

On Idempotent Polynomial Matrices

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ABSTRACT: Some properties of polynomial idempotent matrices are discussed and characterization of polynomial idempotent matrices are obtained.

KEYWORDS: Polynomial Matrix, Polynomial idempotent matrix.

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I. INTRODUCTION

In matrix theory, we come across many special types of matrices and one among them is idempotent matrix. Idempotent matrices are used in econometric analysis. Idempotent matrices play important role in general linear hypothesis theory. The importance of idempotent matrices is being increasingly recognized in the field of distribution of quadratic forms and tests of linear hypothesis [2], [3].

Idempotent and Nilpotent were used by Benjamin Peirce (1809-1880) in 1870 in American Journal of Mathematics (1881). A matrix 'A' that satisfies $A^2 = A$ is called an idempotent matrix and the matrix 'A' that satisfies $A^2 = 0$ is called a nilpotent matrix.

In this paper some properties of idempotent matrices are extended and proved to idempotent polynomial matrices.

II. PRELIMINARIES

In this section we give definitions and some elementary theorems without proof which will be useful in the rest of the paper.

Definition 2.1 [4]

A polynomial matrix is a matrix whose elements are polynomials.

Example 2.2

Let $A(\lambda) = \begin{pmatrix} 1 + \lambda + \lambda^2 & -1 - \lambda^2 \\ 2 + \lambda & 3 - \lambda \end{pmatrix} = A_0 + A_1\lambda + A_2\lambda^2$ be a polynomial matrix.

Definition 2.3[5]

The number of linearly independent rows (columns) of a polynomial matrix $A(\lambda) \in R^{m \times n}(\lambda)$ is called its normal rank.

Definition 2.4[3]

If A is a square matrix of size n . Then the trace of A , $tr(A)$, is the sum of the diagonal entries of A .

Symbolically, $tr(A) = \sum_{i=1}^n A_{ii}$.

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Theorem 2.5[1]

A^T is idempotent if and only if A is idempotent.

Theorem 2.6[1]

$I - A$ is idempotent if and only if A is idempotent.

Theorem 2.7 [1]

If A is a square polynomial matrix. Prove that if A is an idempotent matrix then determinant value of A is equal to either 0 or 1.

Theorem 2.8[1]

If A and B are idempotent matrices and $AB = BA$ then AB is an idempotent matrix.

Theorem 2.9[4]

If A is an idempotent polynomial matrix then $A^r = A$, where r is any positive integer.

Theorem 2.10[1]

Let A and B are idempotent matrices. If $AB = BA = 0$ then $A + B$ is an idempotent matrix.

Theorem 2.11

Any two of the following imply the other one. If $A \in \mathbb{C}^{n \times n}$ then

- (a) A is idempotent
- (b) A is Hermitian
- (c) A is square Hermitian

III. IDEMPOTENT POLYNOMIAL MATRICES

Definition 3.1

A square polynomial matrix $A(\lambda)$ is said to be an idempotent polynomial matrix if $[A(\lambda)]^2 = A(\lambda)$.

Example 3.2

$$A(\lambda) = \begin{pmatrix} 1 + \lambda & -1 - \lambda \\ \lambda & -\lambda \end{pmatrix}.$$

Theorem 3.3

If $A(\lambda)$ is an idempotent polynomial matrix if and only if $I - A(\lambda)$ is Idempotent.

Proof:

Assume that $A(\lambda)$ is idempotent $\Rightarrow [A(\lambda)]^2 = A(\lambda)$. Now show that $I - A(\lambda)$ is an idempotent.

$$\begin{aligned} [I - A(\lambda)][I - A(\lambda)] &= [I - A(\lambda)] \\ &= I - A(\lambda) - A(\lambda) + (A(\lambda))^2 \\ &= I - 2A(\lambda) + A(\lambda) \\ &= I - A(\lambda) \end{aligned}$$

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Hence $I - A(\lambda)$ is an idempotent.

Conversely, assume that $I - A(\lambda)$ is an idempotent. Then

$$\begin{aligned} [I - A(\lambda)]^2 &= I - A(\lambda) \\ \Rightarrow I - 2A(\lambda) + (A(\lambda))^2 &= I - A(\lambda) \\ \Rightarrow -A(\lambda) + (A(\lambda))^2 &= 0 \\ \Rightarrow (A(\lambda))^2 &= A(\lambda) \end{aligned}$$

Therefore $A(\lambda)$ is an idempotent.

Note:

$A(\lambda)$ is an Idempotent Polynomial Matrix then $A(\lambda) - I$ might not be Idempotent.

Example 3.4

Let $A(\lambda) = \begin{pmatrix} \lambda & 1-\lambda \\ \lambda & 1-\lambda \end{pmatrix}$ be an idempotent polynomial matrix.

$$\begin{aligned} I - A(\lambda) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 1-\lambda \\ \lambda & 1-\lambda \end{pmatrix} \\ &= \begin{pmatrix} 1-\lambda & -1+\lambda \\ -\lambda & \lambda \end{pmatrix} \\ [I - A(\lambda)]^2 &= \begin{pmatrix} 1-\lambda & -1+\lambda \\ -\lambda & \lambda \end{pmatrix} \begin{pmatrix} 1-\lambda & -1+\lambda \\ -\lambda & \lambda \end{pmatrix} \\ &= \begin{pmatrix} 1-2\lambda+\lambda^2+\lambda-\lambda^2 & -1+\lambda \\ -\lambda+\lambda-\lambda^2 & \lambda-\lambda^2+\lambda^2 \end{pmatrix} \end{aligned}$$

$I - A(\lambda)$ is an idempotent. Now

$$\begin{aligned} A(\lambda) - I &= \begin{pmatrix} \lambda & 1-\lambda \\ \lambda & 1-\lambda \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \lambda & 1-\lambda \\ \lambda & \lambda \end{pmatrix} \\ [A(\lambda) - I]^2 &= \begin{pmatrix} \lambda & 1-\lambda \\ \lambda & \lambda \end{pmatrix} \begin{pmatrix} \lambda & 1-\lambda \\ \lambda & \lambda \end{pmatrix} \\ &= \begin{pmatrix} \lambda & 2\lambda - 2\lambda^2 \\ 2\lambda^2 & \lambda \end{pmatrix} \end{aligned}$$

Here $[A(\lambda) - I]^2 \neq A(\lambda) - I$. Hence $A(\lambda) - I$ is not an idempotent matrix

Theorem 3.5

If $A(\lambda)$ is a square polynomial matrix. Prove that if $A(\lambda)$ is an idempotent then determinant value of $A(\lambda)$ is equal to either 0 or 1.

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Proof:

Let $A(\lambda)$ be an idempotent polynomial matrix $\Rightarrow [A(\lambda)]^2 = A(\lambda)$.

Taking determinant of both sides of this equation, we get

$$[A(\lambda)]^2 = \det A(\lambda)$$

$$\Rightarrow \det [A(\lambda)A(\lambda)] = \det A(\lambda)$$

$$\Rightarrow [\det [A(\lambda)]][\det [A(\lambda)]] = \det A(\lambda) \text{ by [3]}$$

$$\Rightarrow \det A(\lambda)[\det (A(\lambda)) - 1] = 0$$

$$\Rightarrow \det A(\lambda) = 0 \text{ (or) } \det A(\lambda) = 1$$

Theorem 3.6

The only non-singular idempotent matrix is the identity matrix.

Proof:

$$\begin{aligned} A(\lambda) &= I(\lambda)A(\lambda) = [[A(\lambda)]^{-1}A(\lambda)]A(\lambda) \quad [\because A(\lambda) \text{ is non-singular}] \\ &= [A(\lambda)]^{-1}[A(\lambda)A(\lambda)] \\ &= [A(\lambda)]^{-1}[A(\lambda)]^2 \\ &= [A(\lambda)]^{-1}[A(\lambda)] \\ &= I(\lambda) \end{aligned}$$

Example 3.7

Let $A(\lambda) = \begin{pmatrix} \lambda & 1-\lambda \\ \lambda & 1-\lambda \end{pmatrix}$ be an idempotent polynomial matrix.

$$|A(\lambda)| = \begin{vmatrix} \lambda & 1-\lambda \\ \lambda & 1-\lambda \end{vmatrix}$$

$$\Rightarrow \lambda(1-\lambda) - \lambda(1-\lambda) = 0$$

Example 3.8

Let $A(\lambda) = \begin{pmatrix} \lambda^0 & 0 \\ 0 & \lambda^0 \end{pmatrix} \Rightarrow |A(\lambda)| \neq 0$ be a non-singular idempotent matrix.

Theorem 3.7

If $A(\lambda)$ and $B(\lambda)$ are idempotent matrices and $A(\lambda)B(\lambda) = B(\lambda)A(\lambda)$ then $A(\lambda)B(\lambda)$ is an idempotent.

Proof:

Given $A(\lambda)$ and $B(\lambda)$ are idempotent matrices

$$\Rightarrow [A(\lambda)]^2 = A(\lambda) \text{ and } [B(\lambda)]^2 = B(\lambda). \text{ Consider}$$

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$$\begin{aligned}
 [A(\lambda)B(\lambda)]^2 &= [A(\lambda)B(\lambda)][A(\lambda)B(\lambda)] \\
 &= A(\lambda)[B(\lambda)A(\lambda)]B(\lambda) \\
 &= A(\lambda)[A(\lambda)B(\lambda)]B(\lambda) \\
 &= [A(\lambda)A(\lambda)][B(\lambda)B(\lambda)] \\
 &= [A(\lambda)]^2[B(\lambda)]^2 \\
 &= A(\lambda)B(\lambda)
 \end{aligned}$$

Theorem 3.9

If $A(\lambda)$ is an idempotent polynomial matrix then $[A(\lambda)]^r = A(\lambda)$, where r is any positive integer.

Proof:

Let $A(\lambda)$ be an idempotent polynomial matrix $\Rightarrow [A(\lambda)]^2 = A(\lambda)$. Now we have to prove

$[A(\lambda)]^r = A(\lambda)$, where r is any positive integer. To prove this is by induction.

For $r = 1$, $A(\lambda) = A(\lambda)$. Assume it is true for $< r$, $[A(\lambda)]^{r-1} = A(\lambda)$. Consider,

$$\begin{aligned}
 [A(\lambda)]^r &= [A(\lambda)]^{r-1} A(\lambda) \\
 &= A(\lambda)A(\lambda) \\
 &= [A(\lambda)]^2 \\
 &= A(\lambda)
 \end{aligned}$$

Hence it is valid for all r .

Example 3.10

Let $A(\lambda) = \begin{pmatrix} \lambda & -1 + \lambda \\ -\lambda & 1 - \lambda \end{pmatrix}$ be an idempotent polynomial matrix.

Then $[A(\lambda)]^2 = A(\lambda)$. And $[A(\lambda)]^3 = [A(\lambda)]^2 A(\lambda) = A(\lambda)A(\lambda) = [A(\lambda)]^2 = A(\lambda)$.

Similarly $[A(\lambda)]^r = [A(\lambda)]^2 [A(\lambda)]^{r-2} = \dots = A(\lambda)A(\lambda) = [A(\lambda)]^2 = A(\lambda)$. Hence $[A(\lambda)]^r = A(\lambda)$, where r is any positive integer.

Definition 3.11

A polynomial matrix satisfying $[A(\lambda)]^2 = 0$ is called a nilpotent and that for which $[A(\lambda)]^2 = I$ could be called a unipotent.

Example 3.12

$A(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$ is a nilpotent polynomial matrix and $A(\lambda) = \begin{pmatrix} -\lambda^0 & 0 \\ 0 & \lambda^0 \end{pmatrix}$ is a unipotent polynomial matrix.

Theorem 3.13

If $A(\lambda)$ and $B(\lambda)$ are idempotent matrices and then $A(\lambda) + B(\lambda)$ is an idempotent if

$$A(\lambda)B(\lambda) = B(\lambda)A(\lambda) = 0.$$

Proof:

If $A(\lambda)$ and $B(\lambda)$ are idempotent matrices $\Rightarrow [A(\lambda)]^2 = A(\lambda)$. and $[B(\lambda)]^2 = B(\lambda)$. (3.1)

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And also $A(\lambda) = B(\lambda) \Rightarrow B(\lambda) = A(\lambda) = 0$ (3.2)

Consider

$$\begin{aligned} [A(\lambda) + B(\lambda)]^2 &= [A(\lambda) + B(\lambda)][A(\lambda) + B(\lambda)] \\ &= A(\lambda)A(\lambda) + A(\lambda)B(\lambda) + B(\lambda)A(\lambda) + B(\lambda)B(\lambda) \\ &= [A(\lambda)]^2 + 0 + 0 + [B(\lambda)]^2 \text{ [by (3.2)]} \\ &= A(\lambda) + B(\lambda) \text{ [by (3.1)]} \end{aligned}$$

Hence $A(\lambda) + B(\lambda)$ is an idempotent.

Theorem 3.14

If $A(\lambda)$ is an idempotent and $A(\lambda) + B(\lambda) = I$ then B is an idempotent and $A(\lambda)B(\lambda) = B(\lambda)A(\lambda) = 0$

Proof:

$$\text{If } A(\lambda) \text{ is an idempotent } \Rightarrow [A(\lambda)]^2 = A(\lambda). \tag{3.3}$$

$$\text{and } A(\lambda) + B(\lambda) = I \Rightarrow B(\lambda) = I - A(\lambda). \tag{3.4}$$

Now to prove $B(\lambda)$ is an idempotent and $A(\lambda)B(\lambda) = B(\lambda)A(\lambda) = 0$

Consider

$$\begin{aligned} [B(\lambda)]^2 &= B(\lambda) \cdot B(\lambda) \\ &= [I - A(\lambda)][I - A(\lambda)] \\ &= [I - A(\lambda)] \text{ [}\because I - A(\lambda) \text{ is an idempotent]} \\ &= B(\lambda) \text{ by (3.4)} \end{aligned}$$

$$A(\lambda) B(\lambda) = A(\lambda) [I - A(\lambda)]$$

$$= A(\lambda) - [A(\lambda)]^2$$

$$= A(\lambda) - A(\lambda) \text{ by (3.3)}$$

$$= 0$$

Theorem 3.15

If $A(\lambda) B(\lambda) = A(\lambda)$ and $B(\lambda) A(\lambda) = B(\lambda)$ then $[A(\lambda)]^2 = A(\lambda)$, $[B(\lambda)]^2 = B(\lambda)$

Proof:

$$\begin{aligned} [A(\lambda)]^2 &= [A(\lambda)] [A(\lambda)] \\ &= [A(\lambda)B(\lambda)] [A(\lambda)B(\lambda)] \\ &= A(\lambda) [B(\lambda)A(\lambda)] B(\lambda) \\ &= A(\lambda) B(\lambda) B(\lambda) \\ &= A(\lambda) B(\lambda) \\ &= A(\lambda) \end{aligned}$$

$$\begin{aligned} [B(\lambda)]^2 &= [B(\lambda)] [B(\lambda)] \\ &= [B(\lambda)A(\lambda)] [B(\lambda)A(\lambda)] \\ &= B(\lambda) [A(\lambda)B(\lambda)] A(\lambda) \\ &= B(\lambda) A(\lambda) A(\lambda) \end{aligned}$$

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$$\begin{aligned} &= B(\lambda) A(\lambda) \\ &= B(\lambda) \end{aligned}$$

Result: 3.16

If $A(\lambda)$ is an idempotent polynomial matrix then $rank [A(\lambda)] = tr [A(\lambda)]$.

Example: 3.17

Let $A(\lambda) = \begin{pmatrix} \lambda & -1 + \lambda \\ -\lambda & 1 - \lambda \end{pmatrix}$ be an idempotent polynomial matrix. Then

$$rank [A(\lambda)] = 1 \text{ and } tr [A(\lambda)] = \lambda + 1 - \lambda = 1.$$

Theorem: 3.18

Let $A(\lambda) \in C^{n \times n}(\lambda)$. If

- i) $A(\lambda)$ is idempotent and Hermitian then $A(\lambda)$ is Square Hermitian
- ii) $A(\lambda)$ is Hermitian and Square Hermitian then $A(\lambda)$ is idempotent
- iii) $A(\lambda)$ is Square Hermitian and idempotent then $A(\lambda)$ is Hermitian

Proof:

Proof of (i)

$$\text{If } A(\lambda) \text{ is an idempotent polynomial matrix } \Rightarrow [A(\lambda)]^2 = A(\lambda) \tag{3.5}$$

$$\text{and } A(\lambda) \text{ is Hermitian } \Rightarrow [A(\lambda)]^* = A(\lambda) \tag{3.6}$$

By (3.5) & (3.6), we get $[A(\lambda)]^2 = A(\lambda)$.

Proof of (ii)

$$\text{If } A(\lambda) \text{ is Hermitian and square Hermitian } \Rightarrow [A(\lambda)]^* = A(\lambda)^2.$$

Therefore $[A(\lambda)]^2 = [A(\lambda)]^* = A(\lambda)$ by (3.2). Hence $A(\lambda)$ is idempotent.

Proof of (iii)

$$\text{If } A(\lambda) \text{ is square Hermitian and idempotent } \Rightarrow [A(\lambda)]^* = A(\lambda)^2$$

$$\Rightarrow [A(\lambda)]^* = A(\lambda) \text{ by (3.5) \& (3.6)}$$

$$\Rightarrow A(\lambda) \text{ is Hermitian}$$

IV. CONCLUSION

In this paper some of the properties of polynomial idempotent matrices are derived and we extend some of the properties of idempotent matrices to polynomial idempotent matrices.

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