, where $FL_{a,\alpha}^{*\beta}$ is the dual space of $FL_{a,\alpha}^\beta$. It contains all distributions of compact support. The distributional Fourier-Laplace transform is a function of $f(t, x)$ and is defined as

$$FL\{f(t, x)\} = F(s, p) = \langle f(t, x), e^{-i(st-ipx)} \rangle, \quad (1.3.1)$$

where, for each fixed t ($0 < t < \infty$), x ($0 < x < \infty$), $s > 0$ and $p > 0$, the right hand side of (1.3.1) has a sense as an application of $f(t, x) \in FL_{a,\alpha}^{*\beta}$ to $e^{-i(st-ipx)} \in FL_{a,\alpha}^\beta$.

1.4. Distributional Generalized Fourier-Finite Mellin Transforms ($FM_f T$)

For $f(t, x) \in FM_{f,b,c,\alpha}^{*\beta}$, where $FM_{f,b,c,\alpha}^{*\beta}$ is the dual space of $FM_{f,b,c,\alpha}^\beta$. It contains all distributions of compact support. The Distributional Fourier-Finite Mellin transform is a function of $f(t, x)$ and is defined as

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$$FM_f \{f(t, x)\} = F(s, p) = \left\langle f(t, x), e^{-ist} \left(\frac{a^{2p}}{x^{p+1}} - x^{p-1} \right) \right\rangle \quad (1.4.1)$$

where, for each fixed t ($0 < t < \infty$), x ($0 < x < \infty$), $s > 0$ and $p > 0$, the right hand side of (1.4.1) has a sense as an application of $f(t, x) \in FM_{f,b,c,\alpha}^{*\beta}$ to $e^{-ist} \left(\frac{a^{2p}}{x^{p+1}} - x^{p-1} \right) \in FM_{f,b,c,\alpha}^\beta$.

Here we have generalized the integral transform in distributional sense. The main purpose of this paper is to give the boundedness properties of Fourier-Laplace transform and Fourier-Finite Mellin transform.

The planning of this paper is as follows:

Boundedness theorem for Fourier-Laplace Transform is given in section 2. Section 3 gives the Boundedness theorem for Fourier-Finite Mellin transform. Section 4 concludes the paper.

Notations and terminology as per Zemanian. [9], [10].

II. BOUNDEDNESS THEOREM FOR FOURIER-LAPLACE TRANSFORM

2.1. Theorem (Boundedness)

Let $f(t, x) \in FL_{a,b,\alpha}^*$ and $F(s, p) = FL\{f(t, x)\} = \left\langle f(t, x), e^{-i(st-ix)} \right\rangle$, $a < \text{Re } p < b, s > 0$. Let

$\text{supp } f(t, x) \in S_A \cap S_B$ such that $S_A = \{t : t \in \mathbb{R}^n, |t| \leq A, A > 0\}$ and $S_B = \{x : x \in \mathbb{R}^n, |x| \leq B, B > 0\}$, then

for each $\varepsilon > 0$, $\delta > 0$ there exist a constant $C > 0$ and a non-negative integer n such that

$$|F(s, p)| \leq C \left(1 + |s|^l\right) e^{s|(A+\varepsilon)} \max_{0 \leq q \leq n} e^{-p(B+\delta)}$$

Proof: Suppose that $\text{supp } f(t, x) \in S_A \cap S_B$ and let $\varepsilon > 0$, $\delta > 0$. Choose $\rho \in D(\mathbb{R}^n)$ such that $\rho(t) = 1$ on a neighbourhood of S_A and $\text{supp } \rho \subset S_{A+\varepsilon}$.

Since $f \in FL_{a,b,\alpha}^*$ and in view of the boundedness property of the generalized functions, there exist a constant C and a non-negative integer n such that

$$\begin{aligned} |F(s, p)| &= \left| \left\langle f(t, x), e^{-i(st-ix)} \right\rangle \right| \\ &= \left| \left\langle f(t, x), \rho(t) e^{-i(st-ix)} \right\rangle \right| \\ &\leq C_1 \max_{0 \leq l \leq n} \left[\gamma_{a,b,k,q,l} \left(\rho(t) e^{-i(st-ix)} \right) \right] \\ &\leq C_1 \max_{0 \leq l \leq n} \sup_{I_1} \left| t^k e^{ax} D_t^l D_x^q \rho(t) e^{-i(st-ix)} \right| \end{aligned}$$

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$$\begin{aligned}
 & \leq C_1 \max_{0 \leq l \leq n} \sup_{I_1} \left| t^k \sum_{v=0}^l \binom{l}{v} D_t^{l-v} (\rho(t)) D_t^v e^{-ist} e^{ax} D_x^q e^{-px} \right| \\
 & \leq C_1 \max_{0 \leq l \leq n} \sup_{I_1} \left| t^k \sum_{v=0}^l \binom{l}{v} D_t^{l-v} \rho(t) (-is)^v e^{-ist} e^{ax} (-p)^q e^{-px} \right| \\
 & \leq C_1 t^k e^{|s|(A+\varepsilon)} \max_{0 \leq l \leq n} \left(\sum_{v=0}^l \binom{l}{v} |s|^v \right) |p^q e^{-(p-a)x}| \\
 & \leq C_1 (1+|s|^l) t^k e^{|s|(A+\varepsilon)} \max_{0 \leq q \leq n} |p^q e^{-(p-a)x}| \\
 & \leq C_1 (1+|s|^l) e^{|s|(A+\varepsilon)} C' \max_{0 \leq q \leq n} e^{-p(B+\delta)}, \text{ where } C' = t^k |e^{ax} p^q| \\
 & \leq C (1+|s|^l) e^{|s|(A+\varepsilon)} \max_{0 \leq q \leq n} e^{-p(B+\delta)}, \text{ where } C = C' C_1
 \end{aligned}$$

III. BOUNDEDNESS THEOREM FOR FOURIER-FINITE MELLIN TRANSFORM

3.1. Theorem (Boundedness)

Let $f(t, x) \in FM_{f,b,c,\alpha}^*$ and $F(s, p) = FM_f \{f(t, x)\} = \left\langle f(t, x), e^{-ist} \left(\frac{a^{2p}}{x^{p+1}} - x^{p-1} \right) \right\rangle$,

$a < \text{Re } p < b, s > 0$. Let $\sup f(t, x) \in S_A \cap S_B$ such that $S_A = \{t : t \in \mathbb{R}^n, |t| \leq A, A > 0\}$ and $S_B = \{x : x \in \mathbb{R}^n, |x| \leq B, B > 0\}$, then for each $\varepsilon > 0, \delta > 0$ there exist a constant $C > 0$ and a non-negative

integer n such that $|F(s, p)| \leq C (1+|s|^l) e^{|s|(A+\varepsilon)} \max_{0 \leq q \leq n} [(B+\delta)^{-p} - (B+\delta)^p]$

Proof: Suppose that $\sup f(t, x) \in S_A \cap S_B$ and let $\varepsilon > 0, \delta > 0$. Choose $\rho \in D(\mathbb{R}^n)$ such that $\rho(t) = 1$ on a neighbourhood of S_A and $\text{sup } \rho \subset S_{A+\varepsilon}$.

Since $f \in FM_{f,b,c,\alpha}^*$ and in view of the boundedness property of the generalized functions, there exist a constant C and a non-negative integer n such that

$$|F(s, p)| = \left| \left\langle f(t, x), e^{-ist} \left(\frac{a^{2p}}{x^{p+1}} - x^{p-1} \right) \right\rangle \right|$$

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$$\begin{aligned}
 &= \left| \left\langle f(t, x), \rho(t) e^{-ist} \left(\frac{a^{2p}}{x^{p+1}} - x^{p-1} \right) \right\rangle \right| \\
 &\leq C_1 \max_{\substack{0 \leq l \leq n \\ 0 \leq q \leq n}} \left[\gamma_{a,b,k,q,l} \left(\rho(t) e^{-ist} \left(\frac{a^{2p}}{x^{p+1}} - x^{p-1} \right) \right) \right] \\
 &\leq C_1 \max_{\substack{0 \leq l \leq n \\ 0 \leq q \leq n}} \sup_{I_1} \left| t^k \lambda_{b,c}(x) x^{q+1} D_t^l D_x^q \rho(t) e^{-ist} \left(\frac{a^{2p}}{x^{p+1}} - x^{p-1} \right) \right| \\
 &\leq C_1 \max_{\substack{0 \leq l \leq n \\ 0 \leq q \leq n}} \sup_{I_1} \left| t^k \sum_v \binom{l}{v} D_t^{l-v} (\rho(t)) D_t^v e^{-ist} \lambda_{b,c}(x) x^{q+1} D_x^q \left(\frac{a^{2p}}{x^{p+1}} - x^{p-1} \right) \right| \\
 &\leq C_1 \max_{\substack{0 \leq l \leq n \\ 0 \leq q \leq n}} \sup_{I_1} \left| t^k \sum_v \binom{l}{v} D_t^{l-v} \rho(t) (-is)^v e^{-ist} \lambda_{b,c}(x) x^{q+1} \left\{ a^{2p} P(-p-q) x^{-p-q-1} - P(p-q) x^{p-q-1} \right\} \right|
 \end{aligned}$$

where $P(-p-q)$ is a polynomial in $(-p-q)$ etc.

$$\begin{aligned}
 &\leq C_1 t^k e^{|s|(A+\varepsilon)} \max_{\substack{0 \leq l \leq n \\ 0 \leq q \leq n}} \left(\sum_v \binom{l}{v} |s|^v \right) \left| \lambda_{b,c}(x) \left\{ \left(\frac{a}{x} \right)^{2p} P(-p-q) - P(p-q) \right\} x^p \right| \\
 &\leq C_1 (1+|s|^l) t^k e^{|s|(A+\varepsilon)} \max_{0 \leq q \leq n} \left| \lambda_{b,c}(x) \left\{ \left(\frac{a}{x} \right)^{2p} P(-p-q) - P(p-q) \right\} x^p \right| \\
 &\leq C_1 (1+|s|^l) t^k e^{|s|(A+\varepsilon)} \max_{0 \leq q \leq n} \left| \lambda_{b,c}(x) \left\{ \left(\frac{a}{x} \right)^{2p} P(-p-q) - P(p-q) \right\} x^p \right| \\
 &\leq C_1 (1+|s|^l) t^k e^{|s|(A+\varepsilon)} \max_{0 \leq q \leq n} \left| \lambda_{b,c}(x) a^{2p} P(-p-q) x^{-p} \right| \\
 &\quad - C_1 (1+|s|^l) t^k e^{|s|(A+\varepsilon)} \max_{0 \leq q \leq n} \left| \lambda_{b,c}(x) P(p-q) x^p \right|
 \end{aligned}$$

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$$\leq C_1 (1 + |s|^l) e^{|s|(A+\varepsilon)} C' \max_{0 \leq q \leq n} (B + \delta)^{-p} - C_1 (1 + |s|^l) e^{|s|(A+\varepsilon)} C'' \max_{0 \leq q \leq n} (B + \delta)^p$$

where $C' = t^k |\lambda_{b,c}(x) a^{2p} P(-p-q)|$ and $C'' = t^k |\lambda_{b,c}(x) P(p-q)|$

$$\leq C (1 + |s|^l) e^{|s|(A+\varepsilon)} \max_{0 \leq q \leq n} (B + \delta)^{-p} - C (1 + |s|^l) e^{|s|(A+\varepsilon)} \max_{0 \leq q \leq n} (B + \delta)^p$$

where $C = C' C_1$

$$\leq C (1 + |s|^l) e^{|s|(A+\varepsilon)} \max_{0 \leq q \leq n} [(B + \delta)^{-p} - (B + \delta)^p]$$

IV.CONCLUSION

This paper discussed and proved Boundedness properties for Fourier-Laplace transform and Fourier-Finite Mellin transform.

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