

# Periodic Solutions of First Order Non – Linear Functional Differential Equations

A.P.Saravanan<sup>1</sup>, V.Mohanasundaram<sup>2</sup>

<sup>1&2</sup>Assistant Professor, Dept. of Mathematics, Sasurie College of Engineering, Tiruppur, Tamilnadu, India

**ABSTRACT:** In this paper, we obtain a set of easily verifiable sufficient conditions for the existence of the periodic solutions of the first-order linear functional differential equations with periodic perturbation  $x'(t) = l(x)(t) + f(t, x_1)$ , where  $l : C(\mathbb{R}) \rightarrow C(\mathbb{R})$  is a linear bounded operator. These conditions generalize and improve the known results given in the literature.

## I. INTRODUCTION

Let  $B \subset (C(\mathbb{R}, \mathbb{R}))$  be the Banach space of bounded continuous functions  $x : \mathbb{R} \rightarrow \mathbb{R}$  with the sup norm  $\|x\| = \sup\{|x(t)| : t \in \mathbb{R}\}$  and let

$C_\omega(\mathbb{R}) = \{x \in B \subset (C(\mathbb{R}, \mathbb{R})) : x(t + \omega) = x(t), t \in \mathbb{R}\}$ , where  $\omega > 0$ , Define the norms as follows:

$$\|x\|_0 = \max\{|x(t)| : 0 \leq t \leq \omega\}, \quad \|x\|_1 = \int_0^\omega |x(t)| dt, \quad \forall x \in C_\omega(\mathbb{R}).$$

Consider the first-order functional differential equations

$$x'(t) = l(x)(t) + f(t, x_1), \quad \text{---} \quad (1)$$

and the particular case

$$x'(t) = l(x)(t) + g(t) \quad \text{---} \quad (2)$$

where  $l : C(\mathbb{R}) \rightarrow C(\mathbb{R})$  is a linear bounded operator and  $l(C_\omega(\mathbb{R})) \subseteq C_\omega(\mathbb{R})$ ,  $g \in C_\omega(\mathbb{R})$ , and  $f \in C(\mathbb{R} \times B \subset (C(\mathbb{R}, \mathbb{R})), \mathbb{R})$ , for any  $t \in \mathbb{R}$ ,  $f(t, \cdot)$  mapping a bounded set in  $B \subset (C(\mathbb{R}, \mathbb{R}))$  into a bounded set in  $\mathbb{R}$  and satisfying  $f(t + \omega, \varphi) = f(t, \varphi)$  for  $(t, \varphi) \in \mathbb{R} \times B \subset (C(\mathbb{R}, \mathbb{R}))$ . Everywhere in what follows, we will assume that the operator  $l$  is nontrivial and admits the representation

$$l(x)(t) = -a(t)x(t) + l_1(x)(t) - l_2(x)(t) \quad \text{---} \quad (3)$$

where  $a \in C_\omega(\mathbb{R})$ ,  $l_1, l_2 : C_\omega(\mathbb{R}) \rightarrow C_\omega(\mathbb{R})$  are linear and satisfy the condition

$$l_1(x)(t) \geq 0, \quad l_2(x)(t) \geq 0, \quad \forall t \in [0, \omega] \text{ if } x(t) \geq 0, \quad \forall t \in [0, \omega].$$

## II. PRELIMINARIES

### Definition 2.1

A **functional differential equation** is a differential equation in which the derivative  $x'(t)$  of an unknown function  $x$  has a value at  $t$  that is related to  $x$  as a function of some other function at  $t$ .

A general 1<sup>st</sup> order functional differential equation is  $x'(t) = l(x)(t) + f(t, x_1)$

### Definition 2.2

A differential equation is said to be **linear** if the dependent variable and all its differential co-efficient occur in the degree and are not multiplied together

### Definition 2.3

A solution that periodically depends on the independent variable  $t$ , for a periodic solution  $x(t)$  [in the case of a system,  $x$  is a vector].

There is a number  $\omega \neq 0$ , such that  $x(t + \omega) = x(t)$  for  $t \in \mathbb{R}$

All possible such  $\omega$  are called **periodic of this  $\omega$ -periodic solution**.

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### Definition 2.4

A sequence of vectors  $\{x_n\}$  in a normed space is called a **Cauchy sequence** if for every  $\varepsilon > 0$  there is an integer  $N$  such that

$$\|x_m - x_n\| < \varepsilon \quad \forall \quad m, n \geq N$$

### III. PERIODIC SOLUTIONS OF FIRST ORDER NON – LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

The Case  $a(t) \geq 0$

#### Theorem : 3.1

Assume that  $a(t) \geq 0$  and that for any  $\alpha \in [0, \omega]$ ,

Assume that  $a(t) \geq 0$  and that for any  $\alpha \in [0, \omega]$ ,

$$\int_{\alpha}^{\alpha+\omega} l_1(1)(t) \exp\left(-\int_t^{\alpha+\omega} a(s) ds\right) dt < \frac{1 - e^{-\|a\|_1} \int_{\alpha}^{\alpha+\omega} l_2(1)(t) \exp\left(-\int_t^{\alpha+\omega} a(s) ds\right) dt}{1 + \int_{\alpha}^{\alpha+\omega} l_2(1)(t) \exp\left(-\int_t^{\alpha+\omega} a(s) ds\right) dt} \rightarrow (4)$$

and that for any  $\beta \in [0, \omega]$ ,

$$\int_{\beta}^{\beta+\omega} l_2(1)(t) \exp\left(-\int_t^{\beta+\omega} a(s) ds\right) dt < e^{-\|a\|_1} \min_{0 \leq c \leq 1} \left\{ \frac{2(1+c)e^{-\|a\|_1}}{\sqrt{(1-c)^2 + 4ce^{-\|a\|_1}} + (1-c)} + \frac{1}{c} \left[ 1 - \int_{\beta}^{\beta+\omega} l_1(1)(t) \exp\left(-\int_t^{\beta+\omega} a(s) ds\right) dt \right] \right\} \rightarrow (5)$$

Then equation (1) has atleast one  $\omega$  – periodic solution and equation (2) has a unique  $\omega$  – periodic solution.

#### Proof :

It is sufficient to show that homogenous equation has only a trivial  $\omega$  – periodic solution. Assume the contrary.

Let  $x(t)$  be a non – trivial  $\omega$  – periodic solution of equation .

Then  $\|x\|_0 > 0$ .

$$\text{Let, } x_* = \min \{ x(t) : 0 \leq t \leq \omega \} \quad x^* = \max \{ x(t) : 0 \leq t \leq \omega \} \rightarrow (6)$$

$$\text{Choose } t_* \in [0, \omega] \text{ and } t^* \in [t_*, t_* + \omega] \text{ such that } x(t_*) = x_*, x(t^*) = x^* \rightarrow (7)$$

Since  $l(x)$  is linear, we can let  $x^* = \|x\|_0$ .

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We know that the conditions from chapter-I

$$l(x)(t) = -a(t)x(t) + l_1(x)(t) - l_2(x)(t) \quad \longrightarrow (8)$$

$$\text{and } x'(t) = l(x)(t) \quad \longrightarrow (9)$$

$$[x(t) \exp(-\int_a(s) ds)]' = [l_1(x)(t) - l_2(x)(t)] \exp(-\int_a(s) ds) \quad \longrightarrow (10)$$

These are two possible cases:

**Case (i)**  $x^* = x(t^*) \geq 0$ , Integrating (10) from  $t^*$  to  $t^*$  we get.  $x^* \leq 1$

**Case (ii)**  $m = -x(t^*) > 0$ , Set  $x^* = M$  and  $c = n/M$ , Then  $0 \leq c \leq 1$ ,

Which contradicts equation (3.2) with  $\beta = t^* + \omega$  (or)  $t^*$

$$\int_{\beta}^{\beta+\omega} l_2(1)(t) \exp(-\int_a(s) ds) dt < e^{-\|a\|_1} + \min_{0 \leq c \leq 1} \left\{ \frac{2(1+c)e^{-\|a\|_1}}{\sqrt{(1-c)^2 + 4ce^{-\|a\|_1} + (1-c)}} + \frac{1}{c} \left[ 1 - \int_{\beta}^{\beta+\omega} l_1(1)(t) \exp(-\int_t^{\beta+\omega} a(s) ds) dt \right] \right\}$$

### EXAMPLES

1. Assume that  $a(t) \geq 0$ ,  $l_2(1)(t) \equiv 0$ , and that for any  $\beta \in [0, \omega]$ ,

$$1 - e^{-\|a\|_1} < \int_{\beta}^{\beta+\omega} l_1(1)(t) \exp(-\int_t^{\beta+\omega} a(s) ds) dt < 1 + e^{-\|a\|_1} + 2e^{-\|a\|_1/2} \quad \longrightarrow (11)$$

Then equation (1) has atleast one  $\omega$  – periodic solution and equation (2) has a unique  $\omega$  – periodic solution

2. Assume that  $a(t) \leq 0$  and  $l_2(1)(t) \equiv 0$  that either  $a(t)$  (or)  $l_1(1)(t) \equiv 0$  and that for any  $\beta \in [0, \omega]$

$$\int_{\beta}^{\beta+\omega} l_1(1)(t) \exp(-\int_{\beta}^t |a(s)| ds) dt < 1 + e^{-\|a\|_1} + 2e^{-\|a\|_1/2} \quad \longrightarrow (12)$$

Then equation (1) has atleast one  $\omega$  periodic solution and equation (2) has a unique  $\omega$  periodic solution.

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