

Stochastic Epidemic Model with Poisson Infection and Poisson Removal Rates

G. Ramesh ¹, R. Surendar ²Associate Professor, Department of Mathematics, Government Arts College (Autonomous), Kumbakonam, Thanjavur (Dist), Tamilnadu, India ¹Assistant Professor, Department of Mathematics, Swami Dayananda College of Arts and Science, Manjakkudi, Tiruvarur (Dist), Tamilnadu, India ²

ABSTRACT: In this paper, we have developed a stochastic epidemic model with Poisson infection and Poisson removal rates. Poisson infection rate is depending on the number of persons infected in the system, and Poisson removal rate is depending on the number of persons removed in the system.

AMS Classification: 60G05, 62P10, 65C20, 92B05, 92D30.

KEYWORDS: Epidemics, susceptible, infection rate, removal rate.

I. INTRODUCTION

Mathematical and theoretical biology is an interdisciplinary scientific research field with wide range of applications in biology, medicine and biotechnology. The field may be referred to as mathematical biology or biomathematics to stress the mathematical side, or as theoretical biology to stress the biological side. It includes at least four major subfields: biological mathematical modeling, relational biology/complex systems biology (CSB), bioinformatics and computational bio modeling / bio computing. Mathematical biology aims at the mathematical representation, treatment and modeling of biological processes, using a variety of applied mathematical techniques and tools. It has both theoretical and practical applications in biological, biomedical and biotechnology research. For example, in cell biology, protein interactions are often represented as cartoon models, which, although easy to visualize, do not accurately describe the systems studied. In order to do this, precise mathematical models are required. By describing the systems in a quantitative manner, their behaviour can be better simulated, and hence properties can be predicted that might not be evident to the experimenter. Mathematical biology is a very active and fast growing interdisciplinary area in which mathematical concepts, techniques, and models are applied to a variety of problems in developmental biology and biomedical sciences.

The occurrence of more cases of a disease than would be expected in a community or region during a given time period is called an epidemic. An epidemic is the rapid spread of infectious disease to a large number of people in a given population within a short period of time, usually two weeks or less. An epidemic may be restricted to one location; however, if it spreads to other countries or continents and affects a substantial number of people, it may be termed a pandemic [16].

Epidemiology is one of the most important fields of research. The modeling and analysis of infectious diseases have been done by many authors. The classical theoretical papers on epidemic models by Kermack and Mckendrick [8, 9, 10] have had a major influence in the development of mathematical models. The modeling literature is now extensive and growing very quickly. Bailey [2] gave a good introduction to the variety of problems and models for the spread and control of infectious disease. The article by Hethcote [6] reviewed three basic epidemiological models. Diekmann and Heesterbeek [5] presented a concentrated review to this field and they discussed how to use biological assumptions in constructing models and present applications.

In mathematics and physics, a deterministic system is a system in which no randomness is involved in the development of future states of the system [17]. A deterministic model will thus always produce the same output from a given starting condition or initial state [18]. For example, physical laws that are described by differential equations represent deterministic systems, even though the state of the system at a given point in time may be difficult to describe

International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Vol. 4, Issue 10, October 2015

explicitly. Markov chains and other random walks are not deterministic systems, because their development depends on random choices.

Stochastic means being or having a random variable. A stochastic model is a tool for estimating probability distributions of potential outcomes by allowing for random variation in one or more inputs over time. The random variation is usually based on fluctuations observed in historical data for a selected period using standard time series techniques. Distributions of potential outcomes are derived from a large number of simulations which reflect the random variation in the input.

Basic Concepts

The term epidemiology is the study of health-event, health-characteristic, or health-determinant patterns in a society. Now before we go further in this study, here the basic concepts are introduced which will be useful in the following sections.

Definition 1.1 [3]

An *epidemic* is an unusually large short – term outbreak of a disease, such as Cholera and Plague etc.

The spread of disease depends on:

- The mode of transmission.
- Susceptibility.
- Infections period.
- Resistance.

and many other factors.

Definition 1.2 [11]

Any group of individuals, usually of a single species, occupying a given area at the same time is known as *population*.

The population in an epidemiological model can be classified into three types : Susceptible, Infected and Removal which are denoted by $S(t)$, $I(t)$ and $R(t)$ respectively, where $S(t)$ represent the number of individuals in the population who can be infected, while $I(t)$ represent the number of infected individuals in the population, that is, those with the disease who are actively transmitting it, $R(t)$ denotes the number of individuals removed from the population by recovery, death, immunization or other means [7].

In probability theory, a Poisson process is a stochastic process that counts the number of events and the time points at which these events occur in a given time interval. The time between each pair of consecutive events has an exponential distribution with parameter λ and each of these inter-arrival times is assumed to be independent of other inter-arrival times. The process is named after the Poisson distribution introduced by French mathematician Simeon Denis Poisson [13]. It describes the time of events in radioactive decay [4], telephone calls [15] or request for documents on a web server under certain conditions [1], and many other phenomena, where events occur independently from each other. The number of arrivals $N(t)$ in a finite interval of length t obeys the Poisson (λt) distribution,

$$P\{N(t) = n\} = \frac{(\lambda t)^n}{n!} e^{-\lambda t}.$$
 Moreover, the numbers of arrivals $N(t_1, t_2)$ and $N(t_3, t_4)$ in non-overlapping intervals $(t_1 \leq t_2 \leq t_3 \leq t_4)$ are independent.

An infection rate is an estimate of the rate of progress of a disease, based on proportional measures of the extent of infection at different times.

In probability theory, the probability generating function of a discrete random variable is a power series representation of the probability mass function of the random variable. Probability generating functions are often

employed for their succinct description of the sequence of probabilities $\Pr(X = i)$ in the probability mass function for a random variable X , and to make available the well developed theory of power series with non-negative coefficients.

Definition 1.3 [12]

Consider a random variable X , i.e, a discrete random variable taking non-negative values.

Write $P_k = P(X = k)$, $k = 0, 1, 2, \dots$

The probability generating function of X is defined as

$$G_X(S) = \sum_{k=0}^{\infty} P_k S^k = E(S^X).$$

Note that $G_X(1) = 1$, so the series converges absolutely for $|S| \leq 1$. Also $G_X(0) = P_0$.

Theorem 1.4 [12] (Total and Compound Probability)

Let A_1, A_2, \dots, A_n be a partition of Ω . For any event B ,

$$\Pr(B) = \sum_{j=1}^n \Pr(A_j) \Pr(B/A_j).$$

In this paper, we have developed a stochastic epidemic model with removal with Poisson infection and Poisson removal rates. Poisson infection rate is depending on the number of persons infected in the system, and Poisson removal rate is depending on the number of persons removed in the system.

II. EPIDEMIC MODEL

In this section we study the characterizations of the stochastic epidemic model with removal with Poisson infection and Poisson removal rates.

2.1. Stochastic Epidemic Model with Poisson Infection and Poisson Removal Rates

Let $P_{m,n}(t)$ be the probability that there are m susceptible and n infectives in the population at time t . If N is the total size of the population, then the number of persons in the removed category is $N-m-n$.

Let the probability of susceptible being infected in the time interval $(t, t + \Delta t)$ be $\beta mn \Delta t + O(\Delta t)$, and let the corresponding probability of one infected being removed in the same time interval be $\gamma n \Delta t + O(\Delta t)$.

In this case, let us assume that βmn is the Poisson rate of susceptible becomes infected with parameter λ_I , and γn is the Poisson rate of an infective becomes removed with parameter λ_R .

i.e.,
$$\beta mn = \frac{e^{-\lambda_I} (\lambda_I)^m}{m!}, \quad m \geq 0 \tag{1}$$

and
$$\gamma n = \frac{e^{-\lambda_R} (\lambda_R)^n}{n!}, \quad n \geq 0. \tag{2}$$

Here, λ_I be the rate of infection from susceptible to infected, and λ_R be the rate of removal from infective to removed.

The probability that there is no change in the time interval $(t, t + \Delta t)$ is then given by

$$1 - \beta mn \Delta t - \gamma n \Delta t + O(\Delta t) \tag{3}$$

Now there can be m susceptibles and n infected persons at time $t + \Delta t$ if there are

International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Vol. 4, Issue 10, October 2015

- (i) $m+1$ susceptibles and $n-1$ infectives at time t and if one person has become infected in time Δt , (or)
- (ii) m susceptibles and $n+1$ infectives at time t and if one infected person has been removed in time Δt , (or)
- (iii) m susceptibles and n infectives at time t and if there is no change in time Δt .

We assume that the probability of more than one change in time Δt is $O(\Delta t)$.

Then, using the theorems of total and compound probability, we get

$$P_{m,n}(t + \Delta t) = P_{m+1,n-1}(t) \beta(m+1)(n-1)\Delta t + P_{m,n+1}(t) \gamma(n+1)\Delta t + P_{m,n}(t)(1 - \beta mn \Delta t - \gamma n \Delta t) + O(\Delta t) \quad (4)$$

So that,

$$\begin{aligned} P_{m,n}(t + \Delta t) &= P_{m+1,n-1}(t) \beta(m+1)(n-1)\Delta t + P_{m,n+1}(t) \gamma(n+1)\Delta t \\ &\quad + P_{m,n}(t) - P_{m,n}(t) \beta mn \Delta t - P_{m,n}(t) \gamma n \Delta t + O(\Delta t) \\ P_{m,n}(t + \Delta t) - P_{m,n}(t) &= P_{m+1,n-1}(t) \beta(m+1)(n-1)\Delta t + P_{m,n+1}(t) \gamma(n+1)\Delta t \\ &\quad - P_{m,n}(t) \beta mn \Delta t - P_{m,n}(t) \gamma n \Delta t + O(\Delta t) \\ \frac{P_{m,n}(t + \Delta t) - P_{m,n}(t)}{\Delta t} &= P_{m+1,n-1}(t) \beta(m+1)(n-1) + P_{m,n+1}(t) \gamma(n+1) \\ &\quad - P_{m,n}(t) \beta mn - P_{m,n}(t) \gamma n + \frac{O(\Delta t)}{\Delta t} \end{aligned} \quad (5)$$

Proceeding to the limit as $\Delta t \rightarrow 0$ in (5), we get

$$\begin{aligned} \lim_{\Delta t \rightarrow 0} \frac{P_{m,n}(t + \Delta t) - P_{m,n}(t)}{\Delta t} &= P_{m+1,n-1}(t) \beta(m+1)(n-1) - P_{m,n}(t) \beta mn \\ &\quad + P_{m,n+1}(t) \gamma(n+1) - P_{m,n}(t) \gamma n + 0. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{d}{dt}(P_{m,n}(t)) &= \beta(m+1)(n-1)P_{m+1,n-1}(t) - \beta mn P_{m,n}(t) \\ &\quad + \gamma(n+1)P_{m,n+1}(t) - \gamma n P_{m,n}(t) \end{aligned} \quad (6)$$

Initially, let there be s susceptibles and a infectives. Then we define the probability generating function by

$$\varphi(x, y, t) = \sum_{m=0}^s \sum_{n=0}^{s+a-m} P_{m,n}(t) x^m y^n \quad (7)$$

Multiplying (6) by $x^m y^n$ and summing over n from 0 to $s+a-m$ and m from 0 to s , we get

$$\frac{\partial}{\partial t} \left(\sum_{m=0}^s \sum_{n=0}^{s+a-m} P_{m,n}(t) x^m y^n \right) = \sum_{m=0}^s \sum_{n=0}^{s+a-m} \frac{d}{dt} (P_{m,n}(t)) x^m y^n$$

So that,

$$\frac{\partial \varphi}{\partial t} = \sum_{m=0}^s \sum_{n=0}^{s+a-m} \left[\beta(m+1)(n-1)P_{m+1,n-1}(t) - \beta mn P_{m,n}(t) + \gamma(n+1)P_{m,n+1}(t) - \gamma n P_{m,n}(t) \right] x^m y^n$$

(Using (6) and (7))

$$\begin{aligned}
 &= \sum_{m=0}^s \sum_{n=0}^{s+a-m} (\beta(m+1)(n-1)P_{m+1,n-1}(t))x^m y^n - \sum_{m=0}^s \sum_{n=0}^{s+a-m} \beta mn P_{m,n}(t)x^m y^n \\
 &\quad + \sum_{m=0}^s \sum_{n=0}^{s+a-m} (\gamma(n+1)P_{m,n+1}(t))x^m y^n - \sum_{m=0}^s \sum_{n=0}^{s+a-m} \gamma n P_{m,n}(t)x^m y^n \\
 \frac{\partial \phi}{\partial t} &= \beta \sum_{m=0}^s \sum_{n=0}^{s+a-m} ((m+1)(n-1)P_{m+1,n-1}(t))x^m y^n - \beta \sum_{m=0}^s \sum_{n=0}^{s+a-m} mn P_{m,n}(t)x^m y^n \\
 &\quad + \gamma \sum_{m=0}^s \sum_{n=0}^{s+a-m} ((n+1)P_{m,n+1}(t))x^m y^n - \gamma \sum_{m=0}^s \sum_{n=0}^{s+a-m} n P_{m,n}(t)x^m y^n \tag{8}
 \end{aligned}$$

Now, the definition of probability generating function, we have

$$x \frac{\partial \phi}{\partial x} = \sum_m \sum_n m P_{m,n}(t) x^m y^n ; \tag{9}$$

$$y \frac{\partial \phi}{\partial y} = \sum_m \sum_n n P_{m,n}(t) x^m y^n ; \text{ and} \tag{10}$$

$$xy \frac{\partial^2 \phi}{\partial x \partial y} = \sum_m \sum_n mn P_{m,n}(t) x^m y^n \tag{11}$$

Equation (8) can be written as,

$$\begin{aligned}
 \frac{\partial \phi}{\partial t} &= \beta y^2 \sum_{m=0}^s \sum_{n=0}^{s+a-m} ((m+1)(n-1)P_{m+1,n-1}(t))x^m y^{n-2} - \beta xy \sum_{m=0}^s \sum_{n=0}^{s+a-m} mn P_{m,n}(t)x^{m-1} y^{n-1} \\
 &\quad + \gamma \sum_{m=0}^s \sum_{n=0}^{s+a-m} ((n+1)P_{m,n+1}(t))x^m y^n - \gamma y \sum_{m=0}^s \sum_{n=0}^{s+a-m} n P_{m,n}(t)x^m y^{n-1} \tag{12}
 \end{aligned}$$

Using the relations (9), (10) and (11) in (12), we get

$$\begin{aligned}
 \frac{\partial \phi}{\partial t} &= \beta y^2 \frac{\partial^2 \phi}{\partial x \partial y} - \beta xy \frac{\partial^2 \phi}{\partial x \partial y} + \gamma \frac{\partial \phi}{\partial y} - \gamma y \frac{\partial \phi}{\partial y} \\
 \frac{\partial \phi}{\partial t} &= \beta (y^2 - xy) \frac{\partial^2 \phi}{\partial x \partial y} + \gamma (1 - y) \frac{\partial \phi}{\partial y} \tag{13}
 \end{aligned}$$

We have to solve (13) subject to the initial condition,

$$\phi(x, y, 0) = x^s y^a \tag{14}$$

2.2. Classification of the solution

In this section, to find the classification of the solution of the partial differential equation (13) derived in section 2.1.

Now, the given equation (13) is a linear partial differential equation of the second order in three independent variables x, y, t .

By using the result of section A.4.7 in [14], equation (13) can be re-written as

$$\begin{aligned}
 0 \cdot \phi_{xx} + \beta (y^2 - xy) \phi_{xy} + 0 \cdot \phi_{xt} + \beta (y^2 - xy) \phi_{yx} + 0 \cdot \phi_{yy} + 0 \cdot \phi_{yt} \\
 + 0 \cdot \phi_{tx} + 0 \cdot \phi_{ty} + 0 \cdot \phi_{tt} + 0 \cdot \phi_x + \gamma (1 - y) \phi_y - 1 \cdot \phi_t + 0 \cdot \phi = 0 \tag{15}
 \end{aligned}$$

Let us define the matrix A, is given by

$$A = \begin{bmatrix} \text{coeff. of } u_{xx} & \text{coeff. of } u_{xy} & \text{coeff. of } u_{xz} \\ \text{coeff. of } u_{yx} & \text{coeff. of } u_{yy} & \text{coeff. of } u_{yz} \\ \text{coeff. of } u_{zx} & \text{coeff. of } u_{zy} & \text{coeff. of } u_{zz} \end{bmatrix} \quad (16)$$

Therefore,
$$A = \begin{bmatrix} 0 & \beta(y^2 - xy) & 0 \\ \beta(y^2 - xy) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (\text{Using (15) and (16)})$$

Since, $a_{ij} = a_{ji}$, $A = [a_{ij}]_{3 \times 3}$ is a symmetric matrix of order 3x3.

So that,

$$|A| = \begin{vmatrix} 0 & \beta(y^2 - xy) & 0 \\ \beta(y^2 - xy) & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0 \quad (17)$$

Also, the Eigen values of A is given by the form $|A - \lambda I| = 0$

That is,
$$\begin{vmatrix} -\lambda & \beta(y^2 - xy) & 0 \\ \beta(y^2 - xy) & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = 0$$

$$-\lambda^3 + \lambda\beta^2(y^2 - xy)^2 = 0$$

$$\lambda(\beta^2(y^2 - xy)^2 - \lambda^2) = 0$$

$$\lambda = 0; \quad \lambda^2 - \beta^2(y^2 - xy)^2 = 0$$

Therefore,
$$\lambda = 0; \quad \lambda = \pm \beta(y^2 - xy). \quad (18)$$

In equations (17) and (18), the determinant value of A is zero and one of the Eigen values of A is also zero.

By using case (ii) of section A.4.7 in [14], the solution of the given partial differential equation (13) is of parabolic type.

III. CONCLUSION

Here, we have developed a stochastic epidemic model with Poisson infection and Poisson removal rates. Poisson infection rate is depending on the number of persons infected in the system, and Poisson removal rate is depending on the number of persons removed in the system. This result can be extended to epidemic models with carriers in consideration with vaccination.

REFERENCES

- [1] Arlitt, M.F. and Williamson, C.L., "Internet Web servers: Workload characterization and performance implications", *IEEE/ACM Transactions on Networking*, **5** (5): 631, (1997).
- [2] Bailey, N.T., *The mathematical theory of infectious diseases*, Second Edition, Griffin, London, 1975.
- [3] Burghes, D.N. and Borrie, M.S., *Modeling with differential equations*, Ellis Horwood Ltd., 1981.
- [4] Cannizzaro, F., Greco, G., Rizzo, S. and Sinagra, E., "Results of the measurements carried out in order to verify the validity of the Poisson-exponential distribution in radioactive decay events", *The International Journal of Applied Radiation and Isotopes*, **29** (11): 649, (1978).

International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Vol. 4, Issue 10, October 2015

- [5] Diekmann, O. and Heesterbeek, J.A., *Mathematical epidemics of infectious diseases: Model building, analysis and interpretation*, John Wiley, New York, 2000.
- [6] Hethcote, H.W., "Three basic epidemiological models." In S.A. Levin, editor, *Lect. Notes in Biomathematics*, Springer-Verlag, Heidelberg, **100** (1994), 119-144.
- [7] Kapur, J.N., *Mathematical Models in Biology and Medicine*, East-West Press Private Limited, New Delhi, 1985.
- [8] Kermack, W.O. and Mckendrick, A.G., "Contributions to mathematical theory of epidemics." *Proc. R. Soc. Lond. A.*, **115** (1927), 700-721.
- [9] Kermack, W.O. and Mckendrick, A.G., "Contributions to mathematical theory of epidemics." *Proc. R. Soc. Lond. A.*, **138** (1932), 55-83.
- [10] Kermack, W.O. and Mckendrick, A.G., "Contributions to mathematical theory of epidemics." *Proc. R. Soc. Lond. A.*, **141** (1933), 94-122.
- [11] May, R.M., *Stability and complexity in model ecosystem*, Princeton, New Jersey: Princeton University Press, 1973.
- [12] Medhi, J., *Stochastic Processes*, Wiley Eastern Limited, New Delhi, 1981.
- [13] Poisson, S.D., *Probabilite des jugements en matiere criminelle et en matiere civile, precedees des regles generales du calcul des probabilities*, (Paris, France: Bachelier, 1837), page 206.
- [14] Raisinghania, M.D., *Advanced Differential Equations*, 11th Edition, S. Chand and Company Limited, New Delhi, 2007.
- [15] Willkomm, D., Machiraju, S., Bolot, J. and Wolisz, A., "Primary user behavior in cellular networks and implications for dynamic spectrum access". *IEEE Communications Magazine*, **47** (3): 88, (2009).

Web References

- [16] *Principles of Epidemiology, Second Edition*. Atlanta, Georgia: Centers for Disease Control and Prevention.
- [17] Deterministic system definition at *The Internet Encyclopedia of Science* - (http://www.daviddarling.info/encyclopedia/D/deterministic_system.html).
- [18] Dynamical systems (http://www.scholarpedia.org/article/Dynamical_systems) at Scholarpedia.

BIOGRAPHY



Dr. G. RAMESH, M.Sc., M.Phil., Ph.D., MS(IT)., is working as an Associate Professor of Mathematics at Ramanujan Research Centre, Government Arts College (Autonomous), Kumbakonam, Tamilnadu, India. He has 23 years of experience and he has published 30 research papers in various *National* and *International Journals*. His areas of specializations are *Linear Algebra*, *Graph Theory* and *Stochastic Processes*. He published two Books named *Discrete Mathematics* and *Topology* in Tamil. He has produced two Ph.D., scholars and six scholars are pursuing their Ph.D. Presently, he is doing research in *Artificial Genetic Code Matrices*.



Mr. R. SURENDAR, M.Sc., B.Ed., M.Phil., CILP., is working as an Assistant Professor of Mathematics at Swami Dayananda College of Arts and Science, Manjakkudi, Tiruvarur, Tamilnadu, India. He has 3 years of experience and he has published 4 research papers in various *International Journals*. His areas of specializations are *Linear Algebra* and *Stochastic Processes*. At present he is doing his research in *Stochastic Modelling* as a part time research scholar and apart from this he is publishing various research articles in *Matrix Theory*.