

Some Common Fixed Point Theorems in S-Metric Space

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ABSTRACT: In 2012 S. Sedghi, N. Shobe and A. Aliouche have introduced the concept of S-metric space. In this paper some common fixed point results for two mappings has been established in S-metric space.

KEYWORDS: S-metric space, Fixed point.

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I. INTRODUCTION AND PRELIMINARIES

Definition 1 [1]: Let X be a nonempty set. An S-metric on X is a function $S: X^3 \rightarrow [0, \infty)$ that satisfies the following conditions for all $x, y, z, a \in X$

(S1) $S(x, y, z) = 0$ if and only if $x = y = z$.

(S2) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

The pair (X, S) is called an S-metric space.

The concept of S-metric space is a generalization of a G-metric space [11] and a D*-metric space [14].

Definition 2 [2]: Let X be a nonempty set. A b-metric on X is a function $d: X^2 \rightarrow [0, \infty)$ such that, for a real number $s \geq 1$, the following conditions hold for all $x, y, z \in X$.

(i) $d(x, y) = 0$ if and only if $x = y$.

(ii) $d(x, y) = d(y, x)$.

(iii) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

The pair (X, d) is called a b-metric space.

Definition 3 [1]: Let (X, S) be an S-metric space. For $r > 0$ and $x \in X$, we define the open ball $B_S(x, r)$ and the closed ball $B_S[x, r]$ with centre x and radius r as follows

$$B_S(x, r) = \{y \in X: S(y, y, x) < r\},$$

$$B_S[x, r] = \{y \in X: S(y, y, x) \leq r\}.$$

Definition 4 [1]: Let (X, S) be an S-metric space.

(1) A sequence $\{x_n\}$ in X is said to converge x , if $S(x_n, x_n, x) \rightarrow 0$ as $n \rightarrow \infty$. That is for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $S(x_n, x_n, x) < \varepsilon$.

(2) A sequence $\{x_n\}$ in X is a Cauchy sequence if $S(x_n, x_n, x_m) \rightarrow 0$ as $m, n \rightarrow \infty$. That is for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$, $S(x_n, x_n, x_m) < \varepsilon$.

(3) The S-metric space (X, S) is complete if every Cauchy sequence is a convergent sequence.

Lemma 5 [1]: In an S-metric space, we have $S(x, x, y) = S(y, y, x)$, $\forall x, y \in X$.

Lemma 6 [1]: Let (X, S) be an S-metric space. If $x_n \rightarrow x$ and $y_n \rightarrow y$ then

$$S(x_n, x_n, y_n) \rightarrow S(x, x, y).$$

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II. MAIN RESULTS

Theorem 1: Let U, V be two self-maps on a complete S-metric space (X, S) and

$S(Ux, Ux, Vy) \leq aS(x, x, y) + bS(Ux, Ux, y) + cS(Vy, Vy, x)$, $\forall x, y \in X$ and a, b, c are non-negative reals with, $a + b + c < 1$, $a + 3b < 1$, and $a + 3c < 1$.

Then the mappings U, V have a unique common fixed point in X .

Proof: Let $x_0 \in X$.

Let us define a sequence $\{x_n\}$ in X as

$$x_{2k+1} = Ux_{2k}, \quad x_{2k+2} = Vx_{2k+1}, \quad k = 0, 1, 2, \dots$$

$$\begin{aligned} \text{Now } S(x_{2k+1}, x_{2k+1}, x_{2k+2}) &= S(Ux_{2k}, Ux_{2k}, Vx_{2k+1}) \\ &\leq aS(x_{2k}, x_{2k}, x_{2k+1}) + bS(Ux_{2k}, Ux_{2k}, x_{2k+1}) + \\ &\quad cS(Vx_{2k+1}, Vx_{2k+1}, x_{2k}) \\ &= aS(x_{2k}, x_{2k}, x_{2k+1}) + bS(x_{2k+1}, x_{2k+1}, x_{2k+1}) + \\ &\quad cS(x_{2k+2}, x_{2k+2}, x_{2k}) \end{aligned}$$

$$\begin{aligned} &\leq \\ aS(x_{2k}, x_{2k}, x_{2k+1}) &+ c[2S(x_{2k+2}, x_{2k+2}, x_{2k+1}) + \\ &\quad S(x_{2k}, x_{2k}, x_{2k+1})] \quad [\text{By S2}] \\ &= aS(x_{2k}, x_{2k}, x_{2k+1}) + c[2S(x_{2k+1}, x_{2k+1}, x_{2k+2}) + S(x_{2k}, x_{2k}, x_{2k+1})] \quad [\text{By Lemma 5}] \end{aligned}$$

Thus,

$$S(x_{2k+1}, x_{2k+1}, x_{2k+2}) \leq \frac{a+c}{1-2c} S(x_{2k}, x_{2k}, x_{2k+1}).$$

Let $\frac{a+c}{1-2c} = h_1$. Since $0 \leq a + 3c < 1$, $0 \leq h_1 < 1$.

Thus

$$S(x_{2k+1}, x_{2k+1}, x_{2k+2}) \leq h_1 S(x_{2k}, x_{2k}, x_{2k+1}).$$

Again,

$$\begin{aligned} S(x_{2k+2}, x_{2k+2}, x_{2k+3}) &= S(Vx_{2k+1}, Vx_{2k+1}, Ux_{2k+2}) \\ &= S(Ux_{2k+2}, Ux_{2k+2}, Vx_{2k+1}) \\ &\leq aS(x_{2k+2}, x_{2k+2}, x_{2k+1}) + bS(Ux_{2k+2}, Ux_{2k+2}, x_{2k+1}) + \\ &\quad cS(Vx_{2k+1}, Vx_{2k+1}, x_{2k+2}) \\ &= aS(x_{2k+2}, x_{2k+2}, x_{2k+1}) + bS(x_{2k+3}, x_{2k+3}, x_{2k+1}) + \\ &\quad cS(x_{2k+2}, x_{2k+2}, x_{2k+2}) \\ &\leq aS(x_{2k+2}, x_{2k+2}, x_{2k+1}) + b[2S(x_{2k+3}, x_{2k+3}, x_{2k+2}) + \\ &\quad S(x_{2k+1}, x_{2k+1}, x_{2k+2})] \\ &= aS(x_{2k+1}, x_{2k+1}, x_{2k+2}) + b[2S(x_{2k+2}, x_{2k+2}, x_{2k+3}) + \\ &\quad S(x_{2k+1}, x_{2k+1}, x_{2k+2})] \end{aligned}$$

Thus,

$$S(x_{2k+2}, x_{2k+2}, x_{2k+3}) \leq \frac{a+b}{1-2b} S(x_{2k+1}, x_{2k+1}, x_{2k+2})$$

Let $\frac{a+b}{1-2b} = h_2$. Since $0 \leq a + 3b < 1$, $0 \leq h_2 < 1$.

Then $S(x_{2k+2}, x_{2k+2}, x_{2k+3}) \leq h_2 S(x_{2k+1}, x_{2k+1}, x_{2k+2})$

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$$\begin{aligned} \text{Now, } S(t, t, t^*) &= S(Ut, Ut, Vt^*) \\ &\leq aS(t, t, t^*) + bS(Ut, Ut, t^*) + cS(Vt^*, Vt^*, t) \\ &= aS(t, t, t^*) + bS(t, t, t^*) + cS(t^*, t^*, t) \\ &= aS(t, t, t^*) + bS(t, t, t^*) + cS(t, t, t^*) \end{aligned}$$

Thus, $(1 - a - b - c)S(t, t, t^*) \leq 0$.

Since $0 \leq a + b + c < 1$, $S(t, t, t^*) = 0$.

Thus $t = t^*$.

Thus t is the unique common fixed point of U and V .

Corollary 1.1: Let T be a self-maps on a complete S-metric space (X, S) and

$S(Tx, Tx, Ty) \leq aS(x, x, y) + bS(Tx, Tx, y) + cS(Ty, Ty, x)$, $\forall x, y \in X$, and
 a, b, c are non-negative reals with $a + b + c < 1$, $a + 3b < 1$, $a + 3c < 1$.

Then the mapping T has a unique fixed point in X .

Proof: Setting $U = V = T$ in the above theorem, we get the result.

Theorem 2: Let (X, S) be a complete S-metric space. Let U, V be two self-maps on X , satisfying

$$S(Ux, Ux, Vy) \leq aS(x, x, y) + \frac{bS(Ux, Ux, y)S(Vy, Vy, x)}{1+S(x, x, y)}, \text{ where } 0 \leq a + b < 1.$$

Then U, V have a unique common fixed point in X .

Proof: Let $x_0 \in X$.

Let us define a sequence $\{x_n\}$ in X as

$$x_{2k+1} = Ux_{2k}, \quad x_{2k+2} = Vx_{2k+1}, \quad k = 0, 1, 2, \dots$$

$$\begin{aligned} \text{Now } S(x_{2k+1}, x_{2k+1}, x_{2k+2}) &= S(Ux_{2k}, Ux_{2k}, Vx_{2k+1}) \\ &\leq aS(x_{2k}, x_{2k}, x_{2k+1}) + \frac{bS(Ux_{2k}, Ux_{2k}, x_{2k+1})S(Vx_{2k+1}, Vx_{2k+1}, x_{2k})}{1+S(x_{2k}, x_{2k}, x_{2k+1})} \\ &= aS(x_{2k}, x_{2k}, x_{2k+1}) + \frac{bS(x_{2k+1}, x_{2k+1}, x_{2k+1})S(x_{2k+2}, x_{2k+2}, x_{2k})}{1+S(x_{2k}, x_{2k}, x_{2k+1})} \\ &= aS(x_{2k}, x_{2k}, x_{2k+1}) \end{aligned}$$

Again,

$$\begin{aligned} S(x_{2k+2}, x_{2k+2}, x_{2k+3}) &= S(Vx_{2k+1}, Vx_{2k+1}, Ux_{2k+2}) \\ &= S(Ux_{2k+2}, Ux_{2k+2}, Vx_{2k+1}) \\ &\leq aS(x_{2k+2}, x_{2k+2}, x_{2k+1}) + \frac{bS(Ux_{2k+2}, Ux_{2k+2}, x_{2k+1})S(Vx_{2k+1}, Vx_{2k+1}, x_{2k+2})}{1+S(x_{2k+2}, x_{2k+2}, x_{2k+1})} \\ &= aS(x_{2k+1}, x_{2k+1}, x_{2k+2}) + \frac{bS(x_{2k+3}, x_{2k+3}, x_{2k+1})S(x_{2k+2}, x_{2k+2}, x_{2k+2})}{1+S(x_{2k+2}, x_{2k+2}, x_{2k+1})} \\ &= aS(x_{2k+1}, x_{2k+1}, x_{2k+2}) \end{aligned}$$

Now,

$$S(x_{n+1}, x_{n+1}, x_{n+2}) \leq aS(x_n, x_n, x_{n+1}) \leq a^2S(x_{n-1}, x_{n-1}, x_n) \leq \dots \leq a^{n+1}S(x_0, x_0, x_1).$$

Now,

$$S(x_{n+1}, x_{n+1}, x_{n+2}) \leq aS(x_n, x_n, x_{n+1}) \leq a^2S(x_{n-1}, x_{n-1}, x_n) \leq \dots \leq a^{n+1}S(x_0, x_0, x_1).$$

Now, for $m > n$,

$$S(x_n, x_n, x_m) \leq 2S(x_n, x_n, x_{n+1}) + S(x_m, x_m, x_{n+1})$$

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$$\begin{aligned} &\leq 2[S(x_n, x_n, x_{n+1}) + S(x_{n+1}, x_{n+1}, x_{n+2}) + \dots \dots \dots + S(x_{m-1}, x_{m-1}, x_m)] \\ &\leq 2[a^n + a^{n+1} + \dots \dots \dots + a^{m-1}]S(x_0, x_0, x_1) \\ &\leq \frac{2a^n}{1-a}S(x_0, x_0, x_1) \\ &\rightarrow 0, \text{ as } n \rightarrow \infty \quad [\text{Since } 0 \leq a + b < 1.] \end{aligned}$$

Thus $\{x_n\}$ is a Cauchy sequence.

Since (X, S) is complete, there exists $t \in X$, such that $x_n \rightarrow t$ as $n \rightarrow \infty$.

Now,

$$\begin{aligned} S(Ut, Ut, t) &\leq 2S(Ut, Ut, x_{2k+2}) + S(t, t, x_{2k+2}) \\ &= 2S(Ut, Ut, Vx_{2k+1}) + S(t, t, x_{2k+2}) \\ &\leq 2 \left[aS(t, t, x_{2k+1}) + \frac{bS(Ut, Ut, x_{2k+1})S(Vx_{2k+1}, Vx_{2k+1}, t)}{1+S(t, t, x_{2k+1})} \right] + S(t, t, x_{2k+2}) \\ &= 2 \left[aS(t, t, x_{2k+1}) + \frac{bS(Ut, Ut, x_{2k+1})S(x_{2k+2}, x_{2k+2}, t)}{1+S(t, t, x_{2k+1})} \right] + S(t, t, x_{2k+2}) \end{aligned}$$

Letting $k \rightarrow \infty$ and using the lemma 5 and lemma 6, we get,

$$S(Ut, Ut, t) \leq 0$$

Hence $Ut = t$.

Again,

$$\begin{aligned} S(t, t, Vt) &= S(Ut, Ut, Vt) \\ &\leq aS(t, t, t) + \frac{bS(Ut, Ut, t)S(Vt, Vt, t)}{1+S(t, t, t)} \\ &= 0 \end{aligned}$$

Thus, $S(t, t, Vt) = 0$ and hence $Vt = t$.

Therefore, $Ut = Vt = t$.

i.e. t is a common fixed point of U and V .

For Uniqueness, let $t^* \in X$, such that $Ut^* = Vt^* = t^*$.

Thus,

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$$\begin{aligned}
 S(t, t, t^*) &= S(Ut, Ut, Vt^*) \\
 &\leq aS(t, t, t^*) + \frac{bS(Ut, Ut, t^*)S(Vt^*, Vt^*, t)}{1+S(t, t, t^*)} \\
 &= aS(t, t, t^*) + \frac{bS(t, t, t^*)S(t^*, t^*, t)}{1+S(t, t, t^*)} \\
 &\leq aS(t, t, t^*) + bS(t, t, t^*)
 \end{aligned}$$

Thus, $(1 - a - b)S(t, t, t^*) \leq 0$.

Since $0 \leq a + b < 1$, $S(t, t, t^*) = 0$.

Thus $t = t^*$.

Hence t is the unique common fixed point of U and V .

III. CONCLUSION

Many other common fixed point results for two mappings, satisfying different types of rational inequalities in S-metric space can also be proved similarly.

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