Observations on Ternary Quadratic Diophantine Equation

$$6(x^2+y^2) - 11xy + 3x + 3y + 9 = 72z^2$$

G.Janaki 1, C.Saranya 2

Assistant Professor, Department of Mathematics, Cauvery College for Women, Tiruchirappalli, Tamil Nadu, India 1
Assistant Professor, Department of Mathematics, Cauvery College for Women, Tiruchirappalli, Tamil Nadu, India 2

ABSTRACT: The Ternary Quadratic Diophantine Equation representing non-homogeneous cone given by

$$6(x^2 + y^2) - 11xy + 3x + 3y + 9 = 72z^2$$

is analyzed for its non-zero distinct integer points on it. Six different patterns of integral solutions satisfying the cone under consideration are obtained. A few interesting relations between the solutions and some special number patterns are presented.

KEYWORDS: Ternary non-homogeneous Quadratic, Integral solutions.

I. INTRODUCTION

Ternary quadratic equations are rich in variety. For an extensive review of various problems one may refer [1-7]. In [8], the ternary quadratic Diophantine equation of the form

$$kxy + m(x + y) = z^2$$

has been studied for non-trivial integral solutions. In [9,10], a special Pythagorean triangle problem have been discussed for its integral solutions. In [11,12], ternary quadratic diophantine equations have been discussed for its integral solutions. This communication concerns with yet another interesting ternary quadratic equation

$$6(x^2 + y^2) - 11xy + 3x + 3y + 9 = 72z^2$$

representing a cone for determining its infinitely many non-zero integral points. Also, a few interesting relations among the solutions are presented.

II. NOTATIONS

$$P_{n} = \text{Pronic number of rank 'n'}. $$
$$T_{m,n} = \text{Polygonal number of rank 'n' with sides 'm'}. $$
$$4DF_{n} = \text{Four Dimensional Figurate number of rank 'n'}. $$
$$CS_{n} = \text{Centered Square number of rank 'n'}. $$
$$SO_{n} = \text{Stella octangula number of rank 'n'}. $$
$$O_{n} = \text{Octahedral number of rank 'n'}. $$
$$P_{n}^{5} = \text{Pyramidal number of rank 'n' with sides 'm'}. $$
$$Gno_{n} = \text{Gnomonic number of rank 'n'}. $$
$$Star_{n} = \text{Star number of rank 'n'}. $$

Copyright to IJIRSET

DOI:10.15680/IJIRSET.2016.0502054

2060
III. METHOD OF ANALYSIS

The ternary quadratic diophantine equation to be solved for its non-zero integral solutions is

\[ 6(x^2 + y^2) - 11xy + 3x + 3y + 9 = 72z^2 \] (1)

The substitution of linear transformations \( x = u + v \) and \( y = u - v \)

in (1) leads to,

\[ (u + 3)^2 + 23v^2 = 72z^2 \] (3)

We illustrate below six different patterns of non-zero distinct integer solutions to (1).

**PATTERN : 1**

Assume \( z = z(a, b) = a^2 + 23b^2 \) (4)

where \( a \) and \( b \) are non-zero integers.

and write \( 72 = (7 + i\sqrt{23})(7 - i\sqrt{23}) \) (5)

Using (4) and (5) in (3), and using factorization method,

\[ ((u + 3) + i\sqrt{23}v)((u + 3) - i\sqrt{23}v) = (7 + i\sqrt{23})(7 - i\sqrt{23})[(a + i\sqrt{23}b)(a - i\sqrt{23}b)] \] (6)

Equating the like terms and comparing real and imaginary parts, we get

\[ u = u(a, b) = 7a^2 - 161b^2 - 46ab - 3 \]
\[ v = v(a, b) = a^2 - 23b^2 + 14ab \]

Substituting the above values of \( u \) & \( v \) in equation (2), the corresponding integer solutions of (1) are given by

\[ x = x(a, b) = 8a^2 - 184b^2 - 32ab - 3 \]
\[ y = y(a, b) = 6a^2 - 138b^2 - 60ab - 3 \]
\[ z = z(a, b) = a^2 + 23b^2 \]

**OBSERVATIONS:**

1. \( x(a, 1) + y(a, 1) + z(a, 1) + Pr_{l_d} - T_{s, a} - T_{28, a} - 305Gna_{a} \equiv 0 \) (mod 687)
2. \( 6(y(a, a) - x(a, a)) + 2z(a, a) \) is a perfect square.
3. \( 6x(a, 1) + z(a, 1) - x(a, 1) \cdot y(a, 1) - 240SO_{a} - 52Star_{a} + 9798Gna_{a} \equiv 0 \) (mod 62023)
4. \( x(a, 1) + y(a, 1) - 2z(a, 1) - 2Star_{a} + 40Gna_{a} \equiv 0 \) (mod 416)
5. \( 3x(a, 1) - 4y(a, 1) + z(a, 1) \cdot y(a, 1) - 72(4DF_{a}) + 3O_{a} + 116P_{a}^{5} - 122T_{a, a} - 3240Gna_{a} \equiv 0 \) (mod 7776)
6. Each of the following expressions represents a Nasty number.
   i. \( 3x(a, a) - 5y(a, a) - 14z(a, a) \)
   ii. \( 6[6(y(a, a) - x(a, a)) + 2z(a, a)] \)

**PATTERN : 2**

Instead of (5), we write 72 as

\[ 72 = \frac{(5 + i7\sqrt{23})(5 - i7\sqrt{23})}{16} \] (7)

Following the procedure presented above, the corresponding non-zero distinct integer solutions to (1) are obtained as...
\[
x = x(A, B) = 12A^2 - 276B^2 - 312AB - 3 \\
y = y(A, B) = -2A^2 + 46B^2 - 332AB - 3 \\
z = z(A, B) = 4A^2 + 92B^2
\]

**OBSERVATIONS:**
1. \(3z(A, A) + y(A, A) - x(A, A)\) is a perfect square.
2. \(2y(A, l) + z(A, l) + 332Gno \equiv 0 \pmod{154}\)
3. \(y(A, l) + z(A, l) - x(A, l) + 2T_{12,A} + 14Gno \equiv 0 \pmod{20^2}\)
4. \(x(A, A) - y(A, A) + 4z(A, A) - 16Star - 48Gno \equiv 0 \pmod{32}\)
5. Each of the following expressions represents a Nasty number
   i. \(y(A, A) - x(A, A) - 2z(A, A)\)
   ii. \(6[3z(A, A) + y(A, A) - x(A, A)]\)

**PATTERN : 3**

Equation (3) can be written as
\[
(u + 3)^2 = 72z^2 - 23v^2
\]
Introducing the linear transformations,
\[
z = X + 23T \quad \text{and} \quad v = X + 72T
\]
in (8), we get
\[
X^2 = 1656r^2 + \left(\frac{u + 3}{7}\right)^2
\]
which is satisfied by
\[
T(r, s) = 2rs \\
u(r, s) = 11592r^2 - 7s^2 - 3 \\
X(r, s) = 1656r^2 + s^2
\]

Employing the above values in (9) and using (2), the corresponding integer solutions of (1) are given by
\[
x = x(r, s) = 13248r^2 - 6s^2 + 144rs - 3 \\
y = y(r, s) = 9936r^2 - 8s^2 - 144rs - 3 \\
z = z(r, s) = 1656r^2 + s^2 + 46rs
\]

**OBSERVATIONS:**
1. \(x(r, r) - y(r, r) - 2z(r, r)\) is a perfect square.
2. \(6[x(r, r) - y(r, r) - 2z(r, r)]\) represents a Nasty number.
3. \(6z(r, l) - y(r, l) - 210Gno \equiv 0 \pmod{227}\)
4. \(x(r, l) - y(r, l) - 1656CS - 1800Gno \equiv 0 \pmod{146}\)
5. \(x(r, l) - y(r, l) - 2z(r, l) \equiv 0 \pmod{14^2}\)
6. \(3y(r, l) - 4x(r, l) + 14z(r, l) + 182Gno \equiv 0 \pmod{165}\)
PATTERN : 4
Consider (3) as
\[(u + 3)^2 - 49v^2 = 72(z^2 - v^2)\]  \hspace{1cm} (11)

Write (12) in the form of ratio as
\[
\frac{(u + 3) + 7v}{z + v} = \frac{72(z - v)}{(u + 3) - 7v} = \frac{A}{B}, B \neq 0
\]

which is equivalent to the following two equations
\[
B(u + 3) + (7B - A)v - Az = 0
\]
\[
A(u + 3) + (72B - 7A)v - 72Bz = 0
\]

On employing the method of cross multiplication, we get
\[
u = u(A, B) = 7A^2 - 144AB + 504B^2 - 3
\]
\[
v = v(A, B) = A^2 - 72B^2
\]
\[
z = z(A, B) = A^2 - 14AB + 72B^2
\]

Applying the above values of \(u & v\) in equation (2), the corresponding integer solutions of (1) are given by
\[
x = x(A, B) = 8A^2 - 144AB + 432B^2 - 3
\]
\[
y = y(A, B) = 6A^2 - 144AB + 576B^2 - 3
\]
\[
z = z(A, B) = A^2 - 14AB + 72B^2
\]

OBSERVATIONS:
1. \(y(A, A) - x(A, A) - 2z(A, A)\) represents a Nasty number.
2. \(y(A, 1) - 6z(A, 1) + 30Gno_A \equiv 0 \pmod{111}\)
3. \(x(A, 1) + 7y(A, 1) - 7CS_A + 137Gno_A \equiv 0 \pmod{858}\)
4. \(z(A, 1) - x(A, 1) + Pr_A + Star_A - 356Gno_A \equiv 0 \pmod{587}\)
5. \(x(A, 1) - 6z(A, 1) + 6Gno_A \equiv 0 \pmod{78}\)

Remarkable Note:

Equation (12) can also be expressed in the form of ratio in two different ways as follows.
\[
(i) \frac{(u + 3) + 7v}{8(z + v)} = \frac{9(z - v)}{(u + 3) - 7v} = \frac{A}{B}, B \neq 0
\]
\[
(ii) \frac{(u + 3) + 7v}{9(z + v)} = \frac{8(z - v)}{(u + 3) - 7v} = \frac{A}{B}, B \neq 0
\]

Repeating the analysis as above the corresponding integer solutions along with properties are presented below.

Solutions of (i):
\[
x = x(A, B) = 64A^2 - 144AB + 54B^2 - 3
\]
\[
y = y(A, B) = 48A^2 - 144AB + 72B^2 - 3
\]
\[
z = z(A, B) = 8A^2 - 14AB + 9B^2
\]
OBSERVATIONS:

1. \( 9(y(A,A) - x(A,A)) + 2z(A,A) \) represents a Nasty number.
2. \( 8(y(A,A) - x(A,A)) + 3z(A,A) \) is a perfect square.
3. \( y(A,1) - x(A,1) - z(A,1) + 12CS_A + 5Gno_A = 0 \pmod{4^2} \).

Solutions of (ii):

\[
x = x(A,B) = 72A^2 - 144AB + 48B^2 - 3
\]
\[
y = y(A,B) = 54A^2 - 144AB + 64B^2 - 3
\]
\[
z = z(A,B) = 9A^2 - 14AB + 8B^2
\]

OBSERVATIONS:

1. \( y(A,1) - x(A,1) + 2T_{20,A} + 8Gno_A \equiv 0 \pmod{8} \).
2. \( y(A,1) - 6z(A,1) + 30Gno_A \equiv 0 \pmod{17} \).
3. \( y(A,1) + 2z(A,1) - 12Star_A + 50Gno_A \equiv 0 \pmod{15} \).

IV. CONCLUSION

In this paper, we have presented six different patterns of non-zero distinct integer solutions of the non-homogeneous cone given by \( 6(x^2 + y^2) - 11xy + 3x + 3y + 9 = 72z^2 \).
To conclude, one may search for other patterns of non-zero integer distinct solutions and their corresponding properties for other choices of ternary quadratic diophantine equations.

REFERENCES

BIOGRAPHY

Dr. G. JANAKI received the B.Sc., M.Sc. and M.Phil., degree in Mathematics from Bharathidasan University, Trichy, South India. She completed her Ph.D., Degree from Bharathidasan University/National College. She has published many Papers in International and National level Journals. Her research area is Number Theory.

Ms. C. SARANYA received the B.Sc., M.Sc., And M.Phil., degree in Mathematics from Bharathidasan University, Trichy, South India. Her ongoing research focusing on the subject of Number Theory.