

Left Generalized Derivations and Commutativity Of Prime Rings

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ABSTRACT: In the present paper we have discussed the commutativity of prime rings admitting a left generalized derivation F satisfying $F([x, y]) = [x, y]$, for each pair x, y of elements in a specified subset of R .

KEYWORDS: Prime ring, Derivation, Generalized derivation, Left generalized derivation, Ideal and Commutativity.

I. INTRODUCTION

The study of the commutativity of prime rings with derivations was initiated by E.C.Posner[12]. Hvala [8] studied generalized derivations in prime rings. Daif and Bell [6] established that if in a semiprime ring R there exists a nonzero ideal I of R and a derivation d such that $d([x, y]) = [x, y]$ for all $x, y \in I$, then $I \subseteq Z(R)$. Jaya Subba Reddy et.al[14] studied centralizing and commuting left generalized derivation on prime ring is commutative. J.H. Mayne [11] studied on centralizing mappings of prime rings. M.A.Quadri et.al[13] studied generalized derivations and commutativity of prime ring is a commutative. In this paper we extended some results on left generalized derivations and commutativity of prime ring is a commutative. The proofs of the following results can be seen in [1, Lemma 3] and [11, Lemma 3] respectively.

Preliminaries: Throughout this paper R will denote an associative ring with the center $Z(R)$. For any $x, y \in R$, the symbol $[x, y]$ stands for commutator $xy - yx$ and the symbol xoy denotes the anticommutator $xy + yx$. Recall that a ring R is called prime if for any $a, b \in R$, $aRb = (0)$ implies that either $a = 0$ or $b = 0$. An additive mapping $d: R \rightarrow R$ is called a derivation if $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$. For a fixed $a \in R$, the mapping $I_a: R \rightarrow R$ given by $I_a(x) = [a, x]$ is a derivation which is said to be an inner derivation. An additive function $F_{a,b}: R \rightarrow R$ is called a generalized inner derivation if $F_{a,b}(x) = ax + xb$ for some fixed $a, b \in R$. It is straightforward to note that if $F_{a,b}$ is a generalized inner derivation, then for any $x, y \in R$, $F_{a,b}(xy) = F_{a,b}(x)y + x[y, b]$

$$= F_{a,b}(x)y + xI_b(y)$$

Where I_b is an inner derivation. An additive mapping $F: R \rightarrow R$ is called a generalized derivation associated with a derivation d if $F(xy) = F(x)y + xd(y)$, for all $x, y \in R$. An additive mapping $F: R \rightarrow R$ is called a left generalized derivation associated with a derivation d if $F(xy) = xF(y) + d(x)y$, for all $x, y \in R$. It should be interesting to extend some results concerning these notions to left generalized derivations. Throughout the present paper we shall make extensive use of the following basic commutator identities without any specific mention:

$$[xy, z] = x[y, z] + [x, z]y; [x, yz] = y[x, z] + [x, y]z$$

$$xo(yz) = (xoy)z - y[x, z] = y(xoz) + [x, y]z$$

$$(xy)oz = x(yoz) - [x, z]y = (xoz)y + x[y, z].$$

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II. RELATED WORK

Lemma 1: Let R be a prime ring and I be a nonzero right ideal of R . If d is a nonzero derivation on R , then d is nonzero on I .

Lemma 2: If a prime ring R contains a nonzero commutative right ideal, then R is commutative.

Theorem 1: Let R be a prime ring and I be a nonzero ideal of R . If R admits a left generalized derivation F associated with a nonzero derivation d such that $F([x, y]) = [x, y]$ for all $x, y \in I$, then R is commutative.

Proof: We may assume that F is nonzero, otherwise I is commutative and so is R . For any $x, y \in I$, we have $F([x, y]) = [x, y]$, which gives

$$xF(y) + d(x)y - yF(x) - d(y)x - [x, y] = 0, \text{ for all } x, y \in I. \quad (1)$$

Replacing y by zy in equation (1), we get

$$\begin{aligned} xF(zy) + d(x)zy - zyF(x) - d(zy)x - [x, zy] &= 0 \\ xzF(y) + xd(z)y + d(x)zy - zyF(x) - zd(y)x - d(z)yx - z[x, y] - [x, z]y &= 0 \end{aligned}$$

Using equation (1) to substitute for $yF(x)$ in the above equation, we get

$$xzF(y) + xd(z)y + d(x)zy - zxF(y) - zd(x)y + zd(y)x + z[x, y] - zd(y)x - d(z)yx - z[x, y] - [x, z]y = 0.$$

$$\begin{aligned} xzF(y) - zxF(y) + d(x)zy - zd(x)y + xd(z)y - d(z)yx - [x, z]y &= 0 \\ (xz - zx)F(y) + (d(x)z - zd(x))y + xd(z)y - d(z)xy + d(z)xy - d(z)yx - [x, z]y &= 0 \\ (xz - zx)F(y) + (d(x)z - zd(x))y + (xd(z) - d(z)x)y + d(z)(xy - yx) - [x, z]y &= 0 \end{aligned}$$

$$[x, z]F(y) + [d(x), z]y + [x, d(z)]y + d(z)[x, y] - [x, z]y = 0, \text{ for all } x, y, z \in I. \quad (2)$$

Again replacing z by xz in equation (2), we get

$$\begin{aligned} [x, xz]F(y) + [d(x), xz]y + [x, d(xz)]y + d(xz)[x, y] - [x, xz]y &= 0 \\ x[x, z]F(y) + [x, x]zF(y) + x[d(x), z]y + [d(x), x]zy + [x, xd(z) + d(x)z]y + (xd(z) + d(x)z)[x, y] - x[x, z]y &= 0 \\ - [x, x]zy &= 0 \\ x[x, z]F(y) + x[d(x), z]y + [x, xd(z)]y + [x, d(x)z]y + xd(z)[x, y] + d(x)z[x, y] - x[x, z]y &= 0 \\ x[x, z]F(y) + x[d(x), z]y + x[x, d(z)]y + [x, x]d(z)y + d(x)[x, z]y + [x, d(x)]zy + xd(z)[x, y] + d(x)z[x, y] &= 0 \\ - x[x, z]y &= 0 \\ x[x, z]F(y) + x[d(x), z]y + x[x, d(z)]y + d(x)[x, z]y + xd(z)[x, y] + d(x)z[x, y] - x[x, z]y &= 0 \\ x([x, z]F(y) + [d(x), z]y + [x, d(z)]y + d(z)[x, y] - [x, z]y) + d(x)z[x, y] + &= 0 \end{aligned}$$

$$d(x)[x, z]y = 0, \text{ for all } x, y, z \in I.$$

Using equation (2) in above equation, we get

$$d(x)z[x, y] + d(x)[x, z]y = 0, \text{ for all } x, y, z \in I. \quad (3)$$

We replacing y by yy_1 in equation (3), we get

$$\begin{aligned} d(x)z[x, yy_1] + d(x)[x, z]yy_1 &= 0 \\ d(x)zy[x, y_1] + d(x)z[x, y]y_1 + d(x)[x, z]yy_1 &= 0 \end{aligned}$$

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$$d(x)zy[x, y_1] + (d(x)z[x, y] + d(x)[x, z]y)y_1 = 0$$

Using equation (3), we get

$$d(x)zy[x, y_1] = 0, \text{ for all } x, y, y_1, z \in I.$$

We replace z by zr , $r \in R$ in above equation, we get

$$d(x)zry[x, y_1] = 0, \text{ for all } x, y, y_1, z \in I \text{ and } r \in R.$$

And hence $d(x)IRy[x, y_1] = (0)$, for all $x, y, y_1 \in I$.

Thus, primeness of R forces that for each $x \in I$ either $d(x)I = (0)$ or $y[x, y_1] = 0$. The set of all $x \in I$ for which these two properties hold are additive subgroups of I whose union is I ; therefore, $d(x)I = (0)$ for all $x \in I$ or $y[x, y_1] = 0$, for all $x, y, y_1 \in I$. If $d(x)I = (0)$ for all $x \in I$, then by lemma 1, $d = 0$, a contradiction. On the other hand if $y[x, y_1] = 0$ for all $x, y, y_1 \in I$, this gives $IR[x, y_1] = (0)$, and since R is a prime and $I \neq (0)$. We get $[x, y_1] = 0$, $x, y_1 \in I$. Hence by application of lemma 2, R is commutative.

Theorem 2: Let R be a prime ring and I be a nonzero ideal of R . If R admits a left generalized derivation F associated with a nonzero derivation d such that $F([x, y]) + [x, y] = 0$ for all $x, y \in I$, then R is commutative.

Proof: We may assume that F is nonzero, otherwise I is commutative and so is R . For any $x, y \in I$, we have $F([x, y]) + [x, y] = 0$, which gives

$$xF(y) + d(x)y - yF(x) - d(y)x + [x, y] = 0, \text{ for all } x, y \in I. \tag{4}$$

Replacing y by zy in equation (4), we get

$$\begin{aligned} xF(zy) + d(x)zy - zyF(x) - d(zy)x + [x, zy] &= 0 \\ xzF(y) + xd(z)y + d(x)zy - zyF(x) - zd(y)x - d(z)yx + z[x, y] + [x, z]y &= 0 \end{aligned}$$

Using equation (4) to substitute for $yF(x)$ in the above equation, we obtain

$$xzF(y) + xd(z)y + d(x)zy - zxF(y) - zd(x)y + zd(y)x - z[x, y] - zd(y)x - d(z)yx + z[x, y] + [x, z]y = 0.$$

$$xzF(y) - zxF(y) + d(x)zy - zd(x)y + xd(z)y - d(z)yx + [x, z]y = 0$$

$$(xz - zx)F(y) + (d(x)z - zd(x))y + xd(z)y - d(z)xy + d(z)xy - d(z)yx + [x, z]y = 0$$

$$(xz - zx)F(y) + (d(x)z - zd(x))y + (xd(z) - d(z)x)y + d(z)(xy - yx) + [x, z]y = 0$$

$$[x, z]F(y) + [d(x), z]y + [x, d(z)]y + d(z)[x, y] + [x, z]y = 0, \text{ for all } x, y, z \in I. \tag{5}$$

Again replacing z by xz in equation (5), we get

$$[x, xz]F(y) + [d(x), xz]y + [x, d(xz)]y + d(xz)[x, y] + [x, xz]y = 0$$

$$\begin{aligned} x[x, z]F(y) + [x, x]zF(y) + x[d(x), z]y + [d(x), x]zy + [x, xd(z) + d(x)z]y + [xd(z) + d(x)z][x, y] + x[x, z]y \\ + [x, x]zy = 0 \end{aligned}$$

$$x[x, z]F(y) + x[d(x), z]y + [x, xd(z)]y + [x, d(x)z]y + xd(z)[x, y] + d(x)z[x, y] + x[x, z]y = 0$$

$$\begin{aligned} x[x, z]F(y) + x[d(x), z]y + x[x, d(z)]y + [x, x]d(z)y + d(x)[x, z]y + [x, d(x)]zy + xd(z)[x, y] + d(x)z[x, y] \\ + x[x, z]y = 0 \end{aligned}$$

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$$x[x, z]F(y) + x[d(x), z]y + x[x, d(z)]y + d(x)[x, z]y + xd(z)[x, y] + d(x)z[x, y] + x[x, z]y = 0$$

$$x([x, z]F(y) + [d(x), z]y + [x, d(z)]y + d(z)[x, y] + [x, z]y) + d(x)z[x, y] +$$

$d(x)[x, z]y = 0$, for all $x, y, z \in I$.

Using equation (5) then, we get

$$d(x)z[x, y] + d(x)[x, z]y = 0, \text{ for all } x, y, z \in I. \tag{6}$$

The equation (6) is same as equation (3) in theorem1. Thus, by same argument of theorem 1, we can conclude the result is there. Thus the proof is completed.

Theorem 3: Let R be a prime ring and I be a nonzero ideal of R . If R admits a left generalized derivation F associated with a nonzero derivation d such that $F(xoy) = xoy$ for all $x, y \in I$, then R is commutative.

Proof: For any $x, y \in I$, we have $F(xoy) = xoy$. If $F = 0$, then $xoy = 0$, for all $x, y \in I$. Replacing y by yz and using the fact that $xy = -yx$, we find that $y[x, z] = 0$ for all $x, y, z \in I$ and hence $IR_{[x, z]} = (0)$ for all $x, z \in I$. Since $I \neq (0)$ and R is a prime, we get $[x, z] = 0$ for all $x, z \in I$, and hence by lemma 2, R is commutative. Henceforth we assume that $F \neq 0$. For any $x, y \in I$, we have $F(xoy) = xoy$. This can be written as

$$F(xy + yx) = xoy$$

$$xF(y) + d(x)y + yF(x) + d(y)x - xoy = 0 \text{ for all } x, y \in I \tag{7}$$

Replacing y by xy in equation (7), we get

$$xF(xy) + d(x)xy + xyF(x) + d(xy)x - xoxy = 0$$

$$xxF(y) + xd(x)y + d(x)xy + xyF(x) + xd(y)x + d(x)yx - x(xoy) - [x, x]y = 0$$

$$x(xF(y) + d(x)y + yF(x) + d(y)x - xoy) + d(x)xy + d(x)yx = 0$$

Using equation (7), we get

$$d(x)xy + d(x)yx = 0$$

$$d(x)(xy + yx) = 0$$

$$d(x)(xoy) = 0, \text{ for all } x, y \in I. \tag{8}$$

Replacing y by yz in the above equation, we get

$$d(x)[xoyz] = 0$$

$$d(x)[(xoy)z - y[x, z]] = 0$$

$$d(x)(xoy)z - d(x)y[x, z] = 0$$

Using equation (8) in above equation, we get

$$d(x)y[x, z] = 0, \text{ for all } x, y, z \in I.$$

We replacing y by ry , $r \in R$, in above equation, we get

$$d(x)ry[x, z] = 0, \text{ for all } x, y, z \in I.$$

Hence $d(x)RI[x, z] = (0)$, For all $x, z \in I$ and $r \in R$.

Thus, primeness of R forces that for each $x \in I$ either $d(x) = 0$ or $I[x, z] = (0)$ for all $z \in I$. The set of $x \in I$ for which these two properties hold are additive subgroups of I whose union is I , and therefore $d(x) = 0$ for all $x \in I$ or $I[x, z] = (0)$ for all $x, z \in I$. If $d(x) = 0$ for all $x \in I$, then by lemma 1, $d = 0$, a contradiction. On the other hand if $I[x, z] = (0)$ for all $x, z \in I$, $IR[x, z] = (0)$. Since $I \neq (0)$, we find that $[x, z] = 0$ for all $x, z \in I$. By the application of lemma 2, R is commutative.

Theorem 4: Let R be a prime ring and I be a nonzero ideal of R . If R admits a left generalized derivation F associated with a nonzero derivation d such that $F(xoy) + xoy = 0$ holds for all $x, y \in I$, then R is commutative.

Proof: For any $x, y \in I$, we have $F(xoy) + xoy = 0$. If $F = 0$, then $xoy = 0$, for all $x, y \in I$. Replacing y by yz and using the fact that $xy = -yx$, we find that $y[x, z] = 0$ for all $x, y, z \in I$ and hence $IR_{[x, z]} = (0)$ for all $x, z \in I$. Since $I \neq (0)$ and R is a prime, we get $[x, z] = 0$ for all $x, z \in I$, and hence by lemma 2, R is commutative. Henceforth we assume that $F \neq 0$. For any $x, y \in I$, we have $F(xoy) + xoy = 0$. This can be written as

$$F(xy + yx) + xoy = 0$$

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$$xF(y) + d(x)y + yF(x) + d(y)x + xoy = 0 \text{ for all } x, y \in I \quad (9)$$

Replacing y by xy in equation (9), we get

$$xxF(y) + xd(x)y + d(x)xy + xyF(x) + xd(y)x + d(x)yx + x(xoy) + [x, x]y = 0$$

$$x(xF(y) + d(x)y + yF(x) + d(y)x + xoy) + d(x)xy + d(x)yx = 0$$

Using equation (9), we get

$$d(x)xy + d(x)yx = 0$$

$$d(x)(xoy) = 0, \text{ for all } x, y \in I. \quad (10)$$

The equation (10) is same as equation (8) in theorem 3. Thus, by same argument of theorem 3, we can conclude the result is there. Thus the proof is completed.

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