

$(\sigma, \tau)^*$ -Left Derivations on Prime Left Nearrings

D. Bharathi¹, K. Sreenivasulu²

Professor, Department of Mathematics, S. V. University, Tirupati, Andhra Pradesh, India.¹

Research Scholar, Department of Mathematics, S. V. University, Tirupati, Andhra Pradesh, India.²

ABSTRACT: In this paper, N is a prime left nearring admitting a non-trivial $(\sigma, \tau)^*$ -left derivation d . Now we proved (i) if $d(x)d(y)=d(y)d(x)$, for all $x, y \in N$, then $(N, +)$ is an abelian. (ii) if $d(N) \subseteq Z$, then $(N, +)$ is an abelian. (iii) if $u \in N$ is not a right zero divisor and if $[u, d(u)]_{\sigma, \tau} = 0$, then (u, x) is a constant for every $x \in N$. Also, we proved that if N is 2-torsion free and σ, τ commute with d , then N is a commutative ring.

KEYWORDS: Left nearring, Left derivation, (σ, τ) derivation, $(\sigma, \tau)^*$ -Derivation, 2-torsion free.

I. INTRODUCTION

In this paper N will denote a zero symmetric left nearring with multiplicative centre Z . An element x of N is said to be distributive if $x(y+z)=xy+xz$ for all $x, y, z \in N$. A nearring N is called zero symmetric if $x0=0$ for all $x \in N$ (recall that left distributivity yields $0x=0$). An additive mapping $d:N \rightarrow N$ is said to be a derivation on N if $d(xy)=xd(y)+d(x)y$ for all $x, y \in N$ or equivalently, as noted in [1], that $d(xy)=d(x)y+xd(y)$ for all $x, y \in N$, following [2], an additive mapping $d:N \rightarrow N$ is called a σ -derivation if there exists an automorphism $\sigma:N \rightarrow N$ such that $d(xy)=\sigma(x)d(y)+d(x)y$ for all $x, y \in N$ and we define an additive mapping $d:N \rightarrow N$ is called a (σ, τ) -derivation if there exists automorphism $\sigma, \tau:N \rightarrow N$ such that $d(xy)=\sigma(x)d(y)+\tau(y)d(x)$ for all $x, y \in N$. Further this is a motivation to define an additive mapping $d:N \rightarrow N$ is called a $(\sigma, \tau)^*$ -left derivation if there exist automorphisms $\sigma, \tau:N \rightarrow N$ such that $d(xy)=\sigma(y^*)d(x)+\tau(x)d(y)$ for all $x, y \in N$. In case $\sigma=1$, the identity mapping, d is called τ -derivation. Similarly if $\tau=1$, d is called σ -derivation. It is straightforward that an $(1, 1)$ -derivation is ordinary derivation. For $x, y \in N$, the symbol $[x, y]$ will denote the commutator $xy-yx$ while the symbol (x, y) will denote the additive commutator $x+y-x-y$. Following [2] for $x, y \in N$, the symbol $[x, y]_{\sigma, \tau}$ will denote the (σ, τ) -commutator $\sigma(x)y-\tau(x)$ while (σ, τ) -derivation will be called (σ, τ) -commuting if $[x, d(x)]_{\sigma, \tau}=0$ for all $x \in N$. A nearring N is said to be prime if $aNb=(0)$ implies that $a=0$ or $b=0$. Further an element $x \in N$ for which $d(x)=0$ is called a constant.

The recent results on rings deal with commutativity of prime and semiprime rings admitting suitably constrained derivations. It is natural to look for comparable results on nearrings and this has been done in [3],[4],[5],[6],[2], and [1]. It is our purpose to extend some of these results on prime nearrings admitting suitably constrained $(\sigma, \tau)^*$ -derivation.

Lemma 1: An additive endomorphism d on a nearring N is a $(\sigma, \tau)^*$ -left derivation if and only if $d(xy)=\tau(y)d(x)+\sigma(x^*)d(y)$, for all $x, y \in N$.

Proof : Let d be a $(\sigma, \tau)^*$ -right derivation on a nearring N .

Since $(y+y)x=yx+yx$

$$\begin{aligned} d((y+y)x) &= \sigma(x^*)d(y+y) + \tau(y+y)d(x) \\ &= \sigma(x^*)d(y) + \sigma(x^*)d(y) + \tau(y)d(x) + \tau(y)d(x), \text{ for all } x, y \in N \end{aligned} \tag{1}$$

On the other hand, we have

$$d(yx+yx) = d(yx) + d(yx)$$

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(An ISO 3297: 2007 Certified Organization)

Vol. 6, Special Issue 13, July 2017

$$= \sigma(x^*)d(y) + \tau(y)d(x) + \sigma(x^*)d(y) + \tau(y)d(x), \text{ for all } x, y \in N \quad (2)$$

Combining (1) and (2), we find that

$$\sigma(x^*)d(y) + \tau(y)d(x) = \tau(y)d(x) + \sigma(x^*)d(y)$$

Thus, we have

$$d(yx) = \sigma(x^*)d(y) + \tau(y)d(x), \text{ for all } x, y \in N \quad (3)$$

Conversely, suppose that $d(yx) = \sigma(x^*)d(y) + \tau(y)d(x)$

$$\begin{aligned} d((y+y)x) &= \sigma(x^*+x^*)d(y) + \tau(y)d(x+x) \\ &= \sigma(x^*)d(y) + \tau(y)d(x) + \sigma(x^*)d(y) + \tau(y)d(x), \text{ for all } x, y \in N \end{aligned} \quad (4)$$

Also, $d(yx+yx) = d(yx) + d(yx)$

$$= \sigma(x^*)d(y) + \tau(y)d(x) + \sigma(x^*)d(y) + \tau(y)d(x), \text{ for all } x, y \in N \quad (5)$$

Combining (4) and (5), we get

$$\sigma(x^*)d(y) + \tau(y)d(x) = \tau(y)d(x) + \sigma(x^*)d(y), \text{ for all } x, y \in N \quad \diamond$$

Lemma 2 : Let d be a $(\sigma, \tau)^*$ - left derivation on the nearing N . Then N satisfies the following partial distributive laws:

$$(i) z^*(\sigma(y^*)d(x) + \tau(x)d(y)) = z^*\sigma(y^*)d(x) + z^*\tau(x)d(y)$$

$$(ii) z^*(\tau(x)d(y) + \sigma(y^*)d(x)) = z^*\tau(x)d(y) + z^*\sigma(y^*)d(x)$$

Proof : For $x, y, z \in N$, $d((xy)z) = \sigma(z^*)d(xy) + \tau(xy)d(z)$

$$\begin{aligned} &= \sigma(z^*)[\sigma(y^*)d(x) + \tau(x)d(y)] + \tau(x)\tau(y)d(z) \\ &= \sigma(z^*)\sigma(y^*)d(x) + \sigma(z^*)\tau(x)d(y) + \tau(x)\tau(y)d(z), \text{ for all } x, y, z \in N \end{aligned} \quad (6)$$

On the other hand, we have

$$\begin{aligned} d((x(yz))) &= \sigma(z^*y^*)d(x) + \tau(x)d(yz) \\ &= \sigma(z^*)\sigma(y^*)d(x) + \tau(x)(\sigma(z^*)d(y) + \tau(y)d(z)) \\ &= \sigma(z^*)\sigma(y^*)d(x) + \tau(x)\sigma(z^*)d(y) + \tau(x)\tau(y)d(z), \text{ for all } x, y, z \in N. \end{aligned} \quad (7)$$

Equating (6) and (7), we find that

$$\sigma(z^*)(\sigma(y^*)d(x) + \tau(x)d(y)) = \sigma(z^*)(\sigma(y^*)d(x) + \tau(x)d(y)), \text{ for all } x, y, z \in N.$$

Replace $\sigma(z^*)$ by z^* , we get

$$z^*(\sigma(y^*)d(x) + \tau(x)d(y)) = z^*(\sigma(y^*)d(x) + \tau(x)d(y)), \text{ for all } x, y, z \in N.$$

Similarly, (ii) can be proved. \(\diamond\)

Lemma 3: Let d be a $(\sigma, \tau)^*$ - left derivation on N and suppose $u \in N$ is not a right zero divisor. If $[u, d(u)]_{\sigma, \tau} = 0$, then (u, x) is a constant for every $x \in N$.

Proof : Since $(u+x)u = u^2 + xu$, we get

$$\sigma(u^*)d(u+x) + \tau(u+x)d(u) = \sigma(u^*)d(u) + \tau(u)d(u) + \sigma(u^*)d(x) + \tau(x)d(u)$$

$$\sigma(u^*)d(u) + \sigma(u^*)d(x) + \tau(u)d(u) + \tau(x)d(u) = \sigma(u^*)d(u) + \tau(u)d(u) + \sigma(u^*)d(x) + \tau(x)d(u), \text{ we get}$$

$$\sigma(u^*)d(x) + \tau(u)d(u) = \tau(u)d(u) + \sigma(u^*)d(x), \text{ for all } u \in N \text{ and } x \in N.$$

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Due to $[u, d(u)]_{\sigma, \tau} = 0$, the above expression can be written as

$\sigma(u^*)(d(u)+d(x)) = \sigma(u)(d(u)+d(x))$, for all $u, x \in N$

That is $\sigma(u)(d(u, x)) = 0$, for all $x \in N$.

Since σ is an automorphism of N , $\sigma(u^*)$ is not a right zero divisor.

Thus $d(u, x) = 0$.

Hence (u, x) is constant, for all $x \in N$. ◇

II. MAIN THEOREMS

Theorem 1: Let N have no non-zero divisors of zero. If N admits a non-trivial (σ, τ) -commuting $(\sigma, \tau)^*$ -left derivation d , then $(N, +)$ is abelian.

Proof : Let c be any additive commutator.

By the application of Lemma 3. yields that c is a constant.

For any $x \in N$, cx is also an additive commutator, hence a constant..

Thus, $0 = d(cx) = \sigma(x^*)d(c) + \tau(c)d(x)$,

That is $\tau(c)d(x)$, for all $x \in N$ and additive commutators c .

Since $d(x) \neq 0$ for some $x \in N$, so $\tau(c) = 0$,

And thus $c = 0$ for all additive commutator c .

Hence $(N, +)$ is abelian. ◇

Lemma 4 : Let N be a Prime left nearring.

(i) If z is a non-zero element in Z , then z is not a zero divisor.

(ii) If there exists a non-zero element of z of Z such that $z+z \in Z$, then $(N, +)$ is abelian.

(iii) Let d be a non-trivial $(\sigma, \tau)^*$ - left derivation on N and let τ and d commute. Then $xd(N) = (0)$ or $d(N)x = (0)$, implies $x = 0$.

(iv) If N is 2-torsion free and d is a $(\sigma, \tau)^*$ - left derivation on N such that $d^2 = 0$ and (σ, τ) commute with d , then $d = 0$.

(v) If N admits a non-trivial $(\sigma, \tau)^*$ - left derivation d for which $d(N) \subseteq Z$, then $ce \in Z$ for each constant element c of N .

Proof : (i) and (ii) are already proved [4].

(iii) Let $xd(r) = 0$, for all $r \in N$.

Change r by yz , then we get

$xd(yz) = x\sigma(z^*)d(y) + x\tau(y)d(z) = 0$, for all $y, z \in N$.

$xd(yz) = x\sigma(z^*)d(y) + xd(z)\tau(y) = 0$, for all $y, z \in N$. (Since $\tau d = d \tau$)

Hence $x\sigma(z^*)d(y) = 0$.

Again N is prime and $d(N) \neq 0$, we have $x = 0$.

By arguing as above, we can show that $d(r)x = 0$, for all $r \in N$, this implies that $x = 0$.

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(iv) For an arbitrary $x, y \in N$, we have $d^2(xy) = 0$.

We obtain $2d(\sigma(y^*))d(\tau(x)) = 0$.

Since N is 2-torsion free, so $d(\sigma(y^*))d(N) = (0)$, for each $x \in N$.

Hence $d = 0$, by using (iii) and the fact that σ is an automorphism.

(v) Let c be an arbitrary constant and let x be a non-constant element of N .

Then $\tau(c)d(x) = d(cx) \in Z$ for each non-constant element x of N .

This implies that $y\tau(c)d(x) = \tau(c)d(x)y$, for all $y \in N$.

Since $d(x) \in Z \setminus \{0\}$, it follows that $\tau(c)d(x)y = \tau(c)y d(x)$, for all $y \in N$.

And we conclude that $d(x)(cy - yc) = 0$, for all $y \in N$ and additive commutator c .

Hence, using (i), we get the required result. ◊

Theorem 2: Let N be a prime left nearring admitting a non-trivial $(\sigma, \tau)^*$ -left derivation d for which $d(N) \subseteq Z$. Then $(N, +)$ is abelian. More over, if N is 2-torsion free and σ, τ commute with d , then N is a commutative ring.

Proof : Since $d(N) \subseteq Z$ and d is non-trivial, there exist a nonzero element x in N such that $z = d(x) \in Z \setminus \{0\}$ and $z + z = d(x+x) \in Z$.

Hence $(N, +)$ is abelian by Lemma 4(ii).

Now, suppose that N is 2-torsion free and σ, τ commute with d . Application of Lemma 2(i) yields that,

$$(\sigma(y^*)d(x) + \tau(x)d(y))r = \sigma(y^*)d(x)r + \tau(x)d(y)r, \text{ for all } x, y, r \in N. \tag{8}$$

Since $d(N) \subseteq Z$, it follows that $d(xy) \in Z$, for all $x, y \in N$.

Thus, $d(xy)r = rd(xy)$, for all $x, y, r \in N$ and hence

$$\begin{aligned} (\sigma(y^*)d(x) + \tau(x)d(y))r &= r(\sigma(y^*)d(x) + \tau(x)d(y)) \\ &= r\sigma(y^*)d(x) + r\tau(x)d(y), \text{ for all } x, y, r \in N. \end{aligned} \tag{9}$$

Combine (8) and (9) use the fact that $(N, +)$ is abelian, to get

$$\sigma(y^*)d(x)r - r\sigma(y^*)d(x) = r\tau(x)d(y) - \tau(x)d(y)r, \text{ for all } x, y, r \in N.$$

Since σ is an automorphism and $d(N) \subseteq Z$, the equation (9) can be rearranged to yield

$$\begin{aligned} d(x)\sigma(y^*)r - r d(x)\sigma(y^*) &= d(y)r\tau(x) - d(y)\tau(x)r, \text{ for all } x, y, r \in N \text{ or} \\ d(x)(\sigma(y^*)r - r\sigma(y^*)) &= d(y)(r\tau(x) - \tau(x)r), \text{ for all } x, y, r \in N. \end{aligned} \tag{10}$$

Assume on contrary that N is not commutative and select $r, y \in N$ with $r\tau(x) - \tau(x)r \neq 0$.

Let $y = d(a)$, $a \in N$.

This gives that $\sigma(x) = \sigma(d(a)) = d(\sigma(a)) \in Z$.

Now, this Theorem becomes

$$d(x)(d(\sigma(a^*))r - rd(\sigma(a^*))) = d^2(a)(r\tau(x) - \tau(x)r),$$

That is $d^2(a)(r\tau(x) - \tau(x)r) = 0$ for all $a \in N$.

By the Lemma 1 (i), we see that the central element $d^2(a)$ cannot be a divisor of zero,

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We conclude that $d^2(a)=0$, for all $a \in N$.

But by Lemma 1(iv), this cannot be happen for non-trivial derivation d .

Thus, $r \tau(x) - \tau(x)r=0$, for all $r, y \in N$.

Since τ is an automorphism of N ,

The above expression implies that $rz - zr=0$, for all $r, z \in N$.

Hence N is a commutative ring. ◇

Theorem 3 : Let N be a prime left nearring admitting a non-trivial $(\sigma, \tau)^*$ - left derivation d such that $d(x)d(y)=d(y)d(x)$, for all $x, y \in N$. Then $(N, +)$ is abelian. More over, if N is 2-torsion free and σ, τ commute with d , then N is a commutative ring.

Proof : In view of our hypothesis , we have

$$d(x+x)d(x+y)=d(x+y)d(x+x), \text{ for all } x, y \in N.$$

This can be written has $d(x)d(x)+d(x)d(y)=d(x)d(x)+d(y)d(x)$, for all $x, y \in N$.

Hence $d(x)d(x, y)=0$, for all $x, y \in N$.

That is $d(x)d(c)=0$, for all $x \in N$. And additive commutator c .

Now, by the application of Lemma 4(iii) gives that $d(c)=0$, for all additive commutators c .

Since N is a right nearring and c is an additive commutator , cx is also an additive commutator for any $x \in N$.

Hence $d(cx)=0$, for all $x \in N$ and additive commutator c .

Thus, by the Lemma 4(iii), $c=0$ and hence $(N, +)$ is abelian.

Now, suppose that N is 2-torsion free and σ, τ commute with d .

Then the applications of Lemmas 1 and 2(i) gives that

$$\begin{aligned} d(d(x)y)d(z) &= \sigma(y^*)d^2(x) + \tau(d(x))d(y)d(z) \\ &= \sigma(y^*)d^2(x)d(z) + \tau(d(x))d(y)d(z), \text{ for all } x, y, z \in N \end{aligned}$$

$$\text{This implies that } \sigma(y^*)d^2(x)d(z) = d(d(x)y)d(z) - \tau(d(x))d(y)d(z), \text{ for all } x, y, z \in N. \tag{11}$$

Also since $d(x)d(y)=d(y)d(x)$, for all $x, y, z \in N$, we find that

$$\begin{aligned} d(d(x)y)d(z) &= d(z)d(d(x)y) \\ &= d(z)(\sigma(y^*)d^2(x) + \tau(d(x))d(y)) \\ &= d(z)\sigma(y^*)d^2(x) + d(z)\tau(d(x))d(y) \\ &= d^2(x)d(z)\sigma(y^*) + \tau(d(x))d(y)d(z), \text{ for all } x, y, z \in N. \end{aligned} \tag{12}$$

Combine (11) and (12) , we obtain

$$d^2(x)(d(z)\sigma(y^*) - \sigma(y^*)d(z)) = 0, \text{ for all } x, y, z \in N. \tag{13}$$

Now, change y by yr in (13), we obtain

$$d^2(x)\sigma(y^*)(d(z)\sigma(r^*) - \sigma(r^*)d(z)) = 0, \text{ for all } x, y, z \in N.$$

Thus, $d^2(x)N(d(z)\sigma(r^*) - \sigma(r^*)d(z)) = 0$, for all $r, x, z \in N$.

Since N is prime and τ is an automorphism, $d(z)r - rd(z) = 0$ or $d^2(x) = 0$, for all $x \in N$.

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But the last conclusion is impossible by Lemma 4(iv).

Hence, we have $d(z)r-rd(z)=0$, for all $r, z \in N$.

This implies that $d(N) \subseteq Z$.

Hence N is a commutative ring by Theorem 2. ◇

III. CONCLUSION

By definition, Near ring N is not an abelian under addition. We proved that $(N, +)$ is an abelian if $d(N) \subseteq Z$, where d is a $(\sigma, \tau)^*$ - left derivation and Z is center of N . Also, we proved that N is a commutative ring if N is 2 – torsion free and σ, τ commute with d .

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