

International Journal of Innovative Research in Science, Engineering and Technology

(A High Impact Factor, Monthly, Peer Reviewed Journal)

Visit: www.ijirset.com

Vol. 7, Issue 12, December 2018

Colouring of Finite Sets and Colouring of Edges of Finite Graphs

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ABSTRACT: This paper related to our previous papers dealing with colouring of families of disjoint sets technique, the previous papers assume any family of disjoint sets can contains infinite sets and graphs contains infinite vertices and infinite edges, this paper assume both sets and graphs are finite. Results in this paper use colouring of families of disjoint sets technique for finite sets and finite graphs. Using results in this paper you can compare between colouring of families of disjoint sets technique and colouring of graphs in graph theory.

KEYWORDS: difference in degrees of intersections, difference in degrees of vertices, local maximum , local minimum, type of colouring.

I. INTRODUCTION

In paper [1] we introduced the concept of colouring of families of disjoint sets technique as a topic of set theory. We also introduce definitions of degree of intersection, meeting and standard of meeting, minimum number of families of disjoint sets and maximum number of families of disjoint sets. In paper [2] we modify some definitions and examples of paper [1]. In paper [3] we introduce some concepts in graph theory to introduce new method for colouring edges and vertices of graph. This method has two types first related to vertices of the graph and second related to cycles. This method depend on the theory in our paper "Family of disjoint sets and Its Applications", with suitable changes. In paper [4] we prove minimum number of families of disjoint sets with known degree of intersection (minimum number of colours) and we introduce another proof for maximum number of families of disjoint sets (maximum number of colours). Also in that paper we introduce the concept types of families of disjoint sets (types of colouring) and prove some results related to this concept, and we introduce the concept restricted families of disjoint sets (restricted colouring), restricted colouring is colouring in which we add a condition. Suppose we have small identical cubes, we want to colour these cubes such that any two adjacent small cubes coloured by two different colours, and any big cube consists of twenty seven small identical cubes coloured by twenty seven different colours, how many colours will be used. This problem of cubes can be solved by using restricted colouring. In paper [1] we assume any family of disjoint sets contains finite or infinite sets, and proved many results related to family of disjoint sets, in this paper we assume any family of disjoint sets consists of finite sets, so we can used the results in paper [1]. The colouring which we introduced in paper [1] we called it colouring of families of disjoint sets technique, for this technique of colouring we used the terminology sort of colouring and type of colouring. A sort of colouring is colouring expected not defined and need to be defined, also you can't determine edges or vertices adjacent or not adjacent unless adjacency is defined. Sort of colouring depend on degree of intersection or degree of colouring, different degrees determine different types of colouring, this means when sort of colouring is well defined, it is easy to determine many types using different degrees of intersection or different degrees of colouring. In paper [4] we introduce the concept types of families of disjoint sets (types of colouring) and prove some results related to this concept.

In this paper colouring in graph theory means colouring of edges of graphs. In section three and section four we introduce examples to show how to apply theory of paper [1] to finite sets and finite graphs.

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In section five and section six we introduce the concepts of difference in degrees of intersections at number of intersections and difference in degrees of vertices at number of vertices, we introduce some results and examples related to these concepts.

II. RELATED WORK

Definition 2.1. Let $\chi_1, \chi_2, \chi_3, \dots, \chi_m$ be m families of disjoint sets and $X_{i_1}, X_{i_2}, X_{i_3}, \dots, X_{i_n}$ be n sets such that any two sets belong to two different families of disjoint sets, and X is nonempty set, $X = X_{i_1} \cap X_{i_2} \cap X_{i_3} \cap \dots \cap X_{i_n}$, $X = \{x : x \in X_{ij}, 1 \leq i \leq r, 1 \leq j \leq n, n \leq r, X_{ij} \in \chi_i\}$ then the positive integer n is called degree of intersection of X [1].

Definition 2.2. Let $\chi_1, \chi_2, \chi_3, \dots, \chi_r$ be r families of disjoint sets and X is nonempty set, $X = \chi_1 \cap \chi_2 \cap \chi_3 \cap \dots \cap \chi_r$, then the intersection X is called represent the families $\chi_1, \chi_2, \chi_3, \dots, \chi_r$, and we used the notation $X = \chi_1 \cap \chi_2 \cap \chi_3 \cap \dots \cap \chi_r$, or $X = (\chi_1, \chi_2, \chi_3, \dots, \chi_r)$ [1].

Proposition 2.3. Let $\chi_1, \chi_2, \chi_3, \dots, \chi_r$ be r families of disjoint sets and $X_1, X_2, X_3, \dots, X_w$ be W nonempty intersections, the family of disjoint sets χ_i appear w_i times at W intersections and the family of disjoint sets χ_j appear w_j time at W intersections, then we say the family of disjoint sets χ_i meet the family of disjoint sets χ_j at these W intersections if and only if $w + 1 \leq w_i + w_j$.

Definition 2.4. Let $\chi_1, \chi_2, \chi_3, \dots, \chi_m$ be m families of disjoint sets, $X_1, X_2, X_3, \dots, X_w$ each of degree n , and W be the minimum number of nonempty intersections, such that any two families of disjoint sets meet each other at these W intersections, then these W intersections called standard of meeting for these m families of disjoint sets [1].

Theorem 2.5. Let W intersections each with degree of intersection equal n , m be number of local maximum families of disjoint sets, meet each other at W intersections, then we have $m(\frac{w+1}{2}) \leq wn$ for W odd, and $m(\frac{w}{2} + 1) \leq wn$ for W even [1].

Definition 2.6. Let $\chi_1, \chi_2, \chi_3, \dots, \chi_m$ be m families of disjoint sets, meet each other at $X_1, X_2, X_3, \dots, X_w$ intersections each of degree n , then m is called local maximum families of disjoint sets at W intersections, if any two families of disjoint sets meet each other at these W intersections, and any family of disjoint sets has appearance either equal $\frac{w+1}{2}$ or $\frac{w+3}{2}$ for W odd such that not all families of disjoint sets has appearance equal $\frac{w+3}{2}$, and for W even we have three types, first all families of disjoint sets has appearance equal $\frac{w}{2} + 1$, second appearance equal $\frac{w}{2} + 1$ and $\frac{w}{2} + 2$, not all $\frac{w}{2} + 2$ but at least has one appearance equal $\frac{w}{2} + 2$, third all families of disjoint sets has appearance equal $\frac{w}{2} + 1$, except only one family of disjoint sets has appearance equal $\frac{w}{2}$. Maximum of local maximum families of disjoint sets called maximum families of disjoint sets [1].

Definition 2.7. Let $\chi_1, \chi_2, \chi_3, \dots, \chi_m$ be m families of disjoint sets, $X_1, X_2, X_3, \dots, X_w$ be W nonempty intersections each of degree n , then m families of disjoint sets is called local minimum families of disjoint sets at W intersections, if $m(w-1) \leq nw$ such that $m \neq n$. When $m=n$, then m families of disjoint sets is called local minimum families of disjoint sets at W if $m(w-k) \leq nw < m(w-k+1)$ such that $m \neq n$, where $k = 2, 3, 4, \dots$. Minimum of local minimum families of disjoint sets called minimum families of disjoint sets [1].

Theorem 2.8. Minimum number of families of disjoint sets with degree of intersection n equal $n+1$. [4].

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Theorem 2.9. Maximum number of families of disjoint sets with degree of intersection n equal $2n - 1$. [4].

Remark 2.10. If n is degree of intersection, w is standard of meeting for m families of disjoint sets number of families of disjoint sets then $3 \leq w \leq 2n - 1$ [1].

Proposition 2.11. Let $\chi_1, \chi_2, \chi_3, \dots, \chi_m$ be m families of disjoint sets, $X_1, X_2, X_3, \dots, X_w$ are nonempty different intersections each of degree n , and t_i number of times the family of disjoint sets χ_i appear at these w intersections, then we have

$$\sum_{i=1}^m t_i = wn.$$

Remark 2.12. Let $X_1, X_2, X_3, \dots, X_w$ be standard of meeting for m families of disjoint sets $\chi_1, \chi_2, \chi_3, \dots, \chi_m$

With degree of intersection n , it is trivial case $w = 1$. When $w = 1$ we have $m = n$. Also standard of meeting consist of X_1 and X_2 where $X_1 = (1, 2, 3, \dots, n - 1, n)$ and $X_2 = (1, 2, 3, \dots, n - 2, n - 1, n + 1)$, X_1, X_2 is not trivial. For $w = 2$, if standard of meeting for $n + 1$ families of disjoint sets $\chi_1, \chi_2, \chi_3, \dots, \chi_n, \chi_{n+1}$, because X_1 does not meet X_2 . If X_1 is standard of meeting for n families of disjoint sets $\chi_1, \chi_2, \chi_3, \dots, \chi_n$, also X_1, X_2 is standard of meeting for n families of disjoint sets $\chi_1, \chi_2, \chi_3, \dots, \chi_n$. Since $X_1 = X_2$, not allowed, two intersections have same representation (repeated), then standard of meeting X_1, X_2 means standard of meeting X_1 , and standard of meeting X_1, X_2 is trivial [1].

Definition 2.13. Let $\chi_1, \chi_2, \chi_3, \dots, \chi_m$ be m families of disjoint sets, meet each other at $X_1, X_2, X_3, \dots, X_w$ intersections each of degree n , then m is called local maximum families of disjoint sets at w intersections, if any two families of disjoint sets meet each other at these w intersections, and any family of disjoint sets has appearance either equal $\frac{w+1}{2}$

or $\frac{w+3}{2}$ for w odd such that not all families of disjoint sets has appearance equal $\frac{w+3}{2}$, and for w even we have three

types, first all families of disjoint sets has appearance equal $\frac{w}{2} + 1$, second appearance equal $\frac{w}{2} + 1$ and $\frac{w}{2} + 2$, not all

$\frac{w}{2} + 2$ but at least has one appearance equal $\frac{w}{2} + 2$, third all families of disjoint sets has appearance equal $\frac{w}{2} + 1$, except

only one family of disjoint sets has appearance equal $\frac{w}{2}$. Maximum of local maximum families of disjoint sets called

maximum families of disjoint sets [1].

Definition 2.14. Let $\chi_1, \chi_2, \chi_3, \dots, \chi_m$ be m families of disjoint sets, $X_1, X_2, X_3, \dots, X_w$ be w nonempty intersections each of degree n , then m families of disjoint sets is called local minimum families of disjoint sets at w intersections with t

families of disjoint sets appear at each intersection, if $m(w-1) + t = nw$ where $0 \leq t < w$ such that $m \neq n$. When $m = n$, then

m families of disjoint sets is called local minimum families of disjoint sets at w intersections with t families of disjoint sets appear at each intersection if $m(w-k) + t = nw$ where $0 \leq t < w$, $k = 2, 3, 4, \dots$ and $m(w-k) \leq nw < m(w-k+1)$

such that $m \neq n$. Minimum of all local minimum families of disjoint sets called minimum families of disjoint sets [2].

Remark 2.15. By colouring m families of disjoint sets we mean an assignment of colour to each family of disjoint sets so that no two different families of disjoint sets have the same colour, this means all sets belong to one family of disjoint sets will be in same colour. Two different families of disjoint sets intersect each other, so they have two different colours. The idea of colouring families of disjoint sets have been used in paper [1] and any family of disjoint sets labelled by number or colour.

Example 2.16. Let degree of intersection n equal three, regions of a map are sets with degree of intersection three. Let for m be number families of disjoint sets, (number of colours used to draw map), using the rule $m \leq 2n - 1$,

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substitute $n = 3$, we have $m \leq 5$, this means maximum number of colour to draw a map is 5. If you used six or more than six colours, this means by small effort you can reduce the number of colours to five [1].

Definition 2.17. Colouring edges at vertex is defined by assignment one of m colours to each edge so that adjacent edges have not same colour [3].

Definition 2.18. Degree of colouring edges at vertex is maximum number of edges have different colours at vertex [3].

Definition 2.19. Colouring of vertices at vertex is defined by assignment one of m colours to each vertex so that adjacent vertices have different colours and vertices jointed adjacent edges have different colours [3].

Definition 2.20. Degree of colouring vertices at vertex is maximum number of vertices have different colours at vertex [3].

Definition 2.21. Colouring of edges at cycle of length n is defined by assignment one of m colours to each edge of graph G so that adjacent edges have different colours edges of any cycle of length n have different colours (or cycle of length less than n edges and path of n edges have different colours)[3].

Definition 2.22. Degree of colouring at cycle of length n for edges is maximum number of edges have different colours at cycle of length n [3].

Definition 2.23. Colouring of vertices at cycle of length n is defined by assignment one of m colours to each vertex of graph G so that adjacent vertices have different colours and vertices of any cycle of length n vertices have different colours, (or length less than n vertices and any path of n vertices have different colours) n [3].

Definition 4.24. Degree of colouring at cycle of length n for vertices is maximum number of vertices have different colours at cycle of length n [3].

Definition 2.25. A point is a corner of a map if and only if it belongs to the closures of at least three regions [11].

Definition 2.26. Two regions of a map are adjacent if their respective closures have a common point that is not a corner of the map [11].

Theorem 2.27. The sum of degrees of vertices of a graph G is twice the number of edges [9].

III. FAMILIES OF DISJOINT SETS CONTAIN FINITE SETS

In paper [1] we assume any family of disjoint sets can contains finite or infinite sets, and proved many results related to family of disjoint sets, in this paper we assume any family of disjoint sets consists of finite sets, so we can used the results in paper [1]. Let $\chi_1, \chi_2, \chi_3, \dots, \chi_m$ be m families of disjoint sets, $X_1, X_2, X_3, \dots, X_w$ are nonempty different intersections, if the intersections $X_1, X_2, X_3, \dots, X_w$ have different degrees of intersection, the rule $\sum_{i=1}^m t_i = wn$ can be written

as $\sum_{i=1}^m t_i = \sum_{j=1}^w d_j$ where t_i is number of times the family of disjoint sets χ_i appear at these W intersections and d_j is degrees

of intersection of X_j , or can be written as $m(\frac{w+1}{2}) \leq \sum_{j=1}^w d_j$ if w is odd and as $m(\frac{w}{2} + 1) \leq \sum_{j=1}^w d_j$ if w is even.

Example 3.1. Let $\chi_1, \chi_2, \chi_3, \chi_4$ be 4 families of disjoint sets and X_1, X_2, X_3 be 3 nonempty intersections each of degree 3, where two of families of disjoint sets $\chi_1, \chi_2, \chi_3, \chi_4$ meet each other at intersections X_1, X_2, X_3 , where $X_1 = (1,2,3)$, $X_2 = (1,2,4)$, $X_3 = (1,3,4)$. If $Y_1 = (1,2,3)$, $Y_2 = (1,2,4)$, $Y_3 = (3,4)$, also any two of families of disjoint sets $\chi_1, \chi_2, \chi_3, \chi_4$ meet each other at intersections Y_1, Y_2, Y_3 .

Remark 3.2. The above example can represent a map of countries coloured by four colours (four families of disjoint sets). The four colours labeled by 1,2,3,4, and $X_1 = (1,2,3)$, $X_2 = (1,2,4)$, $X_3 = (1,3,4)$ represent three intersections each of degree three. $Y_1 = (1,2,4)$, $Y_2 = (1,2,3)$, $Y_3 = (3,4)$, represent three intersections two of degree three and one of degree two.

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Example3.3. Suppose we have finite sets partitioned into 4 families of disjoint sets $\chi_1, \chi_2, \chi_3, \chi_4$, and all intersections are ten $X_1, X_2, X_3, \dots, X_{10}$, be one intersection of degree three and rest all of degree two. Using the rule $3 \leq w \leq 2n-1$ we have $3 \leq w \leq 5$. Let be $w = 3$, and $X_1 = (1,2,3)$, $X_2 = (1,4)$, $X_3 = (2,4)$. At these three intersections the family of disjoint sets labeled by 3 does not meet the family of disjoint sets labeled by 4. Let be $w = 4$, and $X_1 = (1,2,3)$, $X_2 = (1,4)$, $X_3 = (2,4)$, $X_4 = (3,4)$. At these four intersections the family of disjoint sets labeled by 2 does not meet the family of disjoint sets labeled by 3. Let be $w = 5$, and $X_1 = (1,2,3)$, $X_2 = (1,4)$, $X_3 = (2,4)$, $X_4 = (3,4)$, $X_5 = (2,3)$. At these five intersections the family of disjoint sets labeled by 1 appear twice, any family of disjoint sets should have appear three times. Therefore in three cases, the number of families of disjoint sets can't be four if we used the rule $m(\frac{w+1}{2}) = \sum_{j=1}^w d_j$

or the rule $m(\frac{w}{2}+1) = \sum_{j=1}^w d_j$, for $w = 3$, we have $4(2) = 8 > 3 + 2 + 2 = 7$, for $w = 4$ we have $4(3) = 12 > 3 + 2 + 2 + 2 = 9$ and for $w = 5$ we have $4(3) = 12 > 3 + 2 + 2 + 2 + 2 = 11$ and in three cases, if the number of families of disjoint sets three, it satisfies the rule, for $w = 3$, we have $3(2) = 6 \leq 3 + 2 + 2 = 7$, for $w = 4$ we have $3(3) = 9 \leq 3 + 2 + 2 + 2 = 9$ and for $w = 5$ we have $3(3) = 9 \leq 3 + 2 + 2 + 2 + 2 = 11$.

Example3.4. Suppose we have finite sets with 15 intersections as follows: 4 intersections each of degree 5, one intersection of degree 4, and the remaining intersections each with degree less than 4. If we want to partitioned these finite sets into m families of disjoint sets $\chi_1, \chi_2, \chi_3, \dots, \chi_m$, what is minimum and maximum value of m .

Solution: For minimum value of m , using the rule $\sum_{i=1}^m t_i = \sum_{j=1}^w d_j$ or $m(\frac{w+1}{2}) \leq \sum_{j=1}^w d_j$ or $m(\frac{w}{2}+1) \leq \sum_{j=1}^w d_j$ we have

$2m \leq 5 + 5 + 5$ or $3m \leq 5 + 5 + 5 + 5 = 20$ therefore $m = 6$ or $m = 7$ of m . Minimum value of m equal 6. For maximum value of m we have $3m \leq 5 + 5 + 5 + 5 + 4$ or we have $2m \leq 5 + 5 + 5$, maximum value of m equal 8.

Example3.5. Suppose we have finite sets with 6 intersections as follows: 5 intersections each of degree 3, one intersection of degree 6. If we want to partitioned these finite sets into m families of disjoint sets $\chi_1, \chi_2, \chi_3, \dots, \chi_m$, what is minimum and maximum value of m .

Solution: For minimum and maximum value of m , using the rule $\sum_{i=1}^m t_i = \sum_{j=1}^w d_j$ or $m(\frac{w+1}{2}) \leq \sum_{j=1}^w d_j$ or $m(\frac{w}{2}+1) \leq \sum_{j=1}^w d_j$ we

have two value m equal 5 and m equal 6, the value 6 is trivial and the value 5 rejected since there is no value less than trivial value.

Example3.6. Suppose we have finite sets with ten intersections, each intersection with degree of intersection three. If we want to partitioned these finite sets into m families of disjoint sets $\chi_1, \chi_2, \chi_3, \dots, \chi_m$, what is minimum and maximum value of m .

Solution: For minimum and maximum value of m , we using the rule $\sum_{i=1}^m t_i = \sum_{j=1}^w d_j$ or $m(\frac{w+1}{2}) \leq \sum_{j=1}^w d_j$ or $m(\frac{w}{2}+1) \leq \sum_{j=1}^w d_j$, we

have for $w = 3$, $2m \leq 3 + 3 + 3$ and for $w = 4$, $3m \leq 3 + 3 + 3 + 3$ therefore $m = 4$ or we can used Theorem 2.10. $m = n + 1 = 3 + 1 = 4$. For maximum value of m , $w = 5$, we have $3m \leq 3(5)$, $m = 5$, or we can used Theorem 2.11. $m = 2n - 1 = 2(3) - 1 = 5$.

Example3.7. Suppose we have finite sets with eleven intersections as follows: 5 intersections each of degree 3, 5 intersections each of degree 4, one intersection of degree 5. If we want to partitioned these finite sets into m families of disjoint sets $\chi_1, \chi_2, \chi_3, \dots, \chi_m$, what is minimum and maximum value of m . The solution explained in Table (I).

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Number of intersections w	minimum families of disjoint sets	maximum families of disjoint sets
3	6	6
4	5	5
5	5	8
6	5	6
7	5	7
8	5	6
9	5	6

Table (I)

IV. COLOURING OF EDGES OF FINITE GRAPHS

In paper [1] we assume any family of disjoint sets can contains finite or infinite sets, and proved many results related to family of disjoint sets , in paper [3] introduced sorts of colouring for finite graphs or infinite graphs used results in paper [1]. In this paper any graphs consists of finite edges, so we can used the results in paper [1]. Let m be number of colour classes, labeled by $1,2,3,\dots,m$ and $v_1, v_2, v_3, \dots, v_w$ be w vertices and these vertices of different degrees, the rule

$\sum_{i=1}^m t_i = wn$ can be written as $\sum_{i=1}^m t_i = \sum_{j=1}^w d_j$ where t_i is number of times the colour class i appear at these w vertices and d_j is degrees of vertex v_j , or can be written as $m(\frac{w+1}{2}) \leq \sum_{j=1}^w d_j$ if w is odd and $m(\frac{w}{2} + 1) \leq \sum_{j=1}^w d_j$ if w is even.

Example4.1. Let $1,2,3,4$ be 4 colours used to coloured edges of finite graph and v_1, v_2, v_3 be 3 vertices each vertex of degree 3, (each vertex with degree of colouring 3) where any two colours of $1,2,3,4$ meet each other at vertices v_1, v_2, v_3 , where $v_1 = (1,2,3)$, $v_2 = (1,2,4)$, $v_3 = (1,3,4)$. If $u_1 = (1,2,4)$, $u_2 = (1,2,3)$, $u_3 = (3,4)$, also be 3 vertices each vertex of degree 3, any two colours of $1,2,3,4$ meet each other at vertices u_1, u_2, u_3 .

Remark4.2. The above example can represent a graph, edges of the graph coloured by four colours. The four colours labeled by $1,2,3,4$, (four edges labeled by $1,2,3,4$,) and $v_1 = (1,2,3)$, $v_2 = (1,2,4)$, $v_3 = (1,3,4)$ represent three vertices each of degree three. $u_1 = (1,2,4)$, $u_2 = (1,2,3)$, $u_3 = (3,4)$, represent three vertices two of degree three and one of degree two.

Example4.3. In finite graph G suppose we have finite edges partitioned into four colour classes labeled by $1,2,3,4$, and all vertices are ten $v_1, v_2, v_3, \dots, v_{10}$, one vertex of degree three (degree of colouring) and all the remaining vertices of degree two. Using the rule $3 \leq w \leq 2n-1$ we have $3 \leq w \leq 5$, where w be number of vertices at which the four colour classes meet. At $w = 3$, let $v_1 = (1,2,3)$, $v_2 = (1,4)$, $v_3 = (2,4)$. At these three vertices the colour class labeled by 3 does not meet the colour class labeled by 4. Let be $w = 4$, and $v_1 = (1,2,3)$, $v_2 = (1,4)$, $v_3 = (2,4)$, $v_4 = (3,4)$. At these four vertices the colour class labeled by 2 does not meet the colour class labeled by 3. Let be $w = 5$, and $v_1 = (1,2,3)$, $v_2 = (1,4)$, $v_3 = (2,4)$, $v_4 = (3,4)$, $v_5 = (2,3)$. At these five vertices the colour class labeled by 1 appear twice, any colour class should have appear three times. Therefore in three cases, the number of colour classes can't be four if we used the rule $m(\frac{w+1}{2}) = \sum_{j=1}^w d_j$ or the rule $m(\frac{w}{2} + 1) = \sum_{j=1}^w d_j$, for $w = 3$, we have $4(2) = 8 > 3 + 2 + 2 = 7$, for $w = 4$ we have $4(3) = 12 > 3 + 2 + 2 + 2 = 9$ and for $w = 5$ we have be $4(3) = 12 > 3 + 2 + 2 + 2 + 2 = 11$ and in the three cases,

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if the number of colour classes three, it satisfies the rule, for $w = 3$, we have $3(2) = 6 \leq 3 + 2 + 2 = 7$, for $w = 4$ we have $3(3) = 9 \leq 3 + 2 + 2 + 2 = 9$ and for $w = 5$ we have $3(4) = 12 \leq 3 + 2 + 2 + 2 + 2 = 11$.

Example 4.4. Suppose we have finite graph with 15 vertices as follows: 4 vertices each of degree 5, one vertex of degree 4, and the remaining vertices each with degree less than 4. If we want to colour edges of this graph by m colour classes labeled by $1, 2, 3, \dots, m$, what is minimum and maximum value of m .

$m(\frac{w}{2} + 1) \leq \sum_{j=1}^w d_j$ we have $2m \leq 5 + 5 + 5$ or $3m \leq 5 + 5 + 5 + 5 = 20$ therefore $m = 6$ or $m = 7$ of m . Minimum value of m equal 6. For maximum value of m we have $3m \leq 5 + 5 + 5 + 5 + 4$ or we have $2m \leq 5 + 5 + 5$, maximum value of m equal 8.

Example 4.5. Suppose we have simple graph on six vertices, with degree sequence $(3, 3, 3, 3, 3, 6)$. If we want to colour edges of this graph by m colour classes labeled by $1, 2, 3, \dots, m$, what is minimum and maximum value of m .

Solution: For minimum and maximum value of m , using the rule $\sum_{i=1}^m t_i = \sum_{j=1}^w d_j$ or $m(\frac{w+1}{2}) \leq \sum_{j=1}^w d_j$ or $m(\frac{w}{2} + 1) \leq \sum_{j=1}^w d_j$ we have two value m equal 5 and m equal 6, the value 6 is trivial and the value 5 is rejected since there is no value less than trivial value.

Example 4.6. Suppose we have simple graph on ten vertices, each vertex with degree three. If we want to colour edges of this graph by m colour classes labeled by $1, 2, 3, \dots, m$, what is minimum and maximum value of m .

Solution: For minimum and maximum value of m , we using the rule $\sum_{i=1}^m t_i = \sum_{j=1}^w d_j$ or $m(\frac{w+1}{2}) \leq \sum_{j=1}^w d_j$ or $m(\frac{w}{2} + 1) \leq \sum_{j=1}^w d_j$, we have for $w = 3$, $2m \leq 3 + 3 + 3$ and for $w = 4$, $3m \leq 3 + 3 + 3 + 3$ therefore $m = 4$ or we can use Theorem 2.8. $m = n + 1 = 3 + 1 = 4$. For maximum value of m , $w = 5$, we have $3m \leq 3(5)$, $m = 5$, or we can use Theorem 2.9. $m = 2n - 1 = 2(3) - 1 = 5$.

Example 4.7. Suppose we have simple graph on eleven vertices, with degree sequence $(3, 3, 3, 3, 3, 4, 4, 4, 4, 4, 5)$. If we want to colour edges of this graph by m colour classes labeled by $1, 2, 3, \dots, m$, what is minimum and maximum value of m . The solution explained in Table (II).

Number of vertices w	minimum colour classes	maximum colour classes
3	6	6
4	5	5
5	5	8
6	5	6
7	5	7
8	5	6
9	5	6

Table (2)

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V. DIFFERENCE IN DEGREE OF INTERSECTIONS

The results in paper [1] suppose infinite sets partition into families of disjoint sets, the results in paper [3] suppose graph of infinite edges partition into edges classes (colour classes), in this paper we use same results in paper [1] for finite sets partition into families of disjoint sets and use same results in paper [3] for finite edges partition into edges classes (colour classes). In case of infinite sets and infinite edges all intersections of degree n and all vertices of degree n , in case of finite sets and finite edges we have different degrees of intersection and different degrees of vertex, with restrictions $3 \leq w \leq 2n-1$ and $n+1 \leq m \leq 2n-1$, where W is number of intersections where m families of disjoint sets any two of them meet each other or W is number of vertices where m edges (colour classes) any two of them meet each other. For the number of families of disjoint sets m or the number of edges (colour classes) m , if we can't count the minimum value of m , $m = n + 1$, then from definition of trivial value of m , we consider the trivial value of m the minimum value of m . In case of infinite sets and infinite edges we deal with families of disjoint sets and edges classes, while in case of finite sets we deal with both families of disjoint sets and sets, and in case of finite edges we deal with both edges and edges classes.

Definition 5.1. Suppose we have finite sets with k intersections, and maximum degree of intersection equal n , these finite sets partitioned into m families of disjoint sets $\chi_1, \chi_2, \chi_3, \dots, \chi_m$ at w intersections $X_1, X_2, X_3, \dots, X_w$, where $w \leq k$, and w intersections are maximum degrees among these k intersections, the value $nw - [\deg(X_1) + \deg(X_2) + \deg(X_3) + \dots + \deg(X_w)]$ is called difference in degrees of intersections at intersections $X_1, X_2, X_3, \dots, X_w$.

Remark 5.2. From Definition 5.1 if D is the difference in degrees of intersection at intersections $X_1, X_2, X_3, \dots, X_w$ then we can write $D = nw - \sum_{j=1}^w d(X_j)$.

Example 5.3. Let $\chi_1, \chi_2, \chi_3, \chi_4, \chi_5$ be 5 families of disjoint sets and X_1, X_2, X_3 be 3 nonempty intersections with maximum degree of intersection 4. $X_1 = (1,2,3,4)$, $X_2 = (1,2,5)$, $X_3 = (3,4,5)$, any two families of disjoint sets meet each other at these three intersections. If D is the difference in degrees of intersection at intersections X_1, X_2, X_3 then

$$D = nw - \sum_{j=1}^w d(X_j) = 3(4) - (4 + 3 + 3) = 2.$$

Remark 5.4. From Theorem 2.5. and Definition 5.1. if k are finite sets partitioned into m maximum families of disjoint sets $\chi_1, \chi_2, \chi_3, \dots, \chi_m$ at w intersections $X_1, X_2, X_3, \dots, X_w$, where $w \leq k$, and D is the difference in degrees of intersection at intersections $X_1, X_2, X_3, \dots, X_w$ then $nw - m(\frac{w+1}{2}) = D$ if W is odd, and $nw - m(\frac{w}{2} + 1) = D$ if W is even.

Proposition 5.5. Let $\chi_1, \chi_2, \chi_3, \dots, \chi_m$ be m families of disjoint sets, $X_1, X_2, X_3, \dots, X_w$ are nonempty different intersections and maximum degree of intersection equal n , and t_i number of times the family of disjoint sets χ_i appear at these W intersections, then we have $\sum_{i=1}^m t_i = \sum_{j=1}^w d(X_j)$, where $d(X_j)$ is degree of the intersection X_j .

Proof: Since the appearance for m families of disjoint sets at W intersections $X_1, X_2, X_3, \dots, X_w$ and summation of all degrees equal $\sum_{j=1}^w d(X_j)$, $d(X_1) + d(X_2) + d(X_3) + \dots + d(X_w) = \sum_{j=1}^w d(X_j)$ each intersection of degree n , then summation of appearance for all m families of disjoint sets equal $\sum_{i=1}^m t_i$, since the appearance for m families of disjoint sets at W intersections $X_1, X_2, X_3, \dots, X_w$, therefore $\sum_{i=1}^m t_i = \sum_{j=1}^w d(X_j)$.

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Proposition 5.6. Let $\chi_1, \chi_2, \chi_3, \dots, \chi_m$ be m families of disjoint sets, $X_1, X_2, X_3, \dots, X_w$ are nonempty different intersections and maximum degree of intersection equal n , and t_i number of times the family of disjoint sets χ_i appear at these W intersections, and D is the difference in degrees of intersection at W intersections then we have $wn - \sum_{i=1}^m t_i = D$.

Proof: From Remark 5.2. we have $D = wn - \sum_{j=1}^w d(X_j)$, and from Proposition 5.5. we have $\sum_{i=1}^m t_i = \sum_{j=1}^w d(X_j)$, therefore $wn - \sum_{i=1}^m t_i = D$.

Proposition 5.7. Suppose we have finite sets with k intersections (these k intersections have different degrees of intersection), and maximum degree of intersection equal n , if $\chi_1, \chi_2, \chi_3, \dots, \chi_m$ is maximum families of disjoint sets any two of them meet each other at these intersections $X_1, X_2, X_3, \dots, X_w$. If the difference in degrees of intersection at intersections $X_1, X_2, X_3, \dots, X_w$ equal $\frac{n(w-1)}{2}$ for w odd, and equal $\frac{n(w-2)}{2}$ for w even, then m is trivial families of disjoint sets.

Proof: From Remark 5.2. we have $D = wn - \sum_{j=1}^w d(X_j)$, and from Proposition 5.5. we have $\sum_{i=1}^m t_i = \sum_{j=1}^w d(X_j)$, therefore we can write have $D = wn - \sum_{i=1}^m t_i$, and $\sum_{i=1}^m t_i = wn - D$, since we given have $D = \frac{n(w-1)}{2}$, for w odd, then we can write and

$m \frac{(w+1)}{2} = \sum_{i=1}^m t_i = wn - D = wn - \frac{n(w-1)}{2} = \frac{n(w+1)}{2}$. Therefore $m = n$, and m is trivial families of disjoint sets.

Also for w even we have $m \left(\frac{w}{2} + 1\right) = \sum_{i=1}^m t_i = wn - D = wn - \frac{n(w-2)}{2} = n \left(\frac{w}{2} + 1\right)$. Therefore $m = n$, and m is trivial families of disjoint sets.

Remark 5.8. From Proposition 5.6. we have $wn - \sum_{i=1}^m t_i = D$, since minimum value of t_i for each family of disjoint sets equal $\frac{w+1}{2}$ for w odd or equal $\frac{w}{2} + 1$ for w even, then $m \frac{(w+1)}{2} \leq wn - D$, for w odd and $m \left(\frac{w}{2} + 1\right) \leq wn - D$, for w even.

Proposition 5.9. Suppose we have finite sets with k intersections (these k intersections have different degrees of intersection), and maximum degree of intersection equal n , if $\chi_1, \chi_2, \chi_3, \dots, \chi_m$ any families of disjoint sets any two of them meet each other at these intersections $X_1, X_2, X_3, \dots, X_w$. If the difference in degrees of intersection at intersections $X_1, X_2, X_3, \dots, X_w$ equal $\frac{n(w-1)}{2}$ for w odd, and equal $\frac{n(w-2)}{2}$ for w even, then m is trivial families of disjoint sets.

Proof: From Remark 5.8. and for w odd we have $m \frac{(w+1)}{2} \leq wn - D$, since $D = \frac{n(w-1)}{2}$, therefore we have $m \frac{(w+1)}{2} \leq n \left(\frac{w+1}{2}\right)$, since $m < n$ is impossible, therefore $m = n$, and m is trivial families of disjoint sets.

For w even we have $m \left(\frac{w}{2} + 1\right) \leq n \left(\frac{w}{2} + 1\right)$, therefore $m = n$, and m is trivial families of disjoint sets.

Remark 5.10. From the relation $\sum_{i=1}^m t_i = wn$, where $\chi_1, \chi_2, \chi_3, \dots, \chi_m$ be m families of disjoint sets, $X_1, X_2, X_3, \dots, X_w$ are nonempty different intersections each of degree n , and t_i number of times the family of disjoint sets χ_i appear at these W intersections, we can write $mt \leq wn$, where t is minimum value of t_i , $1 \leq i \leq m$, $\frac{w+1}{2} \leq t \leq w$ for W odd, or $\frac{w}{2} + 1 \leq t \leq w$ for W even. We know $n(w-1) < mt \leq wn$, therefore write $0 \leq wn - mt \leq (n-1)$.

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VI. DIFFERENCE IN DEGREES OF VERTICES

In this section we introduce results in graph theory analogous to results in section five.

Definition6.1. Suppose in graph G we have finite edges with k vertices, and maximum degree of vertices equal n , these finite edges coloured by m colour classes $1,2,3,\dots,m$ at w vertices $v_1, v_2, v_3, \dots, v_w$, where $w \leq k$, and w vertices are maximum degrees among these k vertices, the value $wn - [\deg(v_1) + \deg(v_2) + \deg(v_3) + \dots + \deg(v_w)]$ is called difference in degrees of vertices at vertices $v_1, v_2, v_3, \dots, v_w$.

Remark6.2. From Definition6.1. if D is the difference in degrees of vertices at vertices $v_1, v_2, v_3, \dots, v_w$ then we can write

$$D = wn - \sum_{j=1}^w d(v_j).$$

Example6.3. Let $1,2,3,4,5$ be 5 colour classes to coloured edges and v_1, v_2, v_3 be 3 vertices with maximum degree of vertices 4. $v_1 = (1,2,3,4)$, $v_2 = (1,2,5)$, $v_3 = (3,4,5)$, any two colour classes meet each other at these three vertices. If D is the difference in degrees of vertices at vertices v_1, v_2, v_3 then $D = wn - \sum_{j=1}^w d(v_j) = 3(4) - (4 + 3 + 3) = 2$.

Remark6.4. From Theorem2.5. and Definition6.1. if k are finite edges partitioned into m maximum colour classes labeled by $1,2,3,\dots,m$ at w vertices $v_1, v_2, v_3, \dots, v_w$, where $w \leq k$, and D is the difference in degrees of vertices at vertices $v_1, v_2, v_3, \dots, v_w$ then $wn - m(\frac{w+1}{2}) = D$ if w is odd, and $wn - m(\frac{w}{2} + 1) = D$ if w is even.

Proposition6.5. Let $1,2,3,\dots,m$ be m colour classes, $v_1, v_2, v_3, \dots, v_w$ be w vertices with maximum degree of vertex equal n , and t_i number of times the colour class i appear at these w vertices, then we have $\sum_{i=1}^m t_i = \sum_{j=1}^w d(v_j)$, where $d(v_j)$ is the degree of the vertex v_j .

Proof: Let the appearance for the colour class i at w vertices $v_1, v_2, v_3, \dots, v_w$ equal t_i and summation of all degrees of vertices equal $\sum_{j=1}^w d(v_j)$, since the appearance for all m colour classes at w vertices $v_1, v_2, v_3, \dots, v_w$, then

$$\sum_{i=1}^m t_i = d(v_1) + d(v_2) + d(v_3) + \dots + d(v_w) = \sum_{j=1}^w d(v_j).$$

Proposition6.6. Let $1,2,3,\dots,m$ be m colour classes and $v_1, v_2, v_3, \dots, v_w$ be w vertices with maximum degree of vertex equal n , and t_i number of times the colour class i appear at these w vertices, and D is the difference in degrees of vertices at vertices $v_1, v_2, v_3, \dots, v_w$ then we have $wn - \sum_{i=1}^m t_i = D$.

Proof: From Remark6.2. we have $D = wn - \sum_{j=1}^w d(v_j)$, and from Proposition6.5. we $\sum_{i=1}^m t_i = \sum_{j=1}^w d(v_j)$, therefore $wn - \sum_{i=1}^m t_i = D$.

Proposition6.7. Suppose we have finite sets with k vertices (these k vertices have different degrees of vertices), and maximum degree of vertex equal n , if $1,2,3,\dots,m$ is maximum number of colour classes any two of these colour classes meet each other at these vertices $v_1, v_2, v_3, \dots, v_w$. If the difference in degrees of vertices at vertices $v_1, v_2, v_3, \dots, v_w$ equal or greater than $\frac{n(w-1)}{2}$ for w odd, and equal or greater than $\frac{n(w-2)}{2}$ for w even, then m is trivial colour classes.

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Proof: From Remark6.2. we have $D = wn - \sum_{j=1}^w d(X_j)$, and from Proposition6.5. we have $\sum_{i=1}^m t_i = \sum_{j=1}^w d(X_j)$, therefore we can write have $D = wn - \sum_{i=1}^m t_i$, and $\sum_{i=1}^m t_i = wn - D$, since we given $D = \frac{n(w-1)}{2}$, for w odd, then we can write and $m \frac{(w+1)}{2} = \sum_{i=1}^m t_i = wn - D = wn - \frac{n(w-1)}{2} = \frac{n(w+1)}{2}$, then $m = n$. If $D > \frac{n(w-1)}{2}$, then $m < n$, impossible $m \leq n$, it is not true therefore $m = n$, and m is trivial colour classes.

Also for w even we have $m \frac{(w+1)}{2} = \sum_{i=1}^m t_i = wn - D = wn - \frac{n(w-2)}{2} = n \frac{(w+1)}{2}$, therefore $m = n$. If $D > \frac{n(w-1)}{2}$, then $m < n$, impossible therefore $m = n$, and m is trivial colour classes.

Remark6.8. From Proposition6.6. we have $wn - \sum_{i=1}^m t_i = D$, since minimum value of t_i for each colour class equal $\frac{w+1}{2}$ for w odd or equal $\frac{w}{2} + 1$ for w even, then $m \frac{(w+1)}{2} \leq wn - D$, for w odd and $m \frac{w}{2} + 1 \leq wn - D$, for w even.

Proposition6.9. Suppose we have finite graph with k vertices (these k vertices have different degrees of vertices), and maximum degree of vertex equal n , if $1, 2, 3, \dots, m$ is number of colour classes any two of these colour classes meet each other at these vertices $v_1, v_2, v_3, \dots, v_w$. If D the difference in degrees of vertices at vertices $v_1, v_2, v_3, \dots, v_w$ and $\frac{n(w-1)}{2} \leq D$ for w odd, and $\frac{n(w-2)}{2} \leq D$ for w even, then

Proof: From Remark6.8. and for w odd we have $m \frac{(w+1)}{2} \leq wn - D$, if $D = \frac{n(w-1)}{2}$, therefore we have $m \frac{(w+1)}{2} \leq n \frac{(w+1)}{2}$, therefore we have $m \leq n$, if $D > \frac{n(w-1)}{2}$, we have $m < n$ it is impossible, therefore $m = n$, and m is trivial colour classes.

For w even we have $m \frac{(w+2)}{2} \leq wn - D$, if $D = \frac{n(w-2)}{2}$, we have $m \frac{(w+1)}{2} \leq n \frac{(w+1)}{2}$, therefore we have $m \leq n$, if $D > \frac{n(w-1)}{2}$, we have $m < n$ it is impossible, therefore $m = n$, and m is trivial colour classes.

Remark6.10. From the relation $\sum_{i=1}^m t_i = wn$, where $1, 2, 3, \dots, m$ be m colour classes, $v_1, v_2, v_3, \dots, v_w$ are different vertices each of degree n , and t_i number of times the colour class i appear at these w vertices, we can write $mt \leq wn$, where t is minimum value of t_i , $1 \leq i \leq m$, $\frac{w+1}{2} \leq t \leq w-1$ for w odd, or $\frac{w}{2} + 1 \leq t \leq w-1$ for w even. We know $n(w-1) < mt \leq wn$, therefore $0 \leq wn - mt \leq (n-1)$.

Proposition 6.11. The number of colour classes used to colour edges of connected graph that is regular of degree two (cycle graph) with even number of vertices, is trivial colouring.

Proof: Since two distinct edges are adjacent if they have a vertex in common, we can label edge joining v_i and v_{i+1} by colour class 1 if i is odd, and label edge joining v_i and v_{i+1} by colour class 2 if i is even. Any vertex of degree two and can be labeled by (1,2), therefore this colouring is trivial colouring.

Proposition 6.12. The number of colour classes used to colour edges of connected graph that is regular of degree two (cycle graph) with odd number of vertices, is nontrivial colouring.

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Proof: Suppose colouring of edges is trivial, we have n edges where n is odd, and we label the edge e_i by colour class 1 if i is odd and by colour class 2 if i is even, where $1 \leq i \leq n$, then the edge e_1 labeled by colour class 1 and the edge e_n labeled by colour class 1, since the edge e_1 adjacent to the edge e_n it is a contradiction.

Proposition 6.13. The number of colour classes used to colour vertices of connected graph that is regular of degree two (cycle graph) with even number of vertices is trivial colouring, and with odd number of vertices is nontrivial colouring.
Proof: Same proof of Proposition 6.12 and Proposition 6.13.

Proposition 6.14. The number of colour classes used to colour the graph wheel W_n is trivial colouring for $n \geq 6$.

Proof: The graph W_n is wheel consists of n vertices, $n-1$ vertices each of degree 3 and one vertex of degree $n-1$. Suppose we have m colour classes any two colour classes of m meet each other at n vertices, and suppose m trivial, if m is trivial then m equal maximum degree of vertex, means $m = n-1$. From Proposition 6.7. m is trivial if D the difference in degrees of vertices at vertices $v_1, v_2, v_3, \dots, v_w$ greater than or equal $\frac{n(w-1)}{2}$ for w odd and greater than or

equal $\frac{n(w-2)}{2}$ for w even, where in this case $w = n$, $D = (n-1)(n-4)$, therefore we have $\frac{(n-1)(n-1)}{2} \leq (n-1)(n-4)$, $n-1 \leq 2n-8$, and we have $7 \leq n$ for n odd. If n even therefore we have $\frac{(n-1)(n-2)}{2} \leq (n-1)(n-4)$, $n-2 \leq 2n-8$, and we have $6 \leq n$.

Example 6.15. The following table, Table III, is an example satisfies Proposition 6.14, by n we mean n vertices of the wheel W_n , is wheel, m is number of colour classes any two meet each other at vertices (vertices of the wheel W_n) $v_1, v_2, v_3, \dots, v_w$.

n	w	m trivial	m local minimum	m local maximum
4	3	3	4 (non trivial)	4 (non trivial)
4	4	3	4 (non trivial)	4 (non trivial)
5	3	4	5 (non trivial)	5 (non trivial)
5	4	4	4 (trivial)	4 (trivial)
5	5	4	4 (trivial)	5 (non trivial)
6	3	5	5 (trivial)	5 (trivial)
6	4	5	5 (trivial)	5 (trivial)
6	5	5	5 (trivial)	5 (trivial)
6	6	5	5 (trivial)	5 (trivial)
7	3	6	6 (trivial)	6 (trivial)
7	4	6	6 (trivial)	6 (trivial)
7	5	6	6 (trivial)	6 (trivial)
7	6	6	6 (trivial)	6 (trivial)
7	7	6	6 (trivial)	6 (trivial)

Table III

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