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An application of Generalized Fourier-Finite Mellin Transform

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ABSTRACT: From the time of Laplace up to the present time, different theories of integral transforms have been proposed by the help of integrals with different kernels and ranges of integrations, chosen suitably. These integral transforms are linear continuous operators with their inverses, transforming a class of functions to another class of functions or sequences. The most useful significance of integral transforms lies in the fact that they transform a class of differential equations into a class of algebraic equations, so that solutions of those differential equations can be obtained easily by algebraic methods and by use of results of integral transforms. The main aim of this paper is to give the application of generalized Fourier-Finite Mellin transform for solving some differential equations. Also we introduced new differential operator and its adjoint operator and solved the Differential equations using Fourier-Finite Mellin transform.

KEYWORDS: Fourier transform, Finite Mellin Transform, Fourier- Finite Mellin transform, Generalized function, Adjoint Operator.

1. INTRODUCTION

The distributions are also known as generalized functions which generalize the idea of classical functions and allow us to extend the concept of derivative not only to all continuous functions but also to the discontinuous functions in the classical sense. The theory of distribution has application in various fields especially in science and engineering where there are many non continuous phenomena which might naturally lead to differential equations whose solutions are distributions. For example, such as the delta distribution is not mathematical function in classical sense. However it is very useful in several applications, therefore the distributions can help us to develop an operational calculus in order to investigate linear ordinary differential equations as well as partial differential equations with constant and variable coefficient through their fundamental solutions [1].

Partial differential equations have big importance in Mathematics and other fields of science. Therefore it is very important to know methods to solve such partial differential equations. One of the most known methods to solve partial differential equations is the integral transform method [2]. The Fourier integral transform is well known for finding the probability densities for sums and differences of random variables. The use of Mellin Integral transform is to derive different properties in statistics and probability densities of single continuous random variable [3]. It is also applicable in solution of fractional differential equations [4].

Many authors studied on various integral transforms separately. However there is much scope in extending double transformation to a certain class of generalized functions. B. N. Bhosale and M. S. Choudhary [5] and S. M. Khairmar et.al. [6] has discussed double transform and their application. Motivated by this we have also defined a new combination of integral transforms in distributional generalized sense namely Fourier-Finite Mellin transform and its applications to some particular types of differential equations by introducing a new differential operator and its Adjoint operator. For this it is necessary to define the conventional Fourier-Finite Mellin transform which is defined as follows:

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$$FM_f \{f(t, x)\} = F(s, p) = \langle f(t, x), K(t, x, s, p) \rangle,$$

$$\text{where, } K(t, x, s, p) = e^{-ist} \left(\frac{a^{2p}}{x^{p+1}} - x^{p-1} \right).$$

Also we have defined the testing function spaces as well as distributional generalized Fourier-Laplace transform in our previous work, which are given as follows:

1.1. The space $FM_{f,b,c,\alpha}$

This space is given by

$$FM_{f,b,c,\alpha} = \left\{ \phi : \phi \in E_+ / \xi_{b,c,k,q,l} \phi(t, x) = 0 \begin{matrix} 0 < t < \infty \\ 0 < x < a \end{matrix} \left| t^k \lambda_{b,c}(x) x^{q+1} D_t^l D_x^q \phi(t, x) \right| \leq C_{lq} A^k k^{k\alpha} \right\} \quad (1.1.1)$$

$$\text{for each } k, l, q = 0, 1, 2, 3, \dots \quad \text{where, } \lambda_{b,c}(x) = \begin{cases} x^{+b} & 0 < x < 1 \\ x^{+c} & 1 < x < a \end{cases}$$

Where the constants A and C_{lq} depend on the testing function ϕ .

1.2. Distributional Generalized Fourier-Finite Mellin Transforms ($FM_f T$)

For $f(t, x) \in FM_{f,b,c,\alpha}^{*\beta}$, where $FM_{f,b,c,\alpha}^{*\beta}$ is the dual space of $FM_{f,b,c,\alpha}^\beta$. It contains all distributions of compact support. The Distributional Fourier-Finite Mellin transform is a function of $f(t, x)$ and is defined as

$$FM_f \{f(t, x)\} = F(s, p) = \left\langle f(t, x), e^{-ist} \left(\frac{a^{2p}}{x^{p+1}} - x^{p-1} \right) \right\rangle \quad (1.2.1)$$

where, for each fixed t ($0 < t < \infty$), x ($0 < x < \infty$), $s > 0$ and $p > 0$, the right hand side of (1.2.1) has a sense as

$$\text{an application of } f(t, x) \in FM_{f,b,c,\alpha}^{*\beta} \text{ to } e^{-ist} \left(\frac{a^{2p}}{x^{p+1}} - x^{p-1} \right) \in FM_{f,b,c,\alpha}^\beta.$$

The paper is organized as follows:

In section 2 new differential operator is introduced. In section 3 another operator is defined which is the adjoint operator of the operator defined in section 2. Section 4 gives applications of generalized Fourier-Finite Mellin Transform to differential equations. Section 5 concludes the paper.

Notations and terminology used as in Zemanian [7], [8].

2. APPLICATION OF FOURIER-FINITE MELLIN TRANSFORM

The kernel of Fourier-Finite Mellin transform is $K(t, x, s, p) = e^{-ist} \left(\frac{a^{2p}}{x^{p+1}} - x^{p-1} \right)$, Multiply by x and differentiate with respect to t and x , we get

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$$\begin{aligned}
 xD_t D_x K(t, x, s, p) &= (-is) e^{-ist} xD_x \left(\frac{a^{2p}}{x^{p+1}} - x^{p-1} \right) \\
 &= (-is) e^{-ist} xD_x (a^{2p} x^{-p-1} - x^{p-1}) \\
 &= (-is) e^{-ist} D_x (a^{2p} x^{-p} - x^p) \\
 &= (-is) e^{-ist} (a^{2p} (-p) x^{-p-1} - p x^{p-1}) \\
 &= ispe^{-ist} \left(\frac{a^{2p}}{x^{p+1}} + x^{p-1} \right) \\
 &= ispK(t, x, s, p)
 \end{aligned}$$

We construct an operator $\Lambda_{t,x} = xD_t D_x + isp$, where $D_t = \frac{d}{dt}$, $D_x = \frac{d}{dx}$

$$\begin{aligned}
 \Lambda_{t,x} K(t, x, s, p) &= [xD_t D_x + isp] K(t, x, s, p) \\
 &= xD_t D_x K(t, x, s, p) + ispK(t, x, s, p) \\
 &= ispK(t, s, s, p) + ispK(t, x, s, p) \\
 &= 2ispK(t, s, s, p) \\
 &= C_0 K(t, s, s, p), \text{ where } C_0 = 2isp \text{ is constant}
 \end{aligned}$$

Continuing in the same way, we have

$$\Lambda_{t,x}^2 K(t, x, s, p) = (C_0)^2 K(t, x, s, p)$$

$$\Lambda_{t,x}^3 K(t, x, s, p) = (C_0)^3 K(t, x, s, p)$$

$$\Lambda_{t,x}^4 K(t, x, s, p) = (C_0)^4 K(t, x, s, p)$$

And so on,

$$\Lambda_{t,x}^k K(t, x, s, p) = (C_0)^k K(t, x, s, p) = (2isp)^k K(t, x, s, p)$$

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Since the operator $\Lambda_{t,x}^k K(t, x, s, p) = (2isp)^k K(t, x, s, p)$ is obviously linear and continuous. We have

$$\begin{aligned} FM_f \left\{ \Lambda_{t,x}^k [f(t, x)] \right\} &= \left\langle \Lambda_{t,x}^k [f(t, x)], K(t, x, s, p) \right\rangle \\ &= \left\langle f(t, x), \Lambda_{t,x}^k K(t, x, s, p) \right\rangle \\ &= \left\langle f(t, x), (2isp)^k K(t, x, s, p) \right\rangle \end{aligned}$$

Therefore we have, $FM_f \left\{ \Lambda_{t,x}^k [f(t, x)] \right\} = \left\langle f(t, x), ((C_0)^k) K(t, x, s, p) \right\rangle$ for all $f \in FM_{f,b,c,\alpha}^{*\beta}$.

3. ADJOINT OPERATOR $\Lambda_{t,x}^*$

We define an operator $\Lambda_{t,x}^* : FM_{f,b,c,\alpha}^{*\beta} \rightarrow FM_{f,b,c,\alpha}^\beta$ using the relation

$$\left\langle \Lambda_{t,x}^* (f(t, x)), \phi(t, x) \right\rangle = \left\langle f(t, x), \Lambda_{t,x} (\phi(t, x)) \right\rangle$$

For all $f \in FM_{f,b,c,\alpha}^{*\beta}$ and $\phi \in FM_{f,b,c,\alpha}^\beta$. The operator $\Lambda_{t,x}^*$ is called the adjoint operator of $\Lambda_{t,x}$. For each

$$k = 1, 2, 3, \dots \text{ we easily get } \left\langle (\Lambda_{t,x}^*)^k (f(t, x)), \phi(t, x) \right\rangle = \left\langle f(t, x), (\Lambda_{t,x}^*)^k (\phi(t, x)) \right\rangle$$

It can be readily shown that if f regular distribution generated by an element in then $\Lambda_{t,x}^* f = \Lambda_{t,x} f$ for each $k = 1, 2, 3, \dots$ we have

$$\begin{aligned} \left\langle (\Lambda_{t,x}^*)^k (f(t, x)), K(t, x, s, p) \right\rangle &= \left\langle f(t, x), (\Lambda_{t,x}^*)^k K(t, x, s, p) \right\rangle \\ &= \left\langle f(t, x), (2isp)^k K(t, x, s, p) \right\rangle \\ &= \left\langle f(t, x), ((C_0)^k) K(t, x, s, p) \right\rangle \\ &= ((C_0)^k) \left\langle f(t, x), K(t, x, s, p) \right\rangle \end{aligned}$$

Thus we arrive at the important result, for each $k = 1, 2, 3, \dots$ we have for $f \in FM_{f,b,c,\alpha}^{*\beta}$

$$FM_f \left\{ (\Lambda_{t,x}^*)^k (f(t, x)) \right\} = (C_0)^k FM_f \left\{ f(t, x) \right\} (s, p)$$

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4. AN APPLICATION OF THE FOURIER-FINITE MELLIN TRANSFORM TO DIFFERENTIAL EQUATIONS

4.1. Solution of $P(\Lambda_{t,x}^*)u = f$

Consider the differential equation $P(\Lambda_{t,x}^*)u = f$ (4.1.1)

Where $f \in FM_{f,b,c,\alpha}^{*\beta}$ and P any polynomial of degree m . Suppose that the equation (4.1.1) possesses a solution u .

Applying the Fourier-Finite Mellin transform to (4.1.1) we have,

$$FM_f [P(\Lambda_{t,x}^*)u] = FM_f (f) \text{ if } FM_f [f] = f^\Lambda \text{ then}$$

$$P(2isp)FM_f [u(t,x)] = f^\Lambda \text{ using } FM_f [(\Lambda_{t,x}^*)^k (f(t,x))] = (2isp)^k FM_f \{f(t,x)\}$$

$$\Rightarrow P(2isp)u^\Lambda = f^\Lambda \text{ i.e. } P((C_0))u^\Lambda = f^\Lambda \tag{4.1.2}$$

where $u^\Lambda = FM_f [u(t,x)]$

If we further assume that the polynomial P is such that for $\epsilon > 0$

$$|P(2isp)| < \epsilon \tag{4.1.3}$$

Then under this assumption (4.1.2) gives

$$u^\Lambda = [P(2isp)]^{-1} f^\Lambda \tag{4.1.4}$$

Applying inversion of Fourier-Finite Mellin transform, we get

$$u = (FM_f)^{-1} \left[\frac{f^\Lambda}{P(2isp)} \right] = (FM_f)^{-1} \left[\frac{f^\Lambda}{P(C_0)} \right]$$

Hence the proof.

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4.2. Solution of differential equation $P(D)u = f$

Consider the differential equation $P(D)u = f$ (4.2.1)

When $f \in FM_{f,b,c,\alpha}^{*\beta}$ and $P(D) = \sum_{|\beta| \leq m} a_\beta D^\beta$

Is a linear differential operator of order m with constant coefficients.

Suppose that the equation (4.2.1) possesses a solution u .

Applying the Fourier-Finite Mellin transform to (4.2.1) and using

$$\begin{aligned} x^q D_t^l D_x^q K(t, x, s, p) &= (-is)^l (-p)^q e^{-ist} \left(\frac{a^{2p}}{x^{p+1}} - x^{p-1} \right) \\ &= (-is)^l (-p)^q K(t, x, s, p) \\ &= (-1)^{l+q} (is)^l (p)^q K(t, x, s, p) \end{aligned}$$

We have $FM_f [P(D)u] = FM_f [f] = f^\wedge$ (say) (4.2.2)

We can reform them to the Fourier-Finite Mellin transform and hence we get

$$P(t, x, s, p)u^\wedge = f^\wedge \tag{4.2.3}$$

where $P(t, x, s, p)$ is polynomial in t, x, s , and p , $u^\wedge = FM_f [u]_s$

Under the assumption that the polynomial P is such that

$$P(x^q D_t^l D_x^q K(t, x, s, p)) > \xi > 0 \text{ for } \xi_1, \xi_2, \xi_3, \dots, \xi_n \in R^n.$$

$$u^\wedge = P[x^q D_t^l D_x^q K(t, x, s, p)]^{-1} f^\wedge \tag{4.2.4}$$

Applying inversion of Fourier-Finite Mellin transform to above equation, we have

$$u = (FM_f)^{-1} \left[\frac{f^\wedge}{P(x^q D_t^l D_x^q K(t, x, s, p))} \right]$$

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5. CONCLUSION

In the present work we have introduced new operator Λ and its adjoint operator Λ^* . And using that the differential equations is solved for Fourier-Finite Mellin transform.

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