

# Asymmetric Laplace-Normal Mixture Model for Wind Shear Data

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**ABSTRACT:**In design and developments of products and processes, innumerable uncertainties are faced by engineers in the ordinary run of research projects and its implementation. Variety of statistical techniques are employed by them to deal with the uncertainties inherent in measured information. While much of the classical statistical analysis is based on Gaussian distributional assumptions, statistical modelling with the Laplace distributions has increasingly gained importance in many applied fields.

Wind shear is known to be an important factor affecting the safety of aircraft during take-off and landing period. A mixture model with Laplace and Normal distributions fitted to the data obtained from a civil airline is reviewed. A useful alternative to the above mixture model with asymmetric Laplace distribution, a skewed version of symmetric Laplace distribution is proposed. The parameter estimation for the Laplace-Normal mixture is also considered.

**KEYWORDS:** Asymmetric Laplace Distribution, EM algorithm, Mixture Distributions, Wind shear.

## I. INTRODUCTION

The technology for simulating aircraft performance has undergone dramatic developments in recent years. Accurate modelling of all the component phases of a flight is a prerequisite if modern simulation tools, such as those discussed in Clema [3], can be best put to use. One factor affecting aircraft performance that has received considerable attention is the occurrence of wind shears during aircraft landing. Jones [5] have presented probabilistic models for this phenomenon. Wind shears are encountered by an aircraft during the approach to landing and their distribution is critical for assessing the effectiveness and safety of aircraft and for training pilots to react correctly when they encounter a wind shear. It is the sudden change of wind velocity and/or direction that results from a variety of meteorological conditions. These include temperature inversions, land and sea breezes, frontal systems, strong surface winds and significantly thunder storms. Severe wind shear is defined as a rapid change in wind causing aircraft airspeed changes of greater than 15 knots, or vertical speed changes of greater than 500 feet per minute.

Laplace distribution is perhaps one of the most applied distributions in engineering. This distribution has found applications in a variety of engineering areas such as image and speech recognition, dynamics of manufacturing companies, ecosystem respiration, ocean engineering, vision, image and signal processing, dynamics of electricity prices, fracture problems, information theory, steam generator inspection, inventory management and quality control etc. They are rapidly becoming distributions of first choice whenever "something" with heavier than Gaussian (Normal) tails is observed in the data. Since Laplace distribution is very popular in engineering, there are real situations where measurements could be modelled by these distributions.

Asymmetric Laplace distribution (see Kotz *et al.* [8]), a skewed version of symmetric Laplace distribution has been found to be useful in a variety of applications. It has got considerable attention among the researchers working in financial modelling, engineering sciences, share market return models, stochastic variance models and time series modelling. Kozubowski and Podgorski [9] presented the important properties of asymmetric laws, derived their representations and obtained explicit forms of their densities and distribution functions. The asymmetric Laplace distribution forms a subclass of geometric stable distributions. They have stability properties and are convenient in applications, as their densities have explicit forms and the estimation procedures are easily implemented. This class of

distributions is well suited for modelling phenomena where the variable of interest results from a large number of independent innovations, while the empirical distributions appears to be asymmetric, peaky and has tails heavier than those allowed by normal distribution.

Mixture distributions have been used in wide range of important practical situations because they provide a powerful way to extend common parametric families of distributions to fit data sets not adequately fit by single common parametric distributions. A finite mixture distribution with two component densities of specified parametric form and with unknown mixing weights  $p, q$  ( $q=1-p$ ) is defined as  $f(x) = pf_1(x) + (1-p)f_2(x)$ ;  $0 < p < 1$  where  $f_1(x)$  and  $f_2(x)$  are densities of the two distributions.

The present paper aims at reviewing the mixture model with Laplace and Normal components that is fitted to wind shear data available in grouped form. An alternative for this mixture with asymmetric Laplace distribution is also presented in this paper. The maximum likelihood estimation for the Laplace and Normal using Expectation-Maximisation (EM) algorithm is also considered.

## II. LAPLACE-NORMAL MIXTURE MODEL

Barndorff-Nielsen [1] proposed the hyperbolic distributions for modelling turbulence encountered by an aircraft. But this model is quite complicated and difficult to handle when parameter estimation is considered. As the Laplace distribution and Normal distribution are limiting cases of hyperbolic distribution (see Barndorff-Nielsen *etal.* [2]), Kanji [7] proposed the mixture of Laplace and Normal as a model for wind shear data. He had worked with 24 sets of data on wind shear collected during the last two minutes of landing of a passenger aircraft. The measurement represents the gradient of airspeed change against its duration. The basic assumption is that a wind shear forms an individual gust that has a strictly defined form specified by its duration and the magnitude of change the air velocity. The 120 seconds in flight before touchdown was split into four bands, the first two of 40 seconds length and the last two of 20 seconds length. The histograms of the data suggested that for the early stage (first 40 seconds) of landing, the Laplace distribution seems to fit the data well, while for the last 20 seconds less peaky Gaussian distribution appears to be appropriate.

If  $x$  is the measurement of wind shear, then the proposed mixture model was

$$p(x; \mu, \sigma, \alpha) = \frac{\alpha}{\sqrt{2}\sigma} e^{-\frac{\sqrt{2}|x-\mu|}{\sigma}} + \frac{(1-\alpha)}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}},$$

being a mixture of Laplace and Gaussian distributions having the same mean and variance.  $\alpha$  is the proportionality constant of the mixture distribution and can take value from 0 to 1.

The estimation procedure starts with the estimation of the mean and the variance for both components in the model and then employs the Chi Square goodness of fit procedure to fit the mixing constant  $\alpha$ . As  $\mu$  and  $\sigma^2$  is estimated by sample mean and variance of the data respectively, the fitting process is a one parameter search. In the formation of Chi Square statistic that was under minimization, the mixing parameter  $\alpha$  was set equal to one or zero for a class interval if the fit of a single Laplace or normal component, respectively for that class interval was found to be adequate. The inference led to the conclusion that wind shear data lose their Laplacian character in the earlier stages to the Gaussian at the end of landing.

The results of fitted model to the data are summarized in Table 1.

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Table1.Results for fitted Laplace Normal mixture model

Number	Chi Square value	Degrees of freedom	$\alpha$
1	12.6843	13	0.878560
2	10.6130	14	0.697230
3	15.0687	13	0.502100
4	31.3989	12	0.334300
5	21.2543	11	0.873401
6	8.7066	12	0.710190
7	8.7738	10	0.450151
8	24.1654	10	0.278040
9	15.2469	10	0.920610
10	14.1079	11	0.732737
11	9.0732	11	0.675140
12	20.3008	10	0.350405
13	12.3751	4	0.841170
14	6.1905	4	0.659165
15	12.3994	4	0.495710
16	10.7194	4	0.247700
17	13.6053	4	0.808500
18	18.2423	4	0.617500
19	13.8217	4	0.486160
20	10.9363	3	0.125650
21	23.5394	4	0.996130
22	17.4993	4	0.672300
23	8.5785	2	0.478110
24	11.7774	3	0.112160

Kanji's mixture model was not completely adequate in providing the description of wind shear measurements in the 24 sets of data used. Jones and McLachlan [6] modified the mixture model of Laplace and Normal distributions in the same context by examining the same twenty- four sets of data, but without the constraint that the component densities should have equal variances. They demonstrated an appropriately modified estimation procedure leading to even better fit than those obtained by Kanji. Without the assumption of equal component variances, by taking  $2\beta^2 = \sigma^2$ , Jones and McLachlan [6] considered the modified mixture model

$$p(x; \mu, \sigma^2, \alpha, \beta) = \frac{\alpha}{2\beta} e^{-\frac{|x-\mu|}{\beta}} + \frac{(1-\alpha)}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

The mixture model is fitted using maximum likelihood by EM algorithm of Dempster et al.[4]. The estimates of parameters  $\alpha, \beta, \sigma^2$  at the (p+1) the stage are

$$\alpha^{(p+1)} = \frac{\sum_{j=1}^g n_j E_j^{(p)} \{ \tau_1(X; \beta^{(p)}) \}}{n}$$

$$\beta^{(p+1)} = \frac{\sum_{j=1}^g n_j E_j^{(p)} \{ \tau_1(X; \beta^{(p)}) / X \}}{\sum_{j=1}^g n_j E_j^{(p)} \{ \tau_1(X; \beta^{(p)}) \}}$$

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$$(\sigma^{p+1})^2 = \frac{\sum_{j=1}^{\vartheta} n_j E_j^{(p)} \{ \tau_2 [X; (\sigma^p)^{(2)}] X^2 \}}{\sum_{j=1}^{\vartheta} n_j E_j^{(p)} \{ \tau_2 [X; (\sigma^p)^2] \}}$$

where

$$\tau_1(x; \beta^{(p)}) = \frac{\alpha^{(p)} \exp\left\{-\frac{|x|}{\beta^{(p)}}\right\}}{2\beta^{(p)} p(x; 0, (\sigma^{(p)})^2, \alpha^{(p)}, \beta^{(p)})}$$

$$\tau_2 = [x; (\sigma^{(p)})^2] = 1 - \tau_1(x; \beta^{(p)})$$

and

$$n = \sum_{j=1}^{\vartheta} n_j$$

In the above equations,  $E_j^{(p)}$  refers to expectation with respect to the density

$$\frac{p[x; 0, (\sigma^{(p)})^2, \alpha^{(p)}, \beta^{(p)}]}{P_j^{(p)}}$$

where

$$P_j^{(p)} = \int_{a_j}^{b_j} p(x; 0, (\sigma^{(p)})^2, \alpha^{(p)}, \beta^{(p)}) dx$$

and  $[a_j, b_j)$  corresponds to the  $j^{\text{th}}$  interval.

The results of modified model to the data are summarized in Table 2.

Table2. Results for Laplace Normal modified mixture model with calculated  $\alpha$ .

Number	$\mu$	$\alpha$	$\beta$	$\sigma^2$	Chi Square value	Degrees of freedom
1	0	0.827	0.140	0.066	7.89	11
2	0	0.672	0.167	0.053	7.72	12
3	0	0.491	0.188	0.068	11.00	12
4	0	0.202	0.261	0.065	16.47	11
5	0	0.835	0.136	0.044	2.21	8
6	0	0.685	0.150	0.049	10.49	11
7	0	0.414	0.172	0.057	8.77	10
8	0	0.164	0.237	0.055	14.36	11
9	0	0.822	0.117	0.057	6.94	9
10	0	0.679	0.130	0.054	7.30	10
11	0	0.646	0.1447	0.057	7.70	10
12	0	0.236	0.200	0.048	13.89	9
13	0	0.730	0.075	0.011	0.77	4
14	0	0.620	0.083	0.015	3.56	4
15	0	0.124	0.142	0.014	4.11	4
16	0.012	0	-	0.019	12.84	4

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17	0	0.494	0.043	0.016	0.42	2
18	0	0.272	0.092	0.011	4.15	4
19	0	0.070	0.180	0.012	6.32	3
20	0.016	0	-	0.017	11.51	4
21	0	0.376	0.082	0.008	2.53	3
22	0	0.592	0.073	0.012	6.55	3
23	0	0.309	0.066	0.013	1.96	2
24	0.020	0	-	0.017	11.76	4

**III. ASYMMETRIC LAPLACE-NORMAL MIXTURE MODEL**

A random variable is said to have an asymmetric Laplace distribution AL ( $\theta, \mu, \sigma$ ) if there are parameters  $\theta \in R, \mu \in R, \text{ and } \sigma \geq 0$  such that its characteristic function is

$$\varphi_X(t) = \frac{e^{i\theta t}}{1 + \frac{1}{2}\sigma^2 t^2 - i\mu t}$$

The probability density function and distribution function of an AL ( $\theta, \mu, \sigma$ ) distribution defined by Kozubowski and Podgorski (2000) are respectively

$$f(x) = \begin{cases} \frac{\sqrt{2}}{\sigma} \frac{\kappa}{1 + \kappa^2} e^{-\frac{\sqrt{2}\kappa|x-\theta|}{\sigma}} & \text{if } x \geq \theta \\ \frac{\sqrt{2}}{\sigma} \frac{\kappa}{1 + \kappa^2} e^{-\frac{\sqrt{2}|x-\theta|}{\sigma\kappa}} & \text{if } x < \theta \end{cases}$$

and

$$F(x) = \begin{cases} 1 - \frac{1}{1 + \kappa^2} e^{-\frac{\sqrt{2}\kappa|x-\theta|}{\sigma}} & \text{if } x \geq \theta \\ \frac{\kappa^2}{1 + \kappa^2} e^{-\frac{\sqrt{2}|x-\theta|}{\sigma\kappa}} & \text{if } x < \theta \end{cases}$$

where

$$\kappa = \frac{\sqrt{2}\sigma}{\mu + \sqrt{\mu^2 + 2\sigma^2}}$$

The characteristic function of AL ( $\mu, \sigma$ ) is

$$\varphi_X(t) = \frac{1}{1 + \frac{1}{2}\sigma^2 t^2 - i\mu t}$$

Here  $\sigma=0$  reduces to an exponential random variable and  $\mu=0, \sigma \neq 0$  reduces to symmetric Laplace distribution with location zero.

Asymmetric Laplace distribution is unimodal with mode equal to zero. It is leptokurtic too. These classes of distributions satisfy many important statistical properties like infinite divisibility, geometric infinite divisibility, self-decomposability, stability with respect to geometric summation, maximum entropy, and finiteness of moments, simplicity etc. and subclasses of geometric stable distributions. Moreover, the asymmetric Laplace model provides

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more flexibility, allowing for asymmetry, peakedness and tail heaviness than normal model, which are common features of many financial data sets.

$$f(x, \mu, \sigma, \theta, \kappa) = \begin{cases} \frac{\sqrt{2}}{\sigma} \frac{\alpha\kappa}{1 + \kappa^2} e^{-\frac{\sqrt{2}\kappa|x-\theta|}{\sigma}} + \frac{1 - \alpha}{\sqrt{2\pi} \sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} & \text{if } x \geq \theta \\ \frac{\sqrt{2}}{\sigma} \frac{\alpha\kappa}{1 + \kappa^2} e^{-\frac{\sqrt{2}|x-\theta|}{\sigma\kappa}} + \frac{1 - \alpha}{\sqrt{2\pi} \sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} & \text{if } x < \theta \end{cases}$$

From the analysis of wind shear data presented above, it reveals that an asymmetric Laplace- normal mixture model may be a good alternative to describe behaviour of the data than the Laplace-normal mixture model.

**IV. CONCLUSION**

Kanji's mixture model approach was adopted to accommodate apparent departures from conventional normal shaped distributions in the form of fairly spiked distributions. The Laplace-Normal mixture model by Jones and McLachlan seems to be the better fit for the wind shear data than those obtained by Kanji. A skewed version of symmetric Laplace distribution, namely asymmetric Laplace distribution can be used to get an alternative model. That is, the asymmetric Laplace – normal mixture, which accommodates the skewed nature inherent in the data, may be appropriate to get a good fit than the other mixture models.

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