Operating Characteristics of M^X/M/1/MWV/BD Queuing Systems

J. Maria Remona 1, S. Vinnarasi 2, Dr. K. Julia Rose Mary 3
M.Sc Student, Department of Mathematics, Nirmala College for Women, Coimbatore, India 1
M.Sc Student, Department of Mathematics, Nirmala College for Women, Coimbatore, India 2
Associate Professor, Department of Mathematics, Nirmala College for Women, Coimbatore, India 3

ABSTRACT: In this paper the batch arrival M^X/M/1 queuing system along with server breakdowns and multiple working vacations is analyzed under exponential distribution. For this model the total probability generating function is obtained through the Chapman-Kolmogrov balance equations satisfied by the steady state system size probabilities. The probability generating function is presented in closed form so that various performance measures are analyzed. Further the expected system size when the server is in vacation(L_v) and the expected system size of the model (L) are also evaluated.

KEYWORDS: Batch Arrival, Multiple Working Vacations, Breakdowns, Total Probability Generating Function, Performance Measures.

I. INTRODUCTION

A classical queuing system may be described as one having a service facility at which units of some customer arrive for service and, whenever there are more units in the system than the service facility can handle simultaneously, a queue or waiting line is developed. The waiting units take their turn for service according to a pre assigned rule and after service they leave the system. The study of classical queuing models are made by Saaty [11], Gross & Harris [4] and Medhi [10].


In N-policy the server does not start his service until there are N-customers in the queue. This policy is introduced by Yadin and Naor [16] and is designed to minimize server switch over’s and to avoid excessive frequent use of setups. Lee et al., [6] successively combined the batch arrival queues with N-policy.

Queuing systems with server classical vacations are characterized by the fact that the idle time of the server may be used for other secondary jobs. Allowing server to take vacation make queuing models more realistic and flexible in studying real world queuing situations. Applications arise naturally in call centers with multi task employees, maintenance activity, production and quality control problems etc..

In N-policy queuing models, with server vacation, as soon as the system empties, the server leaves the system for a vacation of random length. When the server returns from the vacation and finds N or more customers, he immediately starts his service. Otherwise he takes repeated number of vacations until he finds N or more customers. This policy is called a Multiple Vacations Policy.

Most of the classical queuing systems the server may fail and can be repaired. The performance of the system may be affected heavily by these breakdowns and limited repair capacity. Queuing systems with such unreliable stations are the topics of worth investigating from the performance prediction point of view. As a result of breakdowns, service facility becomes inoperative and the units demanding service can be served only when it is restored to operative state. Wang and Huang [13] first proposed Markovian queuing system with removable service station. Ke [5]...
considered the control policy for batch arrival $M^X/M/1$ queuing system under N-policy in which the server is characterized by breakdowns and multiple vacations.

In working vacation queues, the server works at a lower service rate rather than completely stopping service during the vacation period. At the vacation termination epochs, if there are customers in the system, the server will start a new regular busy period. Otherwise, he takes another working vacation which follows multiple working vacations policy.

Servi and Finn [12], introduced a class of semi vacation policies, in which servers work at a lower rate rather than completely stopping primary service during vacation. Such a vacation is called working vacation (WV). Li and Tian [7], Xu et al., [14] considered M/M/1 queue with different working vacation policies. Xu et al., [15] studied the results of Liu et al., [8] to bulk input model $M^X/M/1/MWV$. They have formulated the model as two dimensional Markovian chain and obtained the PGF of the stationary queue length and its stochastic decomposition result using the matrix analysis method. Their concept is motivated to combine the batch arrival queues under server breakdowns and multiple working vacations.

In this paper, with the help of the available literature a batch arrival $M^X/M/1$ queuing system along with server breakdowns and multiple working vacations is analyzed under exponential distribution. The probability generating function (PGF) of the system size is obtained through the Chapman-Kolmogorov balanced equations satisfied by the steady state system size probabilities. The PGF is presented in closed form so that various performance measures are calculated easily.

II. MODEL DESCRIPTION

Consider a batch arrival $M^X/M/1$ queue in which, the arrival stream forms a poisson process and the actual number of customers in any arriving module is a random variable $X$, which may take on any positive integral value $k(\leq +\infty)$ with probability $g_k$. If $\lambda_k$ is the arrival rate of a poisson process of batches of size $k$ then $g_k=\lambda_k/\lambda$, $k=1,2,3,…$ where $\lambda$ is the composite arrival rate of all batches equal to $\sum_{k=1}^{+\infty} \lambda_k$. This total process, which arises from the overlap of the set of Poisson processes with rates{$\lambda_k$, $k=1,2,…$} is a compound poisson process. Let $X(z), E(X)$ and $E(X^2)$ denote the PGF, first and second moments of random variable $X$.

Then $X(z) = \sum_{k=1}^{+\infty} g_k z^k$; $E(X) = \sum_{k=1}^{+\infty} g_k k$.

The server serves the customers at a time with exponential service rate $\mu$ in a regular busy period. Whenever the system becomes empty at service completion instant, the server starts a working vacation during which the service is done at a lower rate. The vacation duration $V$ follows an exponential distribution with parameter $\eta$. During working vacations, arriving customers are served with exponential service rate $\mu_v(\leq \mu)$.

When a vacation terminates and the server finds the system is empty, then the server begins another working vacation. On the other hand, if the server finds the system is not empty at the vacation termination instant, then he switches to a regular service period. The distributions of the service times during regular busy period and working vacation period are both exponential but with different rates $\mu$ and $\mu_v$ respectively. It is assumed that, inter arrival times, service time and working vacation times are mutually independent of each other.

The server is subject to breakdowns at any time while working, with poisson rate $\alpha$. Whenever the system fails, the server is sent immediately for repair at a repair facility where the repair time is an independent and identically distributed random variable $B_r$ following exponential distribution $(1-e^{-B_r})$. The customer, who is just the being served when the server breaks down, joins the head of the waiting line and resumes the service as soon as the server returns from the repair facility. This type of service continues until the system becomes empty again.
This model is denoted by $M^y/M/1/MWV/BD$.

### III. SYSTEM SIZE DISTRIBUTION

Let $N_s(t)$ denote the number of customers in the system at time $t$ and $J(t)$ denote the system state at time $t$.

1. The system is in a working vacation period at time $t$.
2. The system is in regular busy period at time $t$.
3. The system is in break down period at time $t$.

Then $\{N_s(t), J(t)\}$ is a Markov process.

Let $Q_n(t) = \Pr\{N_s(t) = n ; J(t) = 0\}, \quad n \geq 0,$ $P_n(t) = \Pr\{N_s(t) = n ; J(t) = 1\}, \quad n \geq 1,$ and $B_n(t) = \Pr\{N_s(t) = n ; J(t) = 2\}, \quad n \geq 1.$

denote the system size probability at time $t$.

Assuming the steady state system size probabilities as

$$\lim_{t \to \infty} P_n(t) = \Pi_n = \mu V_1 + \mu P_1 \quad \forall n \geq 0,$$

$$\lim_{t \to \infty} Q_n(t) = \eta Q_{n+1} + \mu V Q_{n+1} \quad \forall n \geq 0,$$

$$\lim_{t \to \infty} P_n(t) = \mu P_{n+1} + \Theta Q_n + \mu Q_{n+1} \quad \forall n \geq 0,$$

$$\lim_{t \to \infty} B_n(t) = \alpha P_n + \lambda \sum_{k=0}^{n-1} B_{n-k} \quad \forall n \geq 0,$$

$$\lim_{t \to \infty} B_n(t) = \alpha P_n + \lambda \sum_{k=0}^{n-1} B_{n-k} \quad \forall n \geq 0.$$

The steady state equations satisfied by $P_n$’s, $Q_n$’s and $B_n$’s are given by

$$\lambda Q_0 = \mu V_1 + \mu P_1 \quad (1)$$

$$\lambda + \eta + \mu V \sum_{k=0}^{n} Q_{n-k} S_k = \mu V \sum_{k=0}^{n} Q_{n-k+1} \quad (2)$$

$$\lambda + \mu + \alpha \quad P_1 = \mu P_2 + \eta Q_1 + \beta B_1 \quad (3)$$

$$\lambda + \mu + \alpha \quad P_n = \beta B_n + \lambda \sum_{k=0}^{n} P_{n-k} S_k + \mu P_{n+1} + \eta Q_n \quad \forall n \geq 2$$

$$\lambda + \beta \quad B_1 = \alpha P_1 \quad (5)$$

$$\lambda + \beta \quad B_n = \alpha P_n + \lambda \sum_{k=0}^{n-1} B_{n-k} S_k \quad \forall n \geq 2 \quad (6)$$

To obtain the steady state distribution of the model, the partial PGFs, are defined as

$$Q(z) = \sum_{n=0}^{\infty} Q_n z^n; \quad P(z) = \sum_{n=1}^{\infty} P_n z^n \quad \text{and} \quad B(z) = \sum_{n=1}^{\infty} B_n z^n.$$

By multiplying equation (2) by $z^n$, summing over $n \geq 1$ and then adding with equation (1) we get

$$\sum_{n=0}^{\infty} (\lambda + \eta + \mu V) Q_n z^n + \lambda Q_0 = \lambda \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} Q_{n-k} z^n S_k + \sum_{n=1}^{\infty} \mu Q_n z^n + \mu Q_1 + \mu P_1.$$
since, \( \sum_{n=1}^{\infty} z^n \sum_{k=1}^{\infty} Q_{n-k} g_k = Q(z) X(z) \)

(i.e.,) \( (\lambda z (1 - X(z)) + \mu_V (z - 1) + \eta z) Q(z) = \mu_V (z - 1) Q_0 + Pz \)

where \( P = \eta Q_0 + \mu P_1 \).

The equation, \( h(z) = \lambda z (1 - X(z)) + \mu_V (z - 1) + \eta z \)

\[ = (\lambda + \eta + \mu_V) z - \lambda z X(z) - \mu_v \]

has a unique root \( z_i \), with \( |z_i| < 1 \), by Rouche’s theorem.

For \( f(z) = (\lambda + \eta + \mu_V) z \) and \( g(z) = -\lambda z X(z) - \mu_v \) satisfy the condition \(|g(z)| < |f(z)| \) on \(|z| = 1\) and hence the equations \( f(z) = 0 \) and \( f(z) + g(z) = h(z) = 0 \) have the same number of roots inside the circle \(|z| = 1\). Since \( f(z) \) is linear, the root is real and simple and \( h(0) = -\mu_v \) and \( h(1) = \eta \) are of opposite sign.

Thus by evaluating \( P \) at \( z = z_i \) from equation (7), we get

\[ P = \frac{\mu_v (1 - z_i)}{z_i} \]

The equation, \( h(z) = \lambda z (1 - X(z)) + \mu_V (z - 1) + \eta z \)

has a unique root \( z_i \), with \( |z_i| < 1 \), by Rouche’s theorem. We know that \( h(z)=0 \)

we get, \( \eta z_i = \lambda z_i (1 - X(z_i)) + \mu_V (z_i - 1) \)

(10)

With this, the partial PGF \( Q(z) \) is given by

\[ Q(z) = \frac{\mu_v (z - z_i) Q_0}{z_i (\lambda z (1 - X(z)) + \mu_v (z - 1) + \eta z)} \]

Similarly proceeding for equations (3) and (4) imply,

\[ \sum_{n=2}^{\infty} (\lambda + \mu + \alpha) P_n z^n + (\lambda + \mu + \alpha) P_1 = \sum_{n=1}^{\infty} \beta B_{n-1} z^n + \beta \sum_{k=1}^{\infty} P_{n-k} g_k z^n + \sum_{n=1}^{\infty} \eta Q_n z^n + \sum_{n=1}^{\infty} \mu P_{n-1} z^n + \sum_{n=1}^{\infty} \eta Q_{n-1} z^n + \sum_{n=1}^{\infty} \beta B_{n-1} z^n \]

since, \( \sum_{n=2}^{\infty} z^n \sum_{k=1}^{\infty} P_{n-k} g_k = P(z) X(z) \)

(i.e.,) \( \lambda z (1 - X(z)) + \mu (z - 1) + \alpha z \) \( P(z) = \eta z Q(z) + \beta B(z) - P z \)

(12)

Similarly equations (5) and (6) implies,

\[ \sum_{n=2}^{\infty} (\lambda + \beta) B_{n-1} z^n + (\lambda + \beta) B_{n-1} = \lambda \sum_{n=2}^{\infty} B_{n-1} g_k z^n + \sum_{n=2}^{\infty} \alpha P_{n-1} z^n + \alpha P_{n-1} \]

since, \( \sum_{n=2}^{\infty} z^n \sum_{k=1}^{\infty} B_{n-k-1} g_k = B(z) X(z) \)

thus \( B(z) = \frac{\alpha P(z)}{\lambda (1 - X(z)) + \beta} \)
By substituting the value of P from (9), Q(z) from (11), and B(z) from (13) in P(z) of (12) we have,

\[
P(z) = \eta z \left( \frac{\mu \cdot Q_0(z - z_i)}{\lambda z (1 - X(z)) + \mu(z - 1) + \eta z} \right) \left( \frac{\lambda z (1 - X(z)) + \beta}{\lambda z (1 - X(z)) + \beta \eta z} \right) - z \frac{\mu \cdot (1 - z_i) Q_0}{z_i}
\]

Therefore,

\[
P(z) = \frac{\lambda z \mu \cdot Q_0}{z_i} \left( \frac{(z - 1)z_i(X(z_i) - 1) - (z_i - 1)z(X(z) - 1)}{\lambda z (1 - X(z)) + \mu(z - 1) + \eta z} \right) \left( \frac{\lambda z (1 - X(z)) + \beta}{\lambda z (1 - X(z)) + \beta \eta z} \right)
\]

Thus the total PGF \( P_{MWV}^B(z) \) is obtained by adding Q(z), P(z) and B(z)

(i.e.), \( P_{MWV}^B(z) = Q(z) + P(z) + B(z) \)

\[
P_{MWV}^B(z) = \frac{\mu \cdot (z - 1) Q_0}{z_i (\lambda z (1 - X(z)) + \mu(z - 1))} \left( \frac{\mu(z - z_i) + \lambda z z_i (X(z_i) - X(z))}{\lambda z (1 - X(z)) + \mu(z - 1) + \eta z} \right) \left( \frac{\lambda (1 - X(z)) + \beta}{\lambda (1 - X(z)) + \beta \eta z} \right)
\]

Now by using the normalizing condition \( P_{MWV}^B(1) = 1 \), which yields

\[
1 = \frac{\mu \cdot Q_0}{z_i (\lambda z (1 - X(z)) + \mu(z - 1))} \left( \frac{\mu \beta (1 - z_i) + \lambda z z_i (X(z_i) - 1)(\alpha + \beta)}{\eta} \right)
\]

With the help of the normalizing condition, \( Q_0 \) can be evaluated

(i.e., \( Q_0 = \frac{\eta z_i \left[ \beta \mu (1 - \rho) - \alpha \lambda E(X) \right]}{\mu \left[ (1 - z_i) \beta \mu + \lambda z_i (X(z_i) - 1)(\alpha + \beta) \right]} \)) where \( \rho = \frac{\lambda}{\mu} \) E(X)

By substituting for \( Q_0 \) in equation (15), the PGF is given by

\[
P_{MWV}^B(z) = \frac{\mu \cdot (z - 1) Q_0}{z_i (\lambda z (1 - X(z)) + \mu(z - 1))} \left( \frac{\mu(z - z_i) + \lambda z z_i (X(z_i) - X(z))}{\lambda z (1 - X(z)) + \mu(z - 1) + \eta z} \right) \left( \frac{\lambda (1 - X(z)) + \beta}{\lambda (1 - X(z)) + \beta \eta z} \right)
\]

(i.e.,

\[
P_{MWV}^B(z) = \frac{\mu \cdot (1 - z_i) Q_0}{z_i (\lambda z (1 - X(z)) + \mu(z - 1))} \left( \frac{\lambda (1 - X(z)) + \beta}{\lambda (1 - X(z)) + \beta \eta z} \right)
\]
Thus the total PGF of \( P^B_{MWV}(z) \) is obtained.

**IV. PERFORMANCE MEASURES**

Let the steady state system size probabilities \( P_v, P_{busy} \) and \( P_{sv} \) denote the probability that the server is in vacation, busy period and breakdown state respectively and let \( L_v \) denote the expected system size when the server is in vacation. Then let \( L \) denote the expected system size of \( M^V/M/1/MWV/BD \) queuing system.

(i) The probability that the server is on vacation \( P_v \), is given by
\[
i.e., P_v = \lim_{z \to 1} Q(z) = \frac{\mu(1-z)}{\eta z} = \frac{(1-z)[\beta\mu(1-\rho) - \alpha\lambda E(X)]}{\lambda z_i(X(z_i) - 1)(\alpha + \beta) + \beta\mu(1-z_i)} \quad (18)
\]

(ii) The probability that the server is on regular busy period \( (P_{busy}) \) is given by
\[
i.e., P_{busy} = \lim_{z \to 1} P(z) = \frac{\lambda \mu}{z \eta} \left[ \frac{\beta}{\beta\mu(1-\rho) - \alpha\lambda E(X)} \right] Q_0 = \frac{\lambda \mu [z_i(X(z_i) - 1) + (1-z_i)E(X)]}{\lambda z_i(X(z_i) - 1)(\alpha + \beta) + \beta\mu(1-z_i)} \quad (19)
\]

(iii) The probability that the server is in breakdown state \( (P_{sv}) \) is given by
\[
i.e., P_{sv} = \lim_{z \to 1} B(z) = \frac{\lambda \mu}{z \eta} \left( \frac{\alpha}{\beta\mu(1-\rho) - \alpha\lambda E(X)} \right) Q_0 = \frac{\lambda \mu [z_i(X(z_i) - 1) + (1-z_i)E(X)]}{\lambda z_i(X(z_i) - 1)(\alpha + \beta) + \beta\mu(1-z_i)} \quad (20)
\]

(iv) The expected system size when the server is on vacation,
\[
\text{(i.e.), } L_v = \frac{d}{dz}(Q(z))_{z=1} = \frac{\lambda \mu Q_0}{\eta^2 z_i} [E(X)(1-z_i) + (z_i(X(z_i) - 1)]
\]

(v) Thus the expected system size of the model is given by
\[
i.e., L = \frac{d}{dz}(P^B_{MWV}(z))_{z=1} = \left[ 1 - \frac{a\rho}{\beta(1-\rho)} \right] \\
\left[ 1 + \frac{\alpha z_i(X(z_i) - 1)}{\beta z_i(X(z_i) - 1) + \mu(1-z_i)} \right] \\
\left\{ \frac{\rho \mu}{\beta C^2} \left[ C - a \left( \frac{D}{E(X)} \right)(D-C) + aD \left( \frac{C}{\beta} - \frac{1}{\rho} \right) - 1 \right] + \frac{1 + aD}{\beta C} \left[ \frac{\lambda E(X) + E(X^2)}{2\mu(1-\rho)} + \frac{\lambda E(X) - \mu C}{\eta} + \frac{z_i(\mu - \lambda E(X))}{C} \right] \right\}
\]

where \( C = \mu(1-z_i) + \lambda z_i(X(z_i) - 1) \) and \( D = \lambda z_i(X(z_i) - 1) \)

Thus the expected system size of the model is evaluated.

**V. CONCLUSION**

Thus in this paper we have analyzed the batch arrival \( M^V/M/1 \) queuing system under server breakdowns and multiple working vacations and evaluated the total probability generating function, the expected system size when the server is in vacation, the expected system size of the model and also evaluated various performance measures.
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