

# Distinct Colouring of Edges and Edgewise Points of Intersection in Planer Drawing of $K_n$

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**ABSTRACT:** The regular drawing of  $K_n$  is a plane regular drawing. If all edges of it are coloured with  ${}^n C_2$  distinct colours we will call such colouring as a distinct colouring. Every colour in this drawing has  $({}^n C_2 - 2(n-2)-1)$  non adjacencies with the colours used in the colouring. Hence the sum of all such non adjacencies denoted by  $Tz(n)$  is  $({}^n C_2 - 2(n-2)-1) \cdot {}^n C_2$  which is related to the sum of edgewise crossings over any edge where edgewise crossing for any edge  $e$  is the number of edges which are not adjacent to  $e$  and intersect  $e$  in the regular drawing of  $K_n$ . The sum of all edgewise crossings in regular drawing of  $K_n$ , we denote by  $CR(n)$ .

**KEYWORDS:** Crossing number, Complete Graphs, Edge colouring of graphs

## I. INTRODUCTION

The crossing number  $v(G)$  of a graph  $G$  is the minimum number of pair wise intersection of its edges when  $G$  is drawn in the plane. A proper  $k$ -edge colouring of a simple graph  $G$  is assigning  $k$ -colours to the edges of  $G$  so that any two adjacent edges receive different colours. If for each pair  $t_i, t_j$  of colours there exists adjacent edges with this colours then the colouring is said to be complete. The Achromatic index  $\psi'(G)$  of graph  $G$  is the maximum number of colours used in the proper edge colouring of  $G$  such that the colouring is complete. The number for the complete graph on  $n$  vertices is denoted by  $A(n)$ . The number in general is not known for simple graphs. The relationship above is obtained while studying bounds for achromatic indices of graphs.

## II. PROCESS

In the regular plane drawing of  $K_n$ , whenever two edges intersects each other they miss one opportunity of one mutual adjacency of their colours. This makes one to think of relationship between adjacencies/non adjacencies & edgewise crossings. On the other hand if the drawing of  $K_n$  is done using the elevated edges then edgewise crossings in such drawing becomes zero, Thus leading to think of no algebraic relationship between the number of edge crossings & total mutual adjacencies/non adjacencies of colours. The planer drawing of graphs has the one connection with the colours that is the famous 4 colour theorem. It looks all most certain that any colouring has no connection with the point of intersection of edges of graphs as the processes are not algebraically equivalent, but now we have established some colouring called distinct edge colouring of  $K_n$  which has some algebraic relation with the points of intersections of edges pair wise.

## III. DERIVATION OF THE RELATIONSHIP

Every edge (and its colour) has exactly  $2(n-2)$  distinct colours adjacent to it at its extremities.

$\therefore$  Each colour has exactly  ${}^n C_2 - 2(n-2) - 1$  non adjacencies with the remaining colours which is

$$\begin{aligned} &= \frac{n(n-1)}{2} - 2(n-2) - 1 \\ &= \frac{n(n-1) - 4(n-2) - 2}{2} \\ &= \frac{n^2 - n - 4n + 6}{2} \end{aligned}$$

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$$= \frac{n^2 - 5n + 6}{2}$$

$$= \frac{(n-2)(n-3)}{2}$$

Therefore the total of number of non trivial non adjacencies with respect to the distinct colouring is

$$T_z(n) = \frac{n(n-1)}{2} \left[ \frac{(n-2)(n-3)}{2} \right]$$

$$= \frac{n(n-1)(n-2)(n-3)}{4} \dots \dots \dots (1)$$

Now we will count CR(n)

To begin with, We say  $e = V_i V_j$  is  $T_s$  type edge if  $s = \min \{ |i - j|, n - |i - j| \}$

Now we will prove that for  $K_n$ , the edges of it can be classified as  $T_1, T_2, T_3, \dots, T_{\lfloor \frac{n}{2} \rfloor}$ .

**Proof:** case i: If  $|i - j| \leq \lfloor \frac{n}{2} \rfloor$

Implies  $|i - j| \leq \frac{n}{2}$

Implies  $-|i - j| \geq -\frac{n}{2}$

Implies  $n - |i - j| \geq n - \frac{n}{2}$

Implies  $\frac{n}{2} \leq n - |i - j|$

$\Rightarrow |i - j| \leq n - |i - j|$

Implies  $s = |i - j|$

$\therefore s \leq \lfloor \frac{n}{2} \rfloor$

Case ii)  $|i - j| > \lfloor \frac{n}{2} \rfloor$

Suppose

$n - |i - j| > \frac{n}{2}$

implies  $\frac{n}{2} > |i - j|$

but as  $|i - j| \in \mathbb{Z}^+$

$\therefore \lfloor \frac{n}{2} \rfloor > |i - j|$  if n even

and  $\lfloor \frac{n}{2} \rfloor \geq |i - j|$  if n odd.

Hence in both the above cases

$|i - j| \leq \lfloor \frac{n}{2} \rfloor$

Which contradicts to the case therefore our assumption is wrong,

Hence  $n - |i - j| \leq \lfloor \frac{n}{2} \rfloor$

So for the case we have

$|i - j| > \lfloor \frac{n}{2} \rfloor$  and  $n - |i - j| \leq \lfloor \frac{n}{2} \rfloor$

As both  $i, j \in \{1, 2, 3, \dots, n\}$  so  $0 < |i - j| \leq n - 1$

$\therefore n - |i - j| \in \mathbb{Z}^+$

$\therefore n - |i - j| \leq \lfloor \frac{n}{2} \rfloor$  if n even and  $n - |i - j| \leq \frac{n}{2}$  if n is odd.

$\therefore n - |i - j| \leq \frac{n}{2}$  for all  $n \in \mathbb{N}$

$\therefore |i - j| > \lfloor \frac{n}{2} \rfloor$  and  $n - |i - j| \leq \frac{n}{2}$

Therefore  $s = n - |i - j|$

Hence  $s \leq \lfloor \frac{n}{2} \rfloor$

Therefore edges of  $K_n$  can be classified as  $T_1, T_2, T_3, \dots, T_{\lfloor \frac{n}{2} \rfloor}$

To count CR(n), we will count sum of edgewise crossing every type  $T_i$  and then summing them all will give us Cr(n).

Consider n even.

The simple calculations shows that

There are n edges of type  $T_1$

There are n edges of type  $T_2$

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There are n edges of type  $T_{\frac{n-1}{2}}$

and there are  $\frac{n}{2}$  edges of type  $T_{\lfloor \frac{n}{2} \rfloor}$  i.e.  $T_{\frac{n}{2}}$

For any edge of the type  $T_1$

There are zero edgewise crossings for any edge of the type  $T_2$ ,

There are 1.(n-3) edgewise crossings

For any edge of the type  $T_3$

There are 2. (n-4) edgewise crossings

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For any edge of the type  $T_{\frac{n-1}{2}}$

There are  $(\frac{n}{2} - 2) \cdot (\frac{n}{2})$  edgewise crossings

For any edge of the type  $T_{\frac{n}{2}}$ ,

There are  $(\frac{n}{2} - 1) (\frac{n}{2} - 1)$  edgewise crossings

$\therefore CR(n) = n [1.(n-3) + 2.(n-4) + 3.(n-5) + \dots + (\frac{n}{2} - 2) \cdot (\frac{n}{2})] + \frac{n}{2} (\frac{n}{2} - 1)^2$

$$= n \left[ \sum_{i=1}^{\frac{n}{2}-2} i(n-2-i) \right] + \frac{n \cdot (n-2)^2}{8}$$

$$= n \sum_{i=1}^{\frac{n}{2}-2} (n-2) \cdot i + n \sum_{i=1}^{\frac{n}{2}-2} i^2 + \frac{n \cdot (n-2)^2}{8}$$

$$= n \left[ \frac{(n-2)(\frac{n}{2}-2)(\frac{n}{2}-1)}{2} - \frac{(\frac{n}{2}-2)(\frac{n}{2}-1)}{6} (2((\frac{n}{2}-2)+1)) \right] + \frac{n \cdot (n-2)^2}{8}$$

$$= n \left[ \frac{(n-2)(n-2)(n-4)}{8} - \frac{(n-4)(n-2)(n-3)}{24} \right] + \frac{n \cdot (n-2)^2}{8}$$

$$= [3n(n-2)^2 \cdot (n-4) - n(n-2)(n-3)(n-4) + 3(n-2)^2 \cdot n] \div 24$$

$$= \frac{n(n-2)}{24} [3(n-2)(n-4) - (n-3)(n-4) + 3(n-2)]$$

$$= \frac{n(n-2)}{24} [3(n^2 - 6n + 8) - (n^2 - 7n + 12) + 3(n-2)]$$

$$= \frac{n(n-2)}{24} [3n^2 - 18n + 24 - n^2 + 7n - 12 + 3n - 6]$$

$$= \frac{n(n-2)}{24} [2n^2 - 8n + 6]$$

$$= \frac{n(n-2)}{24} 2(n^2 - 4n + 3)$$

$$= \frac{n(n-1)(n-2)(n-3)}{12} \dots \dots \dots (2)$$

If n is odd then the simple counting shows that

There are n edges of type  $T_1$

There are n edges of type  $T_2$

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There are n edges of type  $T_{\lfloor \frac{n}{2} \rfloor}$  i.e.  $T_{\frac{n-1}{2}}$

For any edge of the type  $T_1$ , There are zero edgewise crossings

For any edge of the type  $T_2$ , There are 1(n-3) edgewise crossings

For any edge of the type  $T_3$ , There are 2(n-4) edgewise crossings.

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For any edge of the type  $T_{\frac{n-1}{2}}$ , There are  $(\frac{n-1}{2} - 1)(n - (\frac{n-1}{2} + 1))$

i.e.  $(\frac{n-3}{2}) (\frac{n-1}{2})$  edgewise crossings

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$$\begin{aligned}
 \therefore CR(n) &= n [1.(n-3)+2.(n-4)+ \dots + \binom{n-3}{2} \binom{n-1}{2}] \\
 &= n \cdot \sum_{i=1}^{\frac{n-3}{2}} i(n-2-i) \\
 &= n \cdot (n-2) \sum_{i=1}^{\binom{n-3}{2}} i - n \cdot \sum_{i=1}^{\binom{n-3}{2}} i^2 \\
 &= n \left[ \frac{(n-2) \binom{n-3}{2} (\frac{n-3}{2} + 1)}{2} - \frac{(\frac{n-3}{2}) (\frac{n-3}{2} + 1) (2 \binom{n-3}{2} + 1)}{6} \right] \\
 &= n \left[ \frac{(n-2)(n-3)(n-1)}{8} - \frac{(n-3)(n-1)(n-2)}{24} \right] \\
 &= n \left[ \frac{3(n-1)(n-2)(n-3) - (n-1)(n-2)(n-3)}{24} \right] \\
 &= \frac{n(n-1)(n-2)(n-3)}{12} \dots \dots \dots (3)
 \end{aligned}$$

Therefore in either cases using (2), (3)

$$CR(n) = \frac{n(n-1)(n-2)(n-3)}{12} \dots \dots \dots (4)$$

Using from equations (1) and (4)

$$Tz(n) = 3 CR(n)$$

## IV. CONCLUSION

The algebraic relationship between the numbers is established which may play crucial role for finding relationship of edgewise crossings & complete colouring of  $K_n$  in future.

## REFERENCES

- [1] J. A. Bondy, and U. S. R. Murty, Graph theory with Applications, The Macmillan Press Ltd. 1976
- [2] Ivan Christov, The crossing number of graph
- [3] F. Harary, Graph Theory, ISBN 81-85015-55-4
- [4] R.E.Jamison, On the edge achromatic numbers of complete graphs, Discrete Math 74 (1989) 99-115
- [5] Liu, C.L., Elements of Discrete Mathematics, Mc Graw Hill, New York (1985)
- [6] West Douglas Brent (1999), Introduction to Graph Theory (2nd ed.), Prentice Hall, ISBN 0-13-014400-2