

A Method for Solving Fuzzy Differential Equations using fourth order Runge-kutta Embedded Heronian Means

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ABSTRACT: In this paper a study has been carried out to find a numerical solution for the first order fuzzy differential equation by using fourth order Runge-kutta method based on embedded heronian mean. The accuracy and applicability of the proposed method is illustrated by solving a fuzzy initial value problem with a trapezoidal fuzzy number.

KEYWORDS: Fuzzy Differential Equations, Runge-kutta fourth Order method, Embedded Heronian mean, Trapezoidal Fuzzy Number

I.INTRODUCTION

In recent years fuzzy differential equations leads a natural way to model dynamical systems under uncertainty. First order linear fuzzy differential equations are one of the simplest fuzzy differential equation, which appears in many applications. The concept of fuzzy derivative was first introduced by S.L.Chang and L.A.Zadeh in [6]. D.Dubois and Prade [7] discussed differentiation with fuzzy features. M.L.puri, D.A.Ralescu [23] and R.Goetschel, W.Voxman [10] mainly contributed towards the differential of fuzzy functions. The fuzzy differential equation and initial value problems were extensively studied by O.Kaleva [15,16] and by S.Seikkala [24]. Recently many research papers are focused on numerical solution of fuzzy initial value problems (FIVPS). Numerical Solution of fuzzy differential equation has been introduced by M.Ma, M.Friedman, A.Kandel [18] through Euler method and S.Abbasbandy and T.Allahviranloo [1] through Taylor's method. Runge -Kutta methods have also been studied by authors [2,21]. V.Nirmala, N.Saveetha, S.Chenthurpandiyam discussed numerical Solution of fuzzy differential equation by Runge-Kutta method with higher order derivative approximations [20]. R.Gethsi sharmila & E.C.Henry Amirtharaj investigate numerical solutions of first order fuzzy initial value problems by non-linear Trapezoidal formulae based on variety of Means [13]. Fourth order Runge-kutta embedded heronian mean was studied by R.Ponalagusamy and S.Senthilkumar [22].

Following the introduction in section 1, this paper is organised as follows: In section 2, some basic results of fuzzy numbers and definitions of fuzzy derivative are given. In section 3, the fuzzy initial value problem is discussed. Section 4 describes the structure of fourth order Runge-kutta embedded heronian mean method. In section 5, the proposed method is applied for solving fuzzy differential equation problems and the numerical examples are given to illustrate the applicability of the proposed method. The conclusion was given in the last section.

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II. PRELIMINARIES

2.1 FUZZY NUMBER

An arbitrary fuzzy number is represented by an ordered pair of functions $(\underline{u}(r), \bar{u}(r))$ for all $r \in [0, 1]$ which satisfy the following conditions.

i) $\underline{u}(r)$ is a bounded left continuous non-decreasing function over $[0, 1]$ with respect to any r .

ii) $\bar{u}(r)$ is a bounded right continuous non-decreasing function over $[0, 1]$ with respect to any r .

iii) $(\underline{u}(r) \leq \bar{u}(r))$ for all $r \in [0, 1]$ then the r -level set is $[u]_r = \{x \mid u(x) \geq r\}; 0 \leq r \leq 1$

Clearly, $[u]_0 = \{x \mid u(x) \geq 0\}$ is compact, which is a closed bounded interval and we denote by $[u]_r = (\underline{u}(r), \bar{u}(r))$.

2.2 TRAPEZOIDAL FUZZY NUMBER:

A trapezoidal fuzzy number u is defined by four real numbers $k < l < m < n$, where the base of the trapezoidal is the interval $[k, n]$ and its vertices at $x = l, x = m$. Trapezoidal fuzzy number will be written as $u = (k, l, m, n)$. The membership function for the trapezoidal fuzzy number $u = (k, l, m, n)$ is defined as the following :

$$u(x) = \begin{cases} \frac{x-k}{l-k} & k \leq x \leq l \\ 1 & l \leq x \leq m \\ \frac{x-n}{m-n} & m \leq x \leq n \end{cases}$$

we have :

- (1) $u > 0$ if $k > 0$;
- (2) $u > 0$ if $l > 0$;
- (3) $u > 0$ if $m > 0$; and
- (4) $u > 0$ if $n > 0$;

Let us denote R_F by the class of all fuzzy subsets of R (i.e. $u : R \rightarrow [0, 1]$) satisfying the following properties:

- (i) $\forall u \in R_F$, u is normal, i.e. $\exists x_0 \in R$ with $u(x_0) = 1$;
- (ii) $\forall u \in R_F$, u is convex fuzzy set (i.e. $u(tx + (1-t)y) \geq \min\{u(x), u(y)\}$, $\forall t \in [0, 1], x, y \in R$);
- (iii) $\forall u \in R_F$, u is upper semi continuous on R ;
- (iv) $\{x \in R; u(x) > 0\}$ is compact, where A denotes the closure of A .

2.3: Definition: (α - Level Set)

Let I be the real interval. A mapping $y : I \rightarrow E$ is called a fuzzy process and its α - level Set is denoted by $[y(t)]_\alpha = [\underline{y}(t; \alpha), \bar{y}(t; \alpha)]$, $t \in I, 0 < \alpha < 1$

2.4: Definition: (Seikkala Derivative)

The Seikkala derivative $y'(t)$ of a fuzzy process is defined by $[y'(t)]_\alpha = [\underline{y}'(t; \alpha), \bar{y}'(t; \alpha)]$ $t \in I, 0 < \alpha \leq 1$ provided that this equation defines a fuzzy number, as in [24]

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2.5:Lemma:

If the sequence of non-negative number $\{W_n\}_{n=0}^m$ satisfy $|W_{n+1}| \leq A|W_n| + B, 0 \leq n \leq N-1$ for the given positive constants A and B , then $|W_n| \leq A^n |W_0| + B \frac{A^n - 1}{A - 1}, 0 \leq n \leq N$

2.6:Lemma:

If the sequence of non-negative numbers $\{W_n\}_{n=0}^m, \{V_n\}_{n=0}^N$ satisfy $|W_{n+1}| \leq |W_n| + A \max\{|W_n|, |V_n|\} + B, |V_{n+1}| \leq |V_n| + A \max\{|W_n|, |V_n|\} + B$ for the given positive constants A and B , then $U_n = |W_n| + |V_n|, 0 \leq n \leq N$

we have, $U_n \leq \bar{A}^n U_0 + B \frac{\bar{A}^n - 1}{\bar{A} - 1}, 0 \leq n \leq N$ where $\bar{A} = 1 + 2A$ and $\bar{B} = 2B$.

2.7:Lemma

Let $F(t, u, v)$ and $G(t, u, v)$ belong to $C^1(R_F)$ and the partial derivatives of F and G be bounded over R_F . then for arbitrarily fixed $r, 0 \leq r \leq 1, D(y(t_{n+1}), y^0(t_{n+1})) \leq h^2 L(1 + 2C)$ where L is a bound of partial derivatives of F and G , and $C = \text{Max} \left\{ \left| G \left[t_N, \underline{y}(t_N; r), \bar{y}(t_{N-1}; r) \right] \right|, r \in [0, 1] \right\} < \infty$

2.8:Theorem

Let $F(t, u, v)$ and $G(t, u, v)$ belong to $C^1(R_F)$ and the partial derivatives of F and G be bounded over R_F then for arbitrarily fixed $r, 0 \leq r \leq 1$, the numerical solutions of $\underline{y}(t_{n+1}; r)$ and $\bar{y}(t_{n+1}; r)$ converge to the exact solutions $\underline{Y}(t_{n+1}; r)$ and $\bar{Y}(t_{n+1}; r)$ uniformly in t .

2.9:Theorem

Let $F(t, u, v)$ and $G(t, u, v)$ belong to $C^1(R_F)$ and the partial derivatives of F and G be bounded over R_F and $2Lh < 1$. Then for arbitrarily fixed $0 \leq r \leq 1$, the iterative numerical solutions of $\underline{y}^{(j)}(t_n; r)$ and $\bar{y}^{(j)}(t_n; r)$ converge to the numerical solutions $\underline{y}(t_n; r)$ and $\bar{y}(t_n; r)$ in $t_0 \leq t_n \leq t_N$, when $j \rightarrow \infty$.

III. FUZZY INITIAL VALUE PROBLEM

Consider a first-order fuzzy initial value differential equation is given by

$$\begin{cases} y'(t) = f(t, y(t)), t \in [t_0, T] \\ y(t_0) = y_0 \end{cases} \tag{3.1}$$

where 'y' is a fuzzy function of 't', $f(t, y)$ is a fuzzy function of the crisp variable 't' and the fuzzy variable y, y' is the fuzzy derivative of 'y' and $y(t_0) = y_0$ is a trapezoidal (or) a trapezoidal shaped fuzzy number.

We denote the fuzzy function 'y' by $y = [\underline{y}, \bar{y}]$. It means that the r-level set of $y(t)$ for $t \in [t_0, T]$ is

$$[y(t)]_r = [\underline{y}(t; r), \bar{y}(t; r)], [y(t_0)]_r = [\underline{y}(t_0; r), \bar{y}(t_0; r)], r \in (0, 1],$$

we write $f(t, y) = [\underline{f}(t, y), \bar{f}(t, y)]$ and

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$$\underline{f}(t, y) = F[t, \underline{y}, \bar{y}], \quad \bar{f}(t, y) = G[t, \underline{y}, \bar{y}],$$

because of $y' = f(t, y)$ we have

$$\underline{f}(t, y(t); r) = F[t, \underline{y}(t; r), \bar{y}(t; r)] \tag{3.2}$$

$$\bar{f}(t, y(t); r) = G[t, \underline{y}(t; r), \bar{y}(t; r)] \tag{3.3}$$

by using the extension principle, we have the membership function

$$f(t, y(t))(s) = \sup\{y(t)(\tau) \mid s = f(t, \tau)\}, \quad s \in R \tag{3.4}$$

so the fuzzy number $f(t, y(t))$ follows that

$$[f(t, y(t))]_r = [\underline{f}(t, y(t); r), \bar{f}(t, y(t); r)], \quad r \in (0, 1) \tag{3.5}$$

where $\underline{f}(t, y(t); r) = \min\{f(t, u) \mid u \in [y(t)]_r\}$ (3.6)

$$\bar{f}(t, y(t); r) = \max\{f(t, u) \mid u \in [y(t)]_r\} \tag{3.7}$$

Definition 3.1 A function $f : R \rightarrow R_f$ is said to be fuzzy continuous function, if for an arbitrary fixed $t_0 \in R$ and $\varepsilon > 0, \delta > 0$ such that $|t - t_0| < \delta \Rightarrow D[f(t), f(t_0)] < \varepsilon$ exists.

The fuzzy function considered are continuous in metric D and the continuity of $f(t, y(t); r)$ guarantees the existence of the definition of $f(t, y(t); r)$ for $t \in [t_0, T]$ and $r \in [0, 1]$ [10]. Therefore, the functions G and F can be definite too.

IV. FOURTH ORDER RUNGE-KUTTA METHOD WITH EMBEDDED HERONIAN MEAN

Fourth order Runge-kutta method with embedded heronian mean is considered for approximating the solution of first order fuzzy initial value problem $y'(t) = f(t, y(t)) ; y(t_0) = y_0$. The basis of all Runge-Kutta methods is to express the difference between the value of 'y' at t_{n+1} and t_n as

$$y_{n+1} - y_n = \sum_{i=0}^m w_i k_i \tag{4.1}$$

where w_i 's are constant for all 'i' and $k_i = hf(t_n + a_i h, y_n + \sum_{j=1}^{i-1} c_{ij} k_j)$ (4.2)

Increasing of the order of accuracy, the Runge-Kutta methods have been accomplished by increasing the number of Taylor's series terms used and thus the number of functional evaluations required[5].The method proposed by Goeken.D and Johnson.O[9] introduces new terms involving higher order derivatives of 'f' in the Runge-Kutta k_i terms($i > 0$) to obtain a higher order of accuracy without a corresponding increase in evaluations of 'f', but with the addition of evaluations of f' .

The Fourth order Runge-Kutta method with embedded heronian mean for step n+1 is given by R.Ponalagusamy and S.Senthilkumar [22].

Consider $y(t_{n+1}) = y(t_n) + \frac{h}{9} [k_1 + 2(k_2 + k_3) + k_4 + \sqrt{k_1 + k_2} + \sqrt{k_2 + k_3} + \sqrt{k_3 + k_4}]$ (4.3)

Where $k_1 = hf(t_n, y(t_n))$ (4.4)

$$k_2 = hf(t_n + a_1 h, y(t_n) + a_1 k_1) \tag{4.5}$$

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$$k_3 = hf(t_n + (a_2 + a_3)h, y(t_n) - a_2k_1 + a_3k_2) \tag{4.6}$$

$$k_4 = hf(t_n + (a_4 + a_5 + a_6)h, y(t_n) - a_4k_1 + a_5k_2 + a_6k_3) \tag{4.7}$$

and the parameters $a_1, a_2, a_3, a_4, a_5, a_6$ are chosen to make y_{n+1} closer to $y(t_{n+1})$. The value of parameters are

$$a_1 = \frac{1}{2}, \quad a_2 = \frac{-1}{48}, \quad a_3 = \frac{25}{48}, \quad a_4 = \frac{-1}{24}, \quad a_5 = \frac{47}{600}, \quad a_6 = \frac{289}{300}$$

V. FOURTH ORDER RUNGE-KUTTA EMBEDDED HERONIAN MEAN FOR SOLVING FUZZY DIFFERENTIAL EQUATIONS

Let the exact solution $[Y(t)]_r = [\underline{Y}(t; r), \bar{Y}(t; r)]$, is approximated by some $[y(t)]_r = [\underline{y}(t; r), \bar{y}(t; r)]$. The grid

points at which the solutions is calculated are $h = \frac{T-t_0}{N}$, $t_i = t_0 + ih; 0 \leq i \leq N$

From 4.3 to 4.7 we define

$$\underline{y}(t_{n+1}, r) - \underline{y}(t_n, r) = \frac{h}{9} \left[\begin{aligned} & \underline{k}_1(t_n, y(t_n, r)) + 2(\underline{k}_2(t_n, y(t_n, r)) + \underline{k}_3(t_n, y(t_n, r))) \\ & + \underline{k}_4(t_n, y(t_n, r)) + \sqrt{\underline{k}_1(t_n, y(t_n, r)) + \underline{k}_2(t_n, y(t_n, r))} \\ & + \sqrt{\underline{k}_2(t_n, y(t_n, r)) + \underline{k}_3(t_n, y(t_n, r))} + \sqrt{\underline{k}_3(t_n, y(t_n, r)) + \underline{k}_4(t_n, y(t_n, r))} \end{aligned} \right] \tag{5.1}$$

where

$$k_1 = hF[t_n, \underline{y}(t_n, r), \bar{y}(t_n, r)] \tag{5.2}$$

$$k_2 = hF[t_n + \frac{h}{2}, \underline{y}(t_n, r) + \frac{1}{2}k_1(t_n, y(t_n, r)), \bar{y}(t_n, r) + \frac{1}{2}\bar{k}_1(t_n, y(t_n, r))] \tag{5.3}$$

$$k_3 = hF[t_n + \frac{h}{2}, \underline{y}(t_n, r) - \frac{1}{48}k_1(t_n, y(t_n, r)) + \frac{25}{48}k_2(t_n, y(t_n, r)), \bar{y}(t_n, r) - \frac{1}{48}\bar{k}_1(t_n, y(t_n, r)) + \frac{25}{48}\bar{k}_2(t_n, y(t_n, r))] \tag{5.4}$$

$$k_4 = hF[t_n + h, \underline{y}(t_n, r) - \frac{1}{24}k_1(t_n, y(t_n, r)) + \frac{47}{600}k_2(t_n, y(t_n, r)) + \frac{289}{300}k_3(t_n, y(t_n, r)), \bar{y}(t_n, r) - \frac{1}{24}\bar{k}_1(t_n, y(t_n, r)) + \frac{47}{600}\bar{k}_2(t_n, y(t_n, r)) + \frac{289}{300}\bar{k}_3(t_n, y(t_n, r))] \tag{5.5}$$

and

$$\bar{y}(t_{n+1}, r) - \bar{y}(t_n, r) = \frac{h}{9} \left[\begin{aligned} & \bar{k}_1(t_n, y(t_n, r)) + 2(\bar{k}_2(t_n, y(t_n, r)) + \bar{k}_3(t_n, y(t_n, r))) \\ & + \bar{k}_4(t_n, y(t_n, r)) + \sqrt{\bar{k}_1(t_n, y(t_n, r)) + \bar{k}_2(t_n, y(t_n, r))} \\ & + \sqrt{\bar{k}_2(t_n, y(t_n, r)) + \bar{k}_3(t_n, y(t_n, r))} + \sqrt{\bar{k}_3(t_n, y(t_n, r)) + \bar{k}_4(t_n, y(t_n, r))} \end{aligned} \right]$$

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(5.6)

where $k_1 = hG[t_n, \underline{y}(t_n, r), \bar{y}(t_n, r)]$ (5.7)

$$k_2 = hG\left[t_n + \frac{h}{2}, \underline{y}(t_n, r) + \frac{1}{2}k_1(t_n, y(t_n, r)), \bar{y}(t_n, r) + \frac{1}{2}\bar{k}_1(t_n, y(t_n, r))\right]$$
 (5.8)

$$k_3 = hG\left[t_n + \frac{h}{2}, \underline{y}(t_n, r) - \frac{1}{48}k_1(t_n, y(t_n, r)) + \frac{25}{48}k_2(t_n, y(t_n, r)), \bar{y}(t_n, r) - \frac{1}{48}\bar{k}_1(t_n, y(t_n, r)) + \frac{25}{48}\bar{k}_2(t_n, y(t_n, r))\right]$$
 (5.9)

$$k_4 = hG\left[t_n + h, \underline{y}(t_n, r) - \frac{1}{24}k_1(t_n, y(t_n, r)) + \frac{47}{600}k_2(t_n, y(t_n, r)) + \frac{289}{300}k_3(t_n, y(t_n, r)), \bar{y}(t_n, r) - \frac{1}{24}\bar{k}_1(t_n, y(t_n, r)) + \frac{47}{600}\bar{k}_2(t_n, y(t_n, r)) + \frac{289}{300}\bar{k}_3(t_n, y(t_n, r))\right]$$
 (5.10)

We define $F[t_n, y(t_n, r)] = \frac{h}{9} \left[\frac{k_1(t_n, y(t_n, r)) + 2(k_2(t_n, y(t_n, r)) + k_3(t_n, y(t_n, r)))}{k_4(t_n, y(t_n, r)) + \sqrt{k_1(t_n, y(t_n, r)) + k_2(t_n, y(t_n, r))}} + \frac{\sqrt{k_2(t_n, y(t_n, r)) + k_3(t_n, y(t_n, r))} + \sqrt{k_3(t_n, y(t_n, r)) + k_4(t_n, y(t_n, r))}}{\sqrt{k_2(t_n, y(t_n, r)) + k_3(t_n, y(t_n, r))} + \sqrt{k_3(t_n, y(t_n, r)) + k_4(t_n, y(t_n, r))}} \right]$ (5.11)

$$G[t_n, y(t_n, r)] = \frac{h}{9} \left[\frac{\bar{k}_1(t_n, y(t_n, r)) + 2(\bar{k}_2(t_n, y(t_n, r)) + \bar{k}_3(t_n, y(t_n, r)))}{\bar{k}_4(t_n, y(t_n, r)) + \sqrt{\bar{k}_1(t_n, y(t_n, r)) + \bar{k}_2(t_n, y(t_n, r))}} + \frac{\sqrt{\bar{k}_2(t_n, y(t_n, r)) + \bar{k}_3(t_n, y(t_n, r))} + \sqrt{\bar{k}_3(t_n, y(t_n, r)) + \bar{k}_4(t_n, y(t_n, r))}}{\sqrt{\bar{k}_2(t_n, y(t_n, r)) + \bar{k}_3(t_n, y(t_n, r))} + \sqrt{\bar{k}_3(t_n, y(t_n, r)) + \bar{k}_4(t_n, y(t_n, r))}} \right]$$
 (5.12)

Therefore we have

$$\begin{aligned} \underline{Y}(t_{n+1}, r) &= \underline{Y}(t_n, r) + F[t_n, Y(t_n, r)] \\ \bar{Y}(t_{n+1}, r) &= \bar{Y}(t_n, r) + G[t_n, Y(t_n, r)] \end{aligned}$$
 (5.13)

And

$$\begin{aligned} \underline{y}(t_{n+1}, r) &= \underline{y}(t_n, r) + F[t_n, y(t_n, r)] \\ \bar{y}(t_{n+1}, r) &= \bar{y}(t_n, r) + G[t_n, y(t_n, r)] \end{aligned}$$
 (5.14)

Clearly $\underline{y}(t; r)$ and $\bar{y}(t; r)$ converge to $\underline{Y}(t; r)$ and $\bar{Y}(t; r)$ whenever $h \rightarrow 0$

VI. NUMERICAL EXAMPLE

Consider fuzzy initial value problem

$$\begin{cases} y'(t) = y(t), & t \geq 0 \\ y(0) = (0.8 + 0.125r, 1.1 - 0.1r) \end{cases}$$
 (6.1)

The exact solution is given by

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$$Y(t, r) = [(0.8 + 0.125r)e^t, (1.1 - 0.1r)e^t]$$

At t=1 we get

$$Y(1, r) = [(0.8 + 0.125r)e, (1.1 - 0.1r)e], 0 \leq r \leq 1$$

The values of exact and approximate solution with h= 0.1 is given in Table :1. The approximate solutions obtained by the proposed method is plotted in Fig :I and the error estimation of the exact and approximate solution is given in Fig II.

Table:1

r	Exact Solution t=1		Approximate Solution (h=0.1)		Error 1	Error 2
	$\underline{Y}(t, r)$	$\overline{Y}(t, r)$	$\underline{y}(t, r)$	$\overline{y}(t, r)$		
0.0	2.174625	2.990110	2.328389	3.037771	1.537639e-001	4.766061e-002
0.1	2.208604	2.962927	2.358353	3.014408	1.497493e-001	5.148036e-002
0.2	2.242583	2.935744	2.388276	2.991028	1.456936e-001	5.528322e-002
0.3	2.276561	2.908562	2.418159	2.967631	1.415977e-001	5.906895e-002
0.4	2.310540	2.881379	2.448002	2.944216	1.374626e-001	6.283730e-002
0.5	2.344518	2.854196	2.477807	2.920784	1.332892e-001	6.658804e-002
0.6	2.378497	2.827013	2.507575	2.897334	1.290783e-001	7.032090e-002
0.7	2.412475	2.799830	2.537306	2.873866	1.248306e-001	7.403563e-002
0.8	2.446454	2.772647	2.567001	2.850379	1.205470e-001	7.773196e-002
0.9	2.480432	2.745465	2.596660	2.826874	1.162281e-001	8.140963e-002
1.0	2.514411	2.718282	2.626286	2.803350	1.118748e-001	8.506835e-002

Fig-I

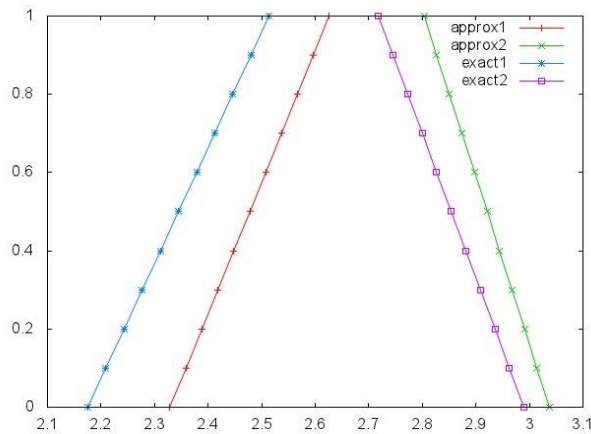
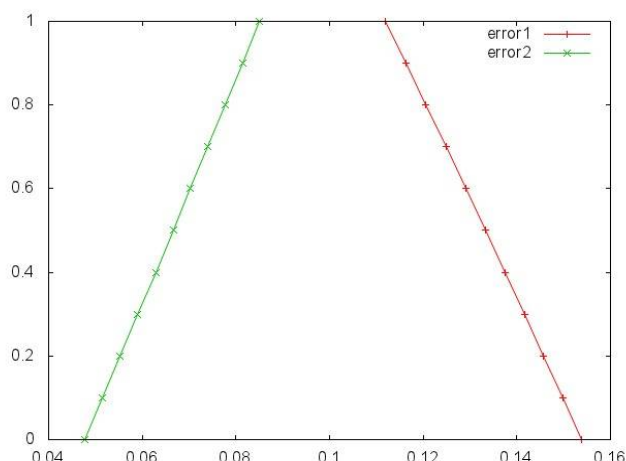


Fig II.

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VI. CONCLUSION

In this paper the fourth order runge-kutta method for embedded heronian mean has been applied for finding the numerical solution of first order fuzzy differential equation using trapezoidal fuzzy number. The accuracy and applicability of the proposed method have been illustrated by a suitable example. From the numerical example it has been observed that by minimizing the step size 'h' the exact solutions at different points of the solution curve and the corresponding approximate solution coincides.

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