Common Fixed Point Theorem on Weak Reciprocally Continuous Mappings

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ABSTRACT: The purpose of this paper is to prove a unique common fixed point theorem in a fuzzy metric space using the concepts of weaker form of weak reciprocally continuous and weakly compatible mappings.

KEYWORDS: Fixed point, self maps, reciprocally continuous S-weak reciprocally continuous mappings, weakly compatible mappings.

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I. INTRODUCTION

The notion of the fuzzy metric spaces was initiated by Zadeh [12], in the recent past many authors developed the theory of fuzzy sets and applications. Grabic [5] obtained the Banach contraction principle in fuzzy version. Kramosil and meshalek introduced the concept of fuzzy metric space. Mishra et al [8] initiated the concept of compatible mappings which was finally generalized by Bijendra Singh and Jain [2] by introducing the notion of weakly compatible mappings in Fuzzy metric space. Pant [9] obtained the concept of reciprocally continuous and proves fixed point theorem. In this paper we prove a unique fixed point theorem using weak reciprocally continuous with S-compatible ( or A-compatible) and weakly compatible mappings.

II. DEFINITIONS AND PRELIMINARIES

Definition 2.1: A binary operation * :[0,1]×[0,1] →[0,1] is called continuous t-norm if * satisfies the following conditions:
   (i) * is commutative and associative
   (ii) * is continuous
   (iii) a*1=a for all a∈[0,1]
   (iv) a*b ≤ c*d whenever a ≤ c and b ≤ d for all a, b, c, d ∈[0,1]

Definition 2.2: A 3-tuple (X, M, *) is said to be Fuzzy metric space if X is an arbitrary set, * is continuous t-norm and M is a fuzzy set on X²×(0,∞) satisfying the following conditions for all x,y,z∈X, s,t>0
(FM-1) M(x,y,0)=0
(FM-2) M(x,y,t)=1 for all t>0 if and only if x=y
(FM-3) M(x,y,t)= M(y,x,t)
(FM-4) M(x,y,t) * M(y,z,s) ≤ M(x,z,t+s)
(FM-5) M(x,y,t) : [0,∞) →[0,1] is left continuous
(FM-6) lim M(x,y,t)→1 as t→∞
The function $M(x, y, t)$ denote the degree of nearness between $x$ and $y$ with respect to $t$. We observe $M(x, y, t) = 1$ when $x = y$ for all $t > 0$.

**Example 2.3:** (Induced fuzzy metric space): Let $(X, d)$ be a metric space defined $a*b = \min\{a, b\}$ for all $x, y \in X$ and $t > 0$,

$$M(x, y, t) = \frac{t}{t + d(x, y)}$$

Then $(X, M, *)$ is a fuzzy metric space. We call this fuzzy metric $M$ induced by metric $d$ the standard fuzzy metric.

From the above example every metric induces a fuzzy metric but there exist no metric on $X$ satisfying (a).

**Definition 2.4:** Let $(X, M, *)$ be a fuzzy metric space then a sequence $<x_n>$ in $X$ is said to be convergent to a point $x \in X$, if for each $\varepsilon > 0, t > 0$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x, t) > 1 - \varepsilon$ for all $n \geq n_0$

**Definition 2.5:** Let $(X, M, *)$ be a fuzzy metric space then a sequence $<x_n>$ in $X$ is said to be Cauchy sequence, if for each $\varepsilon > 0, t > 0$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $n \geq n_0$

**Definition 2.6:** A fuzzy metric space $(X, M, *)$ is said to be complete if every Cauchy sequence is convergent to a point in $X$.

**Lemma 2.7:** For all $x, y \in X$, $M(x, y, .)$ is non-decreasing.

**Lemma 2.8** let $(X, M, *)$ be a fuzzy metric space if there exists $k \in (0, 1)$ such that $M(x, y, kt) \geq M(x, y, t)$ then $x = y$.

**Proposition 2.9:** In the fuzzy metric space $(X, M, *)$ if $a*a \geq a$ for all $a \in [0,1]$ then $a*b = \min\{a, b\}$

**Definition 2.10:** Two self maps $S$ and $T$ of a fuzzy metric space $(X, M, *)$ are said to be compatible mappings if $\lim_{n \to \infty} M(STx_n, TSx_n, t) = 1$, whenever $<x_n>$ is a sequence in $X$ such that $\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = z$ for some $z \in X$.

**Definition 2.11:** Two self maps $S$ and $T$ of a fuzzy metric space $(X, M, *)$ are said to be $S$-compatible mappings if $\lim_{n \to \infty} M(STx_n, TTx_n, t) = 1$, whenever $<x_n>$ is a sequence in $X$ such that $\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = z$ for some $z \in X$.

**Definition 2.12:** Two self maps $S$ and $T$ of a fuzzy metric space $(X, M, *)$ are said to be reciprocally continuous mappings if $\lim_{n \to \infty} M(STx_n, Sz_n, t) = 1$ and $\lim_{n \to \infty} M(TSx_n, Tz_n, t) = 1$, whenever $<x_n>$ is a sequence in $X$ such that $\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = z$ for some $z \in X$.

**Definition 2.13:** Two self maps $S$ and $T$ of a fuzzy metric space $(X, M, *)$ are said to be $S$-weak reciprocally continuous mappings if $\lim_{n \to \infty} M(STx_n, Sz_n) = 1$, whenever $<x_n>$ is a sequence in $X$ such that $\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = z$ for some $z \in X$.

**Definition 2.14:** Two self maps $S$ and $T$ of a fuzzy metric space $(X, M, *)$ are said to be weakly compatible if they commute at their coincidence point. i.e if $Su = Tu$ for some $u \in X$ then $STu = TSu$. 

3.1 Theorem(A): Let A, B, S and T are self maps of a complete Fuzzy metric space \((X, M, \ast)\) with continuous t-norm \(\ast\) defined by \(a \ast b = \min\{a, b\}\), where \(a, b \in [0, 1]\), satisfying the conditions

- \((3.1.1)\) \(A(X) \subseteq T(X)\) and \(B(X) \subseteq S(X)\)
- \((3.1.2)\) S and T are continuous
- \((3.1.3)\) the pairs \((A, S)\) and \((B, T)\) compatible pairs of maps

\[
\begin{align*}
M(Ax, By, kt) &\geq \min\{M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t), M(By, Sx, 2t), M(Ax, Ty, t)\} \\
\text{for all } x, y \text{ in } X, \ k \in (0,1), t > 0
\end{align*}
\]

Now the generalization of the Theorem 3.1 as follows

IV. MAIN RESULT

4.1 Theorem: Let A, B, S and T are self maps of a complete Fuzzy metric space \((X, M, \ast)\) with continuous t-norm \(\ast\) defined by \(a \ast b = \min\{a, b\}\) where \(a, b \in [0, 1]\), satisfying the conditions

- \((4.1.1)\) \(A(X) \subseteq T(X)\) and \(B(X) \subseteq S(X)\)
- \((4.1.2)\) In the pairs \((A, S)\), S-weak reciprocally continuous and S-compatible (or) A-weak reciprocally continuous and A-compatible
- \((4.1.3)\) \((B, T)\) weakly compatible

\[
\begin{align*}
M(Ax, By, kt) &\geq \min\{M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t), M(By, Sx, 2t), M(Ax, Ty, t)\} \\
\text{where for all } x, y \text{ in } X, \ k \in (0,1), t > 0
\end{align*}
\]

Proof:

Let \(x_0\) be any arbitrary point of X from the condition \((4.1.1)\) there exists \(x_1, x_2 \in X\) such that \(Ax_0 = Tx_1\) and \(Bx_1 = Sx_2\).

Inductively we construct a sequence \(\langle x_n \rangle\) and \(\langle y_n \rangle\) in X such that \(y_{2n} = Ax_{2n} = Tx_{2n+1}\) and \(y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}\) for \(n \geq 0\).

By taking \(x = x_{2n}\) \(y = x_{2n+1}\) in the inequality \((4.1.4)\)
\[ M(y_{2n}, y_{2n+1}, kt) \\
= [M(Ax_{2n}, Bx_{2n+1}, kt)] \\
\geq \min \{ M(Sx_{2n}, Tx_{2n+1}, t), M(Ax_{2n}, Sx_{2n}, t), M(Bx_{2n+1}, Tx_{2n+1}, t), M(Bx_{2n+1}, Sx_{2n}, 2t), M(Ax_{2n}, Tx_{2n+1}, t) \} \\
\geq \min \{ M(y_{2n-1}, y_{2n}, t), M(y_{2n}, y_{2n-1}, t), M(y_{2n+1}, y_{2n}, t), M(y_{2n+1}, y_{2n-1}, 2t), M(y_{2n+1}, y_{2n}, t) \} \\
\geq \min \{ M(y_{2n-1}, y_{2n}, t), M(y_{2n+1}, y_{2n-1}, t), M(y_{2n+1}, y_{2n-1}, 2t), 1 \} \\
\geq \min \{ M(y_{2n-1}, y_{2n}, t), M(y_{2n+1}, y_{2n-1}, t), M(y_{2n+1}, y_{2n}, t), M(y_{2n+1}, y_{2n-1}, t), 1 \} \\
\geq \min \{ M(y_{2n-1}, y_{2n}, t), M(y_{2n+1}, y_{2n}, t), 1 \} \\
\geq M(y_{2n-1}, y_{2n}, t) \\
\text{In general } M(y_n, y_{n+1}, kt) \geq M(y_{n-1}, y_n, t) \quad \cdots (1) \]

To prove \( \{y_n\} \) is Cauchy sequence we prove \( M(y_n, y_{n+m}, t) > 1 - \varepsilon \) \( \cdots \) (2) is true for all \( n \geq n_0 \) and for every \( m \in \mathbb{N} \).

Here we use induction method.

From (1) we have
\[
\left[ M(y_n, y_{n+1}, t) \right] \geq M(y_{n-1}, y_n, t/2) \geq M(y_{n-2}, y_{n-1}, t/k) \geq \cdots \geq M(y_1, y_2, t/k_n) \text{ as } n \to \infty
\]
i.e for \( t > 0, \varepsilon \in (0,1) \) we can choose \( n_0 \in \mathbb{N} \) such that \( M(y_n, y_{n+1, t}) > 1 - \varepsilon \)

Thus (2) is true for \( m=1 \).

Suppose (2) is true for \( m \) then we shall show that it is also true for \( m+1 \)
using the definition of fuzzy metric space (1) and (2) we have
\[
M(y_n, y_{n+m}, t) \geq M(y_{n-1}, y_n, t/2) \geq M(y_{n-2}, y_{n-1}, t/k) \geq M(y_{n+1}, y_{n+m}, t/2) \]

Hence (2) is true for \( m+1 \). Thus \( \{y_n\} \) is a Cauchy sequence.

By completeness of \((X, M, *)\), \( \{y_n\} \) converges to some point \( z \) in \( X \)
Therefore \( Ax_{2n} \to z, Tx_{2n+1} \to z, Bx_{2n+1} \to z, Tx_{2n+1} \to z \) as \( n \to \infty \).

**Proof of Theorem 4.1:**

From the Lemma we have \( Ax_{2n} \to z, Tx_{2n+1} \to z, Bx_{2n+1} \to z, Tx_{2n+1} \to z \) as \( n \to \infty \)

Case (i): \text{S-weak reciprocally continuous implies } \lim_{n \to \infty} SAx_n = Sz

Since \text{S-compatibility } \lim_{n \to \infty} M(SAx_{2n}, AAx_{2n}, t) = 1 \text{ implies } \lim_{n \to \infty} SAx_{2n} = \lim_{n \to \infty} AAx_{2n} = Sz

To prove \( Sz = z \)

Put \( x = Ax_{2n}, y = x_{2n+1} \) in the inequality (4.1.4)
\[ M(AAx_{2n}, Bx_{2n+1}, kt) \geq \min \{ M(SAx_{2n}, Tx_{2n+1}, t), M(AAx_{2n}, Sax_{2n}, t), M(Bx_{2n+1}, Tx_{2n+1}, t), M(Bx_{2n+1}, Sax_{2n}, 2t), M(AAx_{2n}, Tx_{2n+1}, t) \} \]
\[ M(Sz, z, kt) \geq \min \{ M(Sz, z, t), M(Sz, Sz, t), M(z, z, t), M(z, Sz, 2t), M(Sz, z, t) \} \]
\[ M(Sz, z, kt) \geq M(Sz, z, t) \]

\[ S_z = z \]

To prove \( A_z = z \) put \( x = z \), \( y = x_{2n+1} \) in the inequality (4.1.4)

\[ M(Az, Bx_{2n+1}, kt) \geq \min \{ M(Sz, Tx_{2n+1}, t), M(Az, Sz, t), M(Bx_{2n+1}, Sz, 2t), M(Az, Tx_{2n+1}, t) \} \]
\[ M(Az, z, kt) \geq \min \{ M(z, z, t), M(Az, z, t), M(z, z, 2t), M(Az, z, t) \} \]
\[ M(Az, z, kt) \geq M(Az, z, t) \]

\[ A_z = z \]
\[ \therefore A_z = S_z = z \]

From the condition \( A(X) \subseteq T(X) \) implies there exists \( u \in X \) such that \( T_u = A_z = z \)

To prove \( B_u = z \) put \( x = z \), \( y = u \) in the inequality (4.1.4)

\[ M(Az, Bu, kt) \geq \min \{ M(Sz, Tu, t), M(Az, Sz, t), M(Bu, Sz, 2t), M(Az, Tu, t) \} \]
\[ M(z, Bu, kt) \geq \min \{ M(z, z, t), M(z, z, 2t), M(Bu, z, 2t), M(z, z, t) \} \]
\[ M(z, Bu, kt) \geq M(Bu, z, t) \]

\[ B_u = z \]

Since the pair \( (B,T) \) is weakly compatible gives \( BT_u = T_B \) implies \( B_z = T_z \).

To prove \( B_z = z \)

Put \( x = z, y = z \) in the inequality (4.1.4)

\[ M(Az, Bz, kt) \geq \min \{ M(Sz, Tz, t), M(Az, Sz, t), M(Bz, Tz, t), M(Bz, Sz, 2t), M(Az, Tz, t) \} \]
\[ M(z, Bz, kt) \geq \min \{ M(z, Bz, t), M(z, z, t), M(Bz, Bz, t), M(Bz, z, 2t), M(z, Bz, t) \} \]
\[ M(z, Bz, kt) \geq M(z, Bz, t) \]

\[ B_z = z \]
\[ \therefore B_z = T_z = z \]

**Case (ii):** Since the pair \((A,S)\) is \(A\)-weak reciprocally continuous which implies \( \lim_{n \to \infty} ASx_n \to Az \)

and since \( A \)-compatibility implies \( \lim_{n \to \infty} (ASx_{2n}, SSx_{2n+1}, t) = 1 \)

therefore \( \lim_{n \to \infty} A Sx_{2n} = \lim_{n \to \infty} SSx_{2n} = A z \)

To prove \( Az = z \) put \( x = Sx_{2n}, y = x_{2n+1} \) in the inequality (4.1.4)
\[ M (A_{2n}, B_{2n}, kt) \geq \min \{ M (S_{2n}, T_{2n+1}, t), M (A_{2n}, S_{2n}, t), \]
\[ M (B_{2n+1}, T_{2n+1}, t), M (B_{2n+1}, S_{2n}, 2t), M (A_{2n}, T_{2n}, t) \} \]
\[ M (A, B, z, kt) \geq \min \{ M (A_{z}, z, t), M (A_{z}, A_{z}, t), M (z, z, t), M (z, A_{z}, 2t), M (A_{z}, z, t) \} \]
\[ M (A, B, z, kt) \geq M (A_{z}, z, t) \]

\( A_{z} = z \)

Since the condition \( A(X) \subseteq T(X) \) implies there exists \( u \in X \) such that \( T_{u} = A_{z} = z \)
To prove \( B_{u} = z \)
Put \( x = x_{2n}, y = u \) in the inequality (4.1.4)
\[ M (A_{2n}, B_{u}, kt) \geq \min \{ M (S_{2n}, T_{u}, t), M (A_{2n}, S_{2n}, t), M (B_{u}, T_{u}, t), M (B_{u}, S_{2n}, 2t), M (A_{2n}, T_{u}, t) \} \]
\[ M (z, B_{u}, kt) \geq \min \{ M (z, z, t), M (z, z, t), M (B_{u}, z, t), M (B_{u}, z, 2t), M (z, z, t) \} \]
\[ M (z, B_{u}, kt) \geq M (B_{u}, z, t) \]

\( B_{u} = z \)

Since the pair \( (B, T) \) is weakly compatible, \( B_{u} = T_{u} \) gives \( B_{z} = T_{z} \)
To prove \( B_{u} = z \)
put \( x = x_{2n}, y = u \) in the inequality (4.1.4)
\[ M (A_{2n}, B_{z}, kt) \geq \min \{ M (S_{2n}, T_{z}, t), M (A_{2n}, S_{2n}, t), M (B_{z}, T_{z}, t), M (B_{z}, S_{2n}, 2t), M (A_{2n}, T_{z}, t) \} \]
\[ M (z, B_{z}, kt) \geq \min \{ M (z, z, t), M (z, z, t), M (B_{z}, B_{z}, t), M (B_{z}, z, 2t), M (z, B_{z}, t) \} \]
\[ M (z, B_{z}, kt) \geq M (B_{z}, z, t) \]

\( B_{z} = z \)

\( B_{z} = T_{z} = z \)

Since the condition \( B(X) \subseteq S(X) \) implies there exists \( v \in X \) such that \( S_{v} = B_{z} = z \)

The pair \( (A, S) \) is \( A \)-compatibility implies
\[ \lim_{n \rightarrow \infty} M (A_{2n}, S_{2n}, t) = 1 \lim_{n \rightarrow \infty} A_{2n} = \lim_{n \rightarrow \infty} S_{2n} \]

Let \( v = \lim_{n \rightarrow \infty} x_{2n} \) then \( A_{v} = S_{v} \) this implies \( A_{z} = S_{z} \)

\( A_{z} = S_{z} = z \).

Since \( A_{z} = B_{z} = S_{z} = T_{z} = z \) implies \( z \) is a common fixed point of \( A, B, S \) and \( T \).

Uniqueness:
Let \( w \) be another common fixed point of above self maps then we have \( A_{w} = B_{w} = S_{w} = T_{w} = w \)
Put \( x = z, y = w \) in the inequality (4.1.4)
\[ M (A_{z}, B_{w}, kt) \geq \min \{ M (S_{z}, T_{w}, t), M (A_{z}, S_{w}, t), M (B_{w}, T_{w}, t), M (B_{w}, S_{z}, 2t), M (A_{z}, T_{w}, t) \} \]
\[ M (z, B_{w}, kt) \geq M (z, B_{w}, t) \]

\( z = w. \)

\textbf{4.1.6 Example:} Let \( X = [-1, 1] \), \( M (x, y, t) = \frac{t}{t + d(x, y)} \) where \( d(x, y) = |x - y| \)
\[ A x = B x = \begin{cases} \frac{1}{10} & \text{if } -1 < x < \frac{1}{6} \\ \frac{1}{6} & \text{if } \frac{1}{6} \leq x < 1 \end{cases} \]

\[ S x = T x = \begin{cases} \frac{1}{8} & \text{if } -1 < x < \frac{1}{6} \\ \frac{1}{3} & \text{if } \frac{1}{6} \leq x < 1 \end{cases} \]

Then \( A (X) = B (X) = \left( \frac{1}{10}, \frac{1}{6} \right) \) while \( S (X) = T (X) = \left( \frac{1}{8} \cup \left[ -\frac{2}{3}, \frac{1}{6} \right] \right) \) so that the conditions

\[ A (X) \subseteq T (X) \text{ and } B (X) \subseteq S (X) \]

are satisfied.

Take a sequence \( x_n = \left( \frac{1}{6} + \frac{1}{n} \right) \) for \( n \geq 1 \), then

\[ \lim_{n \to \infty} A x_n = \lim_{n \to \infty} S x_n = 1/6 \text{ and } \lim_{n \to \infty} AS x_n = 1/8 \text{ also } SA x_n = 1/6. \]

\( \lim_{n \to \infty} M (SA x_n, AA x_n, t) = 1 \) implies \( S \)-compatible

\( \lim_{n \to \infty} SA x_n, S(1/6) \) implies \( S \)-weakly reciprocally continuous

but so that \( \lim_{n \to \infty} M (AS x_n, SA x_n, t) \neq 1. \)

In the pairs \((A, S)\), \( S \)-weakly reciprocally continuous and \( S \)-compatible \((B, T)\) are weakly compatible as they commute at coincident point \( 1/6 \).

Also note that none of the mappings are continuous. Hence \( 1/6 \) is the unique common fixed point of \( A, B, S \) and \( T \).

V. CONCLUSION

Unique fixed point exists for Theorem 4.1 where the none of the mappings are continuous. And also from the Example 4.1.6 we have observed that the pair of \( S \)-compatible Mappings is need not be compatible mappings.

REFERENCES