

Operating Characteristics of $M^X/M/1/MWV/BD$ Queuing Systems

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ABSTRACT: In this paper the batch arrival $M^X/M/1$ queuing system along with server breakdowns and multiple working vacations is analyzed under exponential distribution. For this model the total probability generating function is obtained through the Chapman-Kolmogorov balance equations satisfied by the steady state system size probabilities. The probability generating function is presented in closed form so that various performance measures are analyzed. Further the expected system size when the server is in vacation (L_v) and the expected system size of the model (L) are also evaluated.

KEYWORDS: Batch Arrival, Multiple Working Vacations, Breakdowns, Total Probability Generating Function, Performance Measures.

I. INTRODUCTION

A classical queuing system may be described as one having a service facility at which units of some customer arrive for service and, whenever there are more units in the system than the service facility can handle simultaneously, a queue or waiting line is developed. The waiting units take their turn for service according to a pre assigned rule and after service they leave the system. The study of classical queuing models are made by Saaty [11], Gross & Harris [4] and Medhi [10].

The batch arrival is described as the flow of arrivals in batches. Gaver [3] introduced bulk arrival queues, where the arrivals could be in batch. Choudhury and Templeton [2] and Medhi [9] discuss the subject at great length.

In N-policy the server does not start his service until there are N-customers in the queue. This policy is introduced by Yadin and Naor [16] and is designed to minimize server switch over's and to avoid excessive frequent use of setups. Lee et al., [6] successively combined the batch arrival queues with N-policy.

Queuing systems with server classical vacations are characterized by the fact that the idle time of the server may be used for other secondary jobs. Allowing server to take vacation make queuing models more realistic and flexible in studying real world queuing situations. Applications arise naturally in call centers with multi task employees, maintenance activity, production and quality control problems etc.,.

In N-policy queuing models, with server vacation, as soon as the system empties, the server leaves the system for a vacation of random length. When the server returns from the vacation and finds N or more customers, he immediately starts his service. Otherwise he takes repeated number of vacations until he finds N or more customers. This policy is called a Multiple Vacations Policy.

Most of the classical queuing systems the server may fail and can be repaired. The performance of the system may be affected heavily by these breakdowns and limited repair capacity. Queuing systems with such unreliable stations are the topics of worth investigating from the performance prediction point of view. As a result of breakdowns, service facility becomes inoperative and the units demanding service can be served only when it is restored to operative state. Wang and Huang [13] first proposed Markovian queuing system with removable service station. Ke [5]

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considered the control policy for batch arrival $M^X/M/1$ queuing system under N-policy in which the server is characterized by breakdowns and multiple vacations.

In working vacation queues, the server works at a lower service rate rather than completely stopping service during the vacation period. At the vacation termination epochs, if there are customers in the system, the server will start a new regular busy period. Otherwise, he takes another working vacation which follows multiple working vacations policy.

Servi and Finn [12], introduced a class of semi vacation policies, in which servers work at a lower rate rather than completely stopping primary service during vacation. Such a vacation is called working vacation (WV). Li and Tian [7], Xu et al., [14] considered $M/M/1$ queue with different working vacation policies. Xu et al., [15] studied the results of Liu et al., [8] to bulk input model $M^X/M/1/MWV$. They have formulated the model as two dimensional Markovian chain and obtained the PGF of the stationary queue length and its stochastic decomposition result using the matrix analysis method. Their concept is motivated to combine the batch arrival queues under server breakdowns and multiple working vacations.

In this paper, with the help of the available literature a batch arrival $M^X/M/1$ queuing system along with server breakdowns and multiple working vacations is analyzed under exponential distribution. The probability generating function (PGF) of the system size is obtained through the Chapman-Kolmogorov balanced equations satisfied by the steady state system size probabilities. The PGF is presented in closed form so that various performance measures are calculated easily.

II. MODEL DESCRIPTION

Consider a batch arrival $M^X/M/1$ queue in which, the arrival stream forms a poisson process and the actual number of customers in any arriving module is a random variable X , which may take on any positive integral value $k (< \infty)$ with probability g_k . If λ_k is the arrival rate of a poisson process of batches of size k then $g_k = \lambda_k / \lambda$, $k=1,2,3,\dots$ where λ is the composite arrival rate of all batches equal to $\sum_{i=1}^{\infty} \lambda_i$. This total process, which arises from the overlap of the set of Poisson processes with rates $\{\lambda_k, k=1,2,\dots\}$ is a compound poisson process. Let $X(z), E(X)$ and $E(X^2)$ denote the PGF, first and second moments of random variable X .

$$\text{Then } X(z) = \sum_{k=1}^{\infty} g_k z^k ; E(X^k) = \sum_{n=1}^{\infty} g_n n^k$$

The server serves the customers at a time with exponential service rate μ in a regular busy period. Whenever the system becomes empty at service completion instant, the server starts a working vacation during which the service is done at a lower rate. The vacation duration V follows an exponential distribution with parameter η . During working vacations, arriving customers are served with exponential service rate $\mu_v (\leq \mu)$.

When a vacation terminates and the server finds the system is empty, then the server begins another working vacation. On the other hand, if the server finds the system is not empty at the vacation termination instant, then he switches to a regular service period. The distributions of the service times during regular busy period and working vacation period are both exponential but with different rates μ and μ_v respectively. It is assumed that, inter arrival times, service time and working vacation times are mutually independent of each other.

The server is subject to breakdowns at any time while working, with poisson rate α . Whenever the system fails, the server is sent immediately for repair at a repair facility where the repair time is an independent and identically distributed random variable Br following exponential distribution $(1 - e^{-\beta t})$. The customer, who is just the being served when the server breaks down, joins the head of the waiting line and resumes the service as soon as the server returns from the repair facility. This type of service continues until the system becomes empty again.

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This model is denoted by $M^X/M/1/MWV/BD$.

III. SYSTEM SIZE DISTRIBUTION

Let $N_s(t)$ denote the number of customers in the system at time t and

$$J(t) = \begin{cases} 0 & \text{the system is in a working vacation period at time } t. \\ 1 & \text{the system is in regular busy period at time } t. \\ 2 & \text{the system is in break down period at time } t. \end{cases}$$

Then $\{N_s(t), J(t)\}$ is a Markov process.

$$\text{Let } Q_n(t) = \Pr\{N_s(t) = n ; J(t) = 0\}, \quad n \geq 0,$$

$$P_n(t) = \Pr\{N_s(t) = n ; J(t) = 1\}, \quad n \geq 1,$$

$$\text{and } B_n(t) = \Pr\{N_s(t) = n ; J(t) = 2\}, \quad n \geq 1.$$

denote the system size probability at time t .

Assuming the steady state system size probabilities as

$$P_n = \lim_{t \rightarrow \infty} P_n(t) ; Q_n = \lim_{t \rightarrow \infty} Q_n(t) \text{ and } B_n = \lim_{t \rightarrow \infty} B_n(t) \text{ exists.}$$

The steady state equations satisfied by P_n 's, Q_n 's and B_n 's are given by

$$\lambda Q_0 = \mu_v Q_1 + \mu P_1 \tag{1}$$

$$(\lambda + \eta + \mu_v) Q_n = \lambda \sum_{k=1}^n Q_{n-k} g_k + \mu_v Q_{n+1} ; n \geq 1 \tag{2}$$

$$(\lambda + \mu + \alpha) P_1 = \mu P_2 + \eta Q_1 + \beta B_1 \tag{3}$$

$$(\lambda + \mu + \alpha) P_n = \beta B_n + \lambda \sum_{k=1}^{n-1} P_{n-k} g_k + \mu P_{n+1} + \eta Q_n ; n \geq 2 \tag{4}$$

$$(\lambda + \beta) B_1 = \alpha P_1 \tag{5}$$

$$(\lambda + \beta) B_n = \alpha P_n + \lambda \sum_{k=1}^{n-1} B_{n-k} g_k ; n \geq 2 \tag{6}$$

To obtain the steady state distribution of the model, the partial PGFs, are defined as

$$Q(z) = \sum_{n=0}^{\infty} Q_n z^n ; P(z) = \sum_{n=1}^{\infty} P_n z^n \text{ and } B(z) = \sum_{n=1}^{\infty} B_n z^n .$$

By multiplying equation (2) by z^n , summing over $n \geq 1$ and then adding with equation (1) we get

$$\sum_{n=1}^{\infty} (\lambda + \eta + \mu_v) Q_n z^n + \lambda Q_0 = \lambda \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} Q_{n-k} g_k z^n + \sum_{n=1}^{\infty} \mu_v Q_{n+1} z^n + \mu_v Q_1 + \mu P_1$$

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since, $\sum_{n=1}^{\infty} z^n \sum_{k=1}^n Q_{n-k} g_k = Q(z) X(z)$

$$\text{(i.e.,)} (\lambda z (1 - X(z)) + \mu_v (z - 1) + \eta z) Q(z) = \mu_v (z - 1)Q_0 + Pz \tag{7}$$

$$\text{where } P = \eta Q_0 + \mu P_1.$$

The equation, $h(z) = \lambda z (1 - X(z)) + \mu_v (z - 1) + \eta z$

$$= (\lambda + \eta + \mu_v) z - \lambda z X(z) - \mu_v \tag{8}$$

has a unique root z_1 , with $|z_1| < 1$, by Rouché's theorem.

For $f(z) = (\lambda + \eta + \mu_v) z$ and $g(z) = -\lambda z X(z) - \mu_v$ satisfy the condition $|g(z)| < |f(z)|$ on $|z| = 1$ and hence the equations $f(z) = 0$ and $f(z) + g(z) = h(z) = 0$ have the same number of roots inside the circle $|z| = 1$. Since $f(z)$ is linear, the root is real and simple and $h(0) = -\mu_v$ and $h(1) = \eta$ are of opposite sign.

Thus by evaluating P at $z = z_1$ from equation (7), we get

$$P = \frac{\mu_v (1 - z_1) Q_0}{z_1} \tag{9}$$

The equation, $h(z) = \lambda z (1 - X(z)) + \mu_v (z - 1) + \eta z$

has a unique root z_1 , with $|z_1| < 1$, by Rouché's theorem. We know that $h(z) = 0$

$$\text{we get, } \eta z_1 = \lambda z_1 (1 - X(z_1)) + \mu_v (z_1 - 1) \tag{10}$$

With this, the partial PGF $Q(z)$ is given by

$$Q(z) = \frac{\mu_v (z - z_1) Q_0}{z_1 (\lambda z (1 - X(z)) + \mu_v (z - 1) + \eta z)} \tag{11}$$

Similarly proceeding for equations (3) and (4) imply,

$$\sum_{n=2}^{\infty} (\lambda + \mu + \alpha) P_n z^n + (\lambda + \mu + \alpha) P_1 = \sum_{n=2}^{\infty} \beta B_n z^n + \lambda \sum_{k=1}^{n-1} \sum_{n=2}^{\infty} P_{n-k} g_k z^n + \sum_{n=2}^{\infty} \mu P_{n+1} z^n + \sum_{n=2}^{\infty} \eta Q_n z^n + \mu P_2 + \eta Q_1 + \beta B_1$$

since, $\sum_{n=2}^{\infty} z^n \sum_{k=1}^{n-1} P_{n-k} g_k = P(z) X(z)$

$$\text{(i.e.,)} (\lambda z (1 - X(z)) + \mu (z - 1) + \alpha z) P(z) = \eta z Q(z) + \beta B(z) - Pz \tag{12}$$

Similarly equations (5) and (6) implies,

$$\sum_{n=2}^{\infty} (\lambda + \beta) B_n z^n + (\lambda + \beta) B_1 = \lambda \sum_{k=1}^{n-1} \sum_{n=2}^{\infty} B_{n-k} g_k z^n + \sum_{n=2}^{\infty} \alpha P_n z^n + \alpha P_1$$

since, $\sum_{n=2}^{\infty} z^n \sum_{k=1}^{n-1} B_{n-k} g_k = B(z) X(z)$

$$\text{thus } B(z) = \frac{\alpha P(z)}{\lambda(1 - X(z)) + \beta} \tag{13}$$

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By substituting the value of P from (9), Q(z) from (11), and B(z) from (13) in P(z) of (12) we have,

$$\left(\frac{(\lambda z(1-X(z)) + \mu(z-1) + \alpha z)(\lambda(1-X(z)) + \beta) - \beta z \alpha}{(\lambda(1-X(z)) + \beta)} \right) P(z) = \eta z \frac{\mu_v Q_0 (z - z_1)}{z_1 (\lambda z(1-X(z)) + \mu_v(z-1) + \eta z)} - z \frac{\mu_v(1-z_1)Q_0}{z_1}$$

Therefore,

$$P(z) = \frac{\lambda z \mu_v Q_0}{z_1} \left(\frac{(z-1)z_1(X(z_1)-1) - (z_1-1)z(X(z)-1)}{\lambda z(1-X(z)) + \mu_v(z-1) + \eta z} \right) \left(\frac{\lambda(1-X(z)) + \beta}{[\lambda z(1-X(z)) + \mu(z-1) + \alpha z][\lambda(1-X(z))] + \beta[\lambda z(1-X(z)) + \mu(z-1)]} \right) \quad (14)$$

Thus the total PGF $P_{MWW}^{Br}(z)$ is obtained by adding Q(z), P(z) and B(z)

(i.e.), $P_{MWW}^{Br}(z) = Q(z) + P(z) + B(z)$

$$P_{MWW}^{Br}(z) = \frac{\mu_v(z-1)Q_0}{z_1(\lambda z(1-X(z)) + \mu(z-1))} \left(\frac{\mu(z-z_1) + \lambda z z_1(X(z_1) - X(z))}{\lambda z(1-X(z)) + \mu_v(z-1) + \eta z} \right) \left(\frac{\lambda(1-X(z)) + \beta + \frac{\alpha \lambda z z_1(X(z_1) - X(z))}{\mu(z-z_1) + \lambda z z_1(X(z_1) - X(z))}}{\lambda(1-X(z)) + \beta + \frac{\alpha \lambda z(1-X(z))}{\mu(z-1) + \lambda z(1-X(z))}} \right) \quad (15)$$

Now by using the normalizing condition $P_{MWW}^{Br}(1) = 1$, which yields

$$1 = \left(\frac{\mu_v Q_0}{z_1(-\alpha \lambda E(x) + \beta \mu(1-\rho))} \right) \left\{ \frac{[\mu \beta(1-z_1) + \lambda z_1(X(z_1)-1)(\alpha + \beta)]}{\eta} \right\}$$

With the help of the normalizing condition, Q_0 can be evaluated

$$\text{i.e., } Q_0 = \frac{\eta z_1 [\beta \mu(1-\rho) - \alpha \lambda E(X)]}{\mu_v [(1-z_1)\beta \mu + \lambda z_1(X(z_1)-1)(\alpha + \beta)]} \quad \text{where } \rho = \frac{\lambda}{\mu} E(X) \quad (16)$$

By substituting for Q_0 in equation (15), the PGF is given by

$$P_{MWW}^{Br}(z) = \frac{\mu(1-\rho)(z-1)}{(\lambda z(1-X(z)) + \mu(z-1))} \left(\frac{\mu(z-z_1) + \lambda z z_1(X(z_1) - X(z))}{\lambda z(1-X(z)) + \mu_v(z-1) + \eta z} \right) \left(\frac{\lambda z_1(X(z_1)-1) + \mu(1-z_1)}{\eta} \right) \left(\frac{1 - \frac{\alpha \rho}{\beta(1-\rho)}}{1 + \frac{\alpha \lambda z_1(X(z_1)-1)}{\beta[\lambda z_1(X(z_1)-1) + \mu(1-z_1)]}} \right) \left(\frac{\lambda(1-X(z)) + \beta + \frac{\alpha \lambda z z_1(X(z_1) - X(z))}{\mu(z-z_1) + \lambda z z_1(X(z_1) - X(z))}}{\lambda(1-X(z)) + \beta + \frac{\alpha \lambda z(1-X(z))}{\mu(z-1) + \lambda z(1-X(z))}} \right) \quad (17)$$

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Thus the total PGF of $P_{MWV}^{Br}(z)$ is obtained.

IV. PERFORMANCE MEASURES

Let the steady state system size probabilities P_v , P_{busy} and P_{br} denote the probability that the server is in vacation, busy period and breakdown state respectively and let L_v denote the expected system size when the server is in vacation. Then let L denote the expected system size of $M^X/M/1/MWV/BD$ queuing system.

(i) The probability that the server is on vacation P_v is given by

$$\text{i.e., } P_v = \lim_{z \rightarrow 1} Q(z) = \frac{\mu_v(1-z_1)Q_0}{\eta z_1} = \frac{(1-z_1)[\beta\mu(1-\rho) - \alpha\lambda E(X)]}{\lambda z_1(X(z_1)-1)(\alpha + \beta) + \beta\mu(1-z_1)} \quad (18)$$

(ii) The probability that the server is on regular busy period (P_{busy}) is given by

$$\text{i.e., } P_{busy} = \lim_{z \rightarrow 1} P(z) = \frac{\lambda\mu_v\beta}{z_1\eta} \left(\frac{z_1(X(z_1)-1) + (1-z_1)E(X)}{\beta\mu(1-\rho) - \alpha\lambda E(X)} \right) Q_0 = \frac{\lambda\beta[z_1(X(z_1)-1) + (1-z_1)E(X)]}{\lambda z_1(X(z_1)-1)(\alpha + \beta) + \beta\mu(1-z_1)} \quad (19)$$

(iii) The probability that the server is in breakdown state (P_{br}) is given by

$$\text{i.e., } P_{br} = \lim_{z \rightarrow 1} B(z) = \frac{\lambda\mu_v\alpha}{z_1\eta} \left(\frac{z_1(X(z_1)-1) + (1-z_1)E(X)}{\beta\mu(1-\rho) - \alpha\lambda E(X)} \right) Q_0 = \frac{\lambda\alpha[z_1(X(z_1)-1) + (1-z_1)E(X)]}{\lambda z_1(X(z_1)-1)(\alpha + \beta) + \beta\mu(1-z_1)} \quad (20)$$

(iv) The expected system size when the server is on vacation,

$$\text{(i.e.), } L_v = \frac{d}{dz} (Q(z))_{z=1} = \frac{\lambda\mu_v Q_0}{\eta^2 z_1} [E(X)(1-z_1) + (z_1(X(z_1)-1))] \quad (21)$$

(v) Thus the expected system size of the model is given by

$$\text{i.e., } L = \frac{d}{dz} (P_{MWV}^{Br}(z))_{z=1} = \frac{\left(1 - \frac{\alpha\rho}{\beta(1-\rho)}\right)}{\left(1 + \frac{\alpha\lambda z_1(X(z_1)-1)}{\beta[\lambda z_1(X(z_1)-1) + \mu(1-z_1)]}\right)} \left\{ \frac{\rho\mu}{\beta C^2} \left[C^2 + \alpha \left(z_1 - \frac{D}{\lambda E(X)} \right) (D-C) + \alpha D \left(\frac{C}{\beta} - \frac{1}{\rho} \right) - 1 \right] + \left(1 + \frac{\alpha D}{\beta C} \right) \left[\frac{\lambda(E(X) + E(X^2))}{2\mu(1-\rho)} + \frac{\lambda E(X) - \mu_v}{\eta} + \frac{z_1(\mu - \lambda E(X))}{C} \right] \right\} \quad (22)$$

where $C = \mu(1-z_1) + \lambda z_1(X(z_1)-1)$ and $D = \lambda z_1(X(z_1)-1)$

Thus the expected system size of the model is evaluated.

V. CONCLUSION

Thus in this paper we have analyzed the batch arrival $M^X/M/1$ queuing system under server breakdowns and multiple working vacations and evaluated the total probability generating function, the expected system size when the server is in vacation, the expected system size of the model and also evaluated various performance measures.

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