

# GR-Normality Axioms and Some Functions in Topological Spaces

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**ABSTRACT** : In 2011 & 2012 , respectively, S.Bhattacharya and S.I.Mahmood have defined and studied the concepts of gr-closed sets , gr-open sets , gr-continuous functions,gr-irresolute functions in topological spaces.Later , many authors have used these sets and functions in their research. Quite recently , Navalagi et al in 2018 , have defined and studied various functions and allied regularity axioms via gr-closed sets. In this paper , we define and study the notions of gr-normal spaces , Gr-normal spaces , GR-normal spaces , ( $\delta$ ,gr)-normal spaces , (gr,p)-normal spaces ,  $\delta$ g-normal spaces ,pre-Gr-open functions and pre-Gr-closed functions via gr-closed sets and gr-open sets . Also, we define and study various ultra normality axioms in this paper.

**KEYWORDS**: regular open sets, regular closed sets, g-closed sets and rg-closed sets , gr-closed sets,gr-continuity , gr-irresoluteness

## I. INTRODUCTION

Levine [12] generalized the closed set to generalized closed set (g-closed set) in topology for the first time. Since then it is noticed that some of the weaker forms of closed sets have been generalized. In 1993 , Palaniappan and Rao [23] have introduced and studied the notions of rg-closed sets ,rg-open sets , rg-continuous functions, rg-irresoluteness .In 2011 , S.Bhattacharya (Halder) [7] introduced the concept of generalized regular closed sets in topology and then in 2012, S.I.Mahmood [15] has defined and studied the concepts of gr-continuity , gr-irresoluteness , perfect gr-continuity and  $T_{gr}$  -spaces . The purpose of this paper is to investigate and study the notions of gr-normal spaces , Gr-normal spaces , GR-normal spaces , ( $\delta$ ,gr)-normal spaces , (gr,p)-normal spaces ,  $\delta$ g-normal spaces ,pre-Gr-open functions and pre-Gr-closed functions via gr-closed sets and gr-open sets . Also, we define and study various ultra normality axioms in this paper.

## II. PRELIMINARIES

Throughout this paper  $(X,\tau),(Y,\sigma),(Z,\gamma)$  (or simply X,Y and Z) always means topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of space X. We denote the closure of A and the interior of A by  $Cl(A)$  and  $Int(A)$  respectively.

**Definition 2.1[26]** : The subset of A of X is said to be :

- (i) regular open (in brief,r-open) if  $A = IntCl(A)$ .
- (ii) regular closed (in brief,r-closed) if  $A = ClInt(A)$ .

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**Definition 2.2 :** The subset of  $A$  of  $X$  is said to be:

- (i) Semiopen [13] set if  $A \subset Cl Int(A)$ .
- (ii) preopen [16] set, if  $A \subset Int Cl(A)$ .
- (iii) semi-pre open[3]set, if  $A \subset Cl(Int(Cl(A)))$
- (iv) b-open [4] if  $A \subset ClInt(A) \cup IntCl(A)$ .
- (v)  $\delta$ -closed [27] if  $A = \delta Cl(A)$ , where  $\delta Cl(A) = \{x \in X : IntCl(U) \cap A \neq \emptyset, U \text{ is open set and } x \in U\}$

The complement of preopen ( resp. semipre-open, b-open,  $\delta$ -closed ) set is called preclosed [11]( resp.semipre-closed [3], b-closed [4],  $\delta$ -open [27] ) set of a space  $X$ .

**Definition 2.3 [11]:** The intersection of all pre-closed sets of  $X$  containing subset  $A$  is called the pre-closure of  $A$  and is denoted by  $pCl(A)$ .

**Definition 2.4[10]:** The intersection of all regular closed(resp.  $\delta$ -closed) sets containing set  $A$  is called the regular closure[23](resp. $\delta$ -closure [27]) of  $A$  and is denoted by  $rCl(A)$  (resp. $\delta Cl(A)$ ).

**Definition 2.5[4]:** The intersection of all b-closed sets containing set  $A$  is called the b- closure of  $A$  and is denoted by  $bCl(A)$ .

Similarly,  $pInt(A)$ ,  $rInt(A)$ ,  $bInt(A)$ ,  $\delta Int(A)$  can be defined.

**Definition 2.6:** A function  $f : X \rightarrow Y$  is said to be :

- (i) pre-continuous [16] if the inverse image of each open set of  $Y$  is preopen in  $X$ .
- (ii) pre-irresolute [24] if the inverse image of each pre-closed set of  $Y$  is pre-closed in  $X$ .
- (iii) gb-irresolute [1]if the inverse image of each gb-closed set of  $Y$  is gb-closed in  $X$ .

**Definition 2.7 :** A subset  $A$  of a space  $X$  is called :

- (i) generalized closed set ( in brief, g-closed) set [12] if  $Cl(A) \subset U$ , whenever  $A \subset U$  and  $U$  is open.
- (ii) regular generalized closed ( in brief, rg-closed) set [23] if  $Cl(A) \subset U$ , whenever  $A \subset U$  and  $U$  is regular open.
- (iii) gp-closed [22] set if  $pCl(A) \subset U$ , whenever  $A \subset U$  and  $U$  is open.
- (iv) generalized regular closed ( in brief, gr-closed) set [ 7] if  $rCl(A) \subset U$ , whenever  $A \subset U$  and  $U$  is open.
- (v) generalized b- closed ( in brief, gb-closed) set [1] if  $bCl(A) \subset U$ , whenever  $A \subset U$  and  $U$  is open.
- (vi) regular b-closed ( in brief, rb-closed ) set [21] if  $rCl(A) \subset U$ , whenever  $A \subset U$  and  $U$  is b-open in  $X$ .
- (vii)  $\delta g$ -closed [10] if  $\delta Cl(A) \subset U$ , whenever  $A \subset U$  and  $U$  is open in  $X$ .

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The complement of a g-closed resp. rg-closed, gr-closed, gb-closed, rb-closed,  $\delta$ g-closed) set in X is called g-open set (resp. rg-open, gr-open, gb-open, rb-open,  $\delta$ g-open) set in X.

**Lemma 2.8 [7]:** A subset A of a space X is gr-open in X if and only if  $F \subset rInt(A)$  whenever  $F \subset A$  and F is r-closed set in X.

**Definition 2.9 [20]:** A subset A of a space X is called gr-clopen if it is both gr-open and gr-closed.

**Definition 2.10 [15]:** The intersection of all gr-closed sets of X containing a set A is called the gr-closure of A and is denoted by  $grCl(A)$ .

Note that, if A is a gr-closed set then  $grCl(A) = A$ .

**Definition 2.11 [1]:** The intersection of all gb-closed sets of X containing a set A is called the gb-closure of A and is denoted by  $gbCl(A)$ .

Note that, if A is a gb-closed set then  $gbCl(A) = A$ .

**Definition 2.12:** A function  $f : X \rightarrow Y$  is said to be :

- (i) gr-continuous [15] if the inverse image of each closed set of Y is gr-closed in X.
- (ii) gr-irresolute [15] if the inverse image of each gr-closed set of Y is gr-closed in X.
- (iii) gb-irresolute [1] if the inverse image of each gb-closed set of Y is gb-closed in X.
- (iv) strongly continuous [14] if the inverse image of each subset of Y is open / closed in X.
- (v) g-continuous [6] if the inverse image of each open set of Y is g-open in X.

**Definition 2.13 [20]:** A space X is called gr-regular if for each gr-closed set of X and each  $x \notin F$ , there exist disjoint open sets U and V of X such that  $x \in U$  and  $F \subset V$ .

**Definition 2.14 [17]:** A space X is called g-normal if for each pair of disjoint g-closed sets A and B of X, there exist disjoint open sets U and V of X such that  $A \subset U$  and  $B \subset V$ .

**Definition 2.15 [18]:** A space X is called p-normal if for each pair of disjoint closed sets A and B of X, there exist disjoint preopen sets U and V of X such that  $A \subset U$  and  $B \subset V$ .

**Definition 2.16 [19]:** A space X is called pre-normal if for each pair of disjoint preclosed sets A and B of X, there exist disjoint preopen sets U and V of X such that  $A \subset U$  and  $B \subset V$ .

## III. PROPERTIES OF GR-NORMAL SPACES

In view of Definition 2.7, we have the following :

**Lemma 3.1:** Let A be a subset of a space X. Then,

- (i) if A is gr-closed set then it is g-closed.
- (ii) if A is g-closed set then it is rg-closed.
- (iii) if A is gr-closed set then it is gb-closed.
- (iv) if A is r-closed set then it is gr-closed.
- (v) if A is  $\delta$ -closed set then it is  $\delta$ g-closed.
- (vi) if A is  $\delta$ g-closed set then it is gr-closed.

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We, define the following :

**Definition 3.2 :** A function  $f : X \rightarrow Y$  is said to be gr-closed if for each closed set  $F$  of  $X$ ,  $f(F)$  is gr-closed in  $Y$ .

**Definition 3.3 :** A function  $f : X \rightarrow Y$  is called pre-Gr-closed if the image of each gr-closed set of  $X$  is preclosed in  $Y$ .

**Definition 3.4 :** A function  $f : X \rightarrow Y$  is said to be g-gr-closed if for each g-closed set  $F$  of  $X$ ,  $f(F)$  is gr-closed in  $Y$ .

**Definition 3.5 :** A function  $f : X \rightarrow Y$  is said to be gr-g-closed if the image of each gr-closed set  $F$  of  $X$ ,  $f(F)$  is g-closed in  $Y$ .

**Definition 3.6:** A function  $f : X \rightarrow Y$  is said to be always gr-closed if the image of each gr-closed set of  $X$  is gr-closed in  $Y$ .

Clearly, every always gr-closed function is g-gr-closed.

**Definition 3.7 :** A function  $f : X \rightarrow Y$  is said to be strongly-gr-closed if for each closed set  $F$  of  $X$ ,  $f(F)$  is gr-closed in  $Y$ .

**Definition 3.8 :** A function  $f : X \rightarrow Y$  is said to be always g-closed if the image of each g-closed set of  $X$  is g-closed in  $Y$ .

We have the following implications.

- (i) Every strongly-gr-closed function is pre-Gr-closed function.

We prove the following.

**Theorem 3.9:** A surjective function  $f : X \rightarrow Y$  is pre-Gr-closed if and only if for each subset  $B$  of  $Y$  and each gr-open set  $U$  of  $X$  containing  $f^{-1}(B)$ , there exists preopen set  $V$  of  $Y$  such that  $B \subset V$  and  $f^{-1}(V) \subset U$ .

**Proof:** Necessity: Suppose  $f$  is pre-Gr-closed. Let  $B$  be any subset of  $Y$  and  $U$  is gr-open set of  $X$  containing  $f^{-1}(B)$ . Put  $V = Y - f(X - U)$ . Then,  $V$  is preopen in  $Y$ ,  $B \subset V$  and  $f^{-1}(V) \subset U$ .

Sufficiency : Let  $F$  be any gr-closed set of  $X$ . Put  $B = Y - f(F)$ , then we have  $f^{-1}(B) \subset X - F$  and  $X - F$  is gr-open set of  $X$ . There exists preopen set  $V$  of  $Y$ , such that  $B = Y - f(F) \subset V$  and  $f^{-1}(V) \subset X - F$ . Therefore, we obtain  $f(F) = Y - V$  and hence  $f(F)$  is preclosed in  $Y$ . This shows that  $f$  is pre-Gr-closed.

**Theorem 3.10:** A surjective function  $f : X \rightarrow Y$  is gr-g-closed if and only if for each subset  $B$  of  $Y$  and each gr-open set  $U$  of  $X$  containing  $f^{-1}(B)$ , there exists g-open set  $V$  of  $Y$  such that  $B \subset V$  and  $f^{-1}(V) \subset U$ .

**Proof:** Necessity: Suppose  $f$  is gr-g-closed. Let  $B$  be any subset of  $Y$  and  $U$  is gr-open set of  $X$  containing  $f^{-1}(B)$ . Put  $V = Y - f(X - U)$ . Then,  $V$  is g-open in  $Y$ ,  $B \subset V$  and  $f^{-1}(V) \subset U$ .

Sufficiency : Let  $F$  be any gr-closed set in  $X$ . Put  $B = Y - f(F)$ , then we have  $f^{-1}(B) \subset X - F$  and  $X - F$  is gr-open set of  $X$ . There exists g-open set  $V$  of  $Y$ , such that  $B = Y - f(F) \subset V$  and  $f^{-1}(V) \subset X - F$ . Therefore, we obtain  $f(F) = Y - V$  and hence  $f(F)$  is g-closed in  $Y$ . This shows that  $f$  is gr-g-closed.

**Theorem 3.11 :** A surjective function  $f : X \rightarrow Y$  is always -gr-closed if and only if for each subset  $B$  of  $Y$  and each gr-open set  $U$  of  $X$  containing  $f^{-1}(B)$ , there exists gr-open set  $V$  of  $Y$  such that  $B \subset V$  and  $f^{-1}(V) \subset U$ .

**Proof:** Necessity: Suppose  $f$  is always gr-closed. Let  $B$  be any subset of  $Y$  and  $U$  is gr-open set of  $X$  containing  $f^{-1}(B)$ . Put  $V = Y - f(X - U)$ . Then,  $V$  is gr-open in  $Y$ ,  $B \subset V$  and  $f^{-1}(V) \subset U$ .

Sufficiency : Let  $F$  be any gr-closed set of  $X$ . Put  $B = Y - f(F)$ , then we have  $f^{-1}(B) \subset X - F$  and  $X - F$  is gr-open set of  $X$ . There exists gr-open set  $V$  of  $Y$ , such that  $B = Y - f(F) \subset V$  and  $f^{-1}(V) \subset X - F$ . Therefore, we obtain  $f(F) = Y - V$  and hence  $f(F)$  is gr-closed in  $Y$ . This shows that  $f$  is always gr-closed.

We,define the following.

**Definition 3.12:** A space  $X$  is called gr-normal if for each pair of disjoint gr-closed sets  $A$  and  $B$  of  $X$ , there exist disjoint open sets  $U$  and  $V$  of  $X$  such that  $A \subset U$  and  $B \subset V$ .

Clearly, every g-normal space is gr-normal space since by Lemma-3.1.

**Definition 3.13 :** A space  $X$  is called gb-normal if for each pair of disjoint gb-closed sets  $A$  and  $B$  of  $X$ , there exist disjoint open sets  $U$  and  $V$  of  $X$  such that  $A \subset U$  and  $B \subset V$ .

Clearly, every gb-normal space is gr-normal space since by Lemma-3.1.

We,prove the following.

**Theorem 3.14:** For a space  $X$ , the following are equivalent :

- (i)  $X$  is gr-normal .
- (ii) For every gr-closed set  $A$  and every gr-open set  $U$  containing  $A$  ,there exists an open set

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$V$  such that  $A \subset V \subset Cl(V) \subset U$ .

**Theorem 3.15 :** For a space  $X$ , the following are equivalent :

(i)  $X$  is gr-normal .

(ii) For every gr-closed set  $A$  and every gr-open set  $U$  containing  $A$  , there is a clopen set  $V$  such that  $A \subset V \subset U$  .

**Proof :** (i) $\rightarrow$ (ii). Let  $A$  be gr-closed set and  $U$  be gr-open set containing  $A$ . Now, we have  $A \cap (X-U) = \emptyset$  , hence there exist disjoint open sets  $G$  and  $H$  such that  $A \subset G$  and  $(X-U) \subset H$ .

If we put  $V = Cl(G)$  , then  $V$  is clopen set satisfying  $A \subset V \subset U$  .

(ii) $\rightarrow$ (i) . Obvious.

We, define the following.

**Definition 3.16 :** A space  $X$  is called Gr-normal if for each pair of disjoint closed sets  $A$  and  $B$  of  $X$ , there exist disjoint gr-open sets  $U$  and  $V$  of  $X$  such that  $A \subset U$  and  $B \subset V$  .

**Definition 3.17 :** A space  $X$  is called Gb-normal if for each pair of disjoint closed sets  $A$  and  $B$  of  $X$ , there exist disjoint gb-open sets  $U$  and  $V$  of  $X$  such that  $A \subset U$  and  $B \subset V$  .

Clearly, every Gr-normal space is Gb-normal space since by Lemma-3.1.

We, prove the following.

**Theorem 3.18:** For a space  $X$ , the following are equivalent :

(i)  $X$  is Gr-normal .

(ii) For every closed set  $A$  and every open set  $U$  containing  $A$  , there is a gr-clopen set  $V$  such that  $A \subset V \subset U$  .

**Proof :** (i) $\rightarrow$ (ii). Let  $A$  be closed set and  $U$  be open set containing  $A$ . Now, we have

$A \cap (X-U) = \emptyset$  , hence there exist disjoint gr-open sets  $G$  and  $H$  such that  $A \subset G$  and  $(X-U) \subset H$ . If we put  $V = grCl(G)$  , then  $V$  is gr-clopen set satisfying  $A \subset V \subset U$  .

(ii) $\rightarrow$ (i) . Obvious.

We,prove the following.

**Theorem 3.19:** Let  $f:X \rightarrow Y$  be gr-continuous ,gr-closed surjection.If  $X$  is gr-normal then  $Y$  is Gr-normal.

**Proof :** Obvious.

**Definition 3.20 :** A space  $X$  is called GR-normal if for each pair of disjoint gr-closed sets  $A$  and  $B$  of  $X$ , there exist disjoint gr-open sets  $U$  and  $V$  of  $X$  such that  $A \subset U$  and  $B \subset V$  .

**Definition 3.21 :** A space  $X$  is called  $(gr, \delta)$ -normal if for each pair of disjoint gr-closed sets  $A$  and  $B$  of  $X$ , there exist disjoint  $\delta$ -open sets  $U$  and  $V$  of  $X$  such that  $A \subset U$  and  $B \subset V$  .

**Definition 3.22 :** A space  $X$  is called  $(\delta, gr)$ -normal if for each pair of disjoint  $\delta$ -closed sets  $A$  and  $B$  of  $X$ , there exist disjoint gr-open sets  $U$  and  $V$  of  $X$  such that  $A \subset U$  and  $B \subset V$  .

Clearly, every  $(gr, \delta)$ -normal space is GR-normal space since by Lemma-3.1 , and

every GR-normal space is  $(\delta, gr)$ -normal space since by Lemma-3.1.

**Definition 3.23 :** A space  $X$  is called  $(gr, p)$ - normal if for each pair of disjoint gr-closed sets  $A$  and  $B$  , there exist disjoint preopen sets  $U$  and  $V$  of  $X$  such that  $A \subset U$  and  $B \subset V$  .

**Definition 3.24 :** A space  $X$  is called  $(p, gr)$ - normal if for each pair of disjoint pre-closed sets  $A$  and  $B$  , there exist disjoint gr-open sets  $U$  and  $V$  of  $X$  such that  $A \subset U$  and  $B \subset V$  .

Routine proofs of the following are omitted :

**Lemma 3.25 :** Let  $f:X \rightarrow Y$  be gr-irresolute , pre-Gr-closed surjection. If  $X$  is GR-normal then  $Y$  is  $(gr, p)$ -normal.

**Proof :** Obvious.

**Lemma 3.26:** Let  $f:X \rightarrow Y$  be precontinuous , pre-Gr-closed surjection.If  $X$  is  $(p, gr)$ -normal then  $Y$  is  $p$ -normal.

**Proof :** Obvious.

**Lemma 3.27:** Let  $f:X \rightarrow Y$  be pre-irresolute , pre-Gr-closed surjection.If  $X$  is  $(p, gr)$ -normal , then  $Y$  is pre-normal.

**Proof :** Obvious.

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We, define the following.

**Definition 3.28:** A function  $f: X \rightarrow Y$  is called pre- $\delta g$ -continuous if the inverse image of each  $\delta$ -closed set of  $Y$  is  $\delta g$ -closed in  $X$ .

**Definition 3.29:** A function  $f: X \rightarrow Y$  is called pre- $\delta g$ -closed if the image of each  $\delta$ -closed set of  $X$  is  $\delta g$ -closed in  $Y$ .

**Definition 3.30 :** A function  $f: X \rightarrow Y$  is called  $\delta g$ -closed if the image of each closed set of  $X$  is  $\delta g$ -closed in  $Y$ .

We, recall the following.

**Definition 3.31[10]:** A function  $f: X \rightarrow Y$  is called  $\delta g$ -continuous if the inverse image of each closed set of  $Y$  is  $\delta g$ -closed in  $X$ .

Clearly, every  $\delta g$ -continuous function is gr-continuous.

**Definition 3.32[10]:** A function  $f: X \rightarrow Y$  is called  $\delta g$ -irresolute if the inverse image of each  $\delta g$ -closed set of  $Y$  is  $\delta g$ -closed in  $X$ .

We, prove the following.

**Theorem 3.33:** A surjective function  $f: X \rightarrow Y$  is pre- $\delta g$ -closed if and only if for each subset  $B$  of  $Y$  and each  $\delta$ -open set  $U$  of  $X$  containing  $f^{-1}(B)$ , there exists  $\delta g$ -open set  $V$  of  $Y$  such that  $B \subset V$  and  $f^{-1}(V) \subset U$ .

**Proof:** Necessity: Suppose  $f$  is pre- $\delta g$ -closed. Let  $B$  be any subset of  $Y$  and  $U$  is  $\delta$ -open set of  $X$  containing  $f^{-1}(B)$ . Put  $V = Y - f(X - U)$ . Then,  $V$  is  $\delta g$ -open in  $Y$ ,  $B \subset V$  and  $f^{-1}(V) \subset U$ .

Sufficiency: Let  $F$  be any  $\delta$ -closed set of  $X$ . Put  $B = Y - f(F)$ , then we have  $f^{-1}(B) \subset X - F$  and  $X - F$  is  $\delta$ -open set of  $X$ . There exists  $\delta g$ -open set  $V$  of  $Y$ , such that  $B = Y - f(F) \subset V$  and  $f^{-1}(V) \subset X - F$ . Therefore, we obtain  $f(F) = Y - V$  and hence  $f(F)$  is  $\delta g$ -closed in  $Y$ . This shows that  $f$  is pre- $\delta g$ -closed.

We, define the following.

**Definition 3.34:** A space  $X$  is called  $\delta g$ -normal if for each pair of disjoint  $\delta g$ -closed sets  $A$  and  $B$  of  $X$ , there exist disjoint open sets  $U$  and  $V$  of  $X$  such that  $A \subset U$  and  $B \subset V$ .

Clearly, every gr-normal space is  $\delta g$ -normal

**Definition 3.35 :** A space  $X$  is called  $(\delta, \delta g)$ -normal if for each pair of disjoint  $\delta$ -closed sets  $A$  and  $B$  of  $X$ , there exist disjoint  $\delta g$ -open sets  $U$  and  $V$  of  $X$  such that  $A \subset U$  and  $B \subset V$ .

**Theorem 3.36 :** If  $f: X \rightarrow Y$  is pre- $\delta g$ -continuous closed injection and  $Y$  is a  $\delta g$ -normal, then  $X$  is  $\delta g$ -normal.

**Theorem 3.37:** If  $f: X \rightarrow Y$  is  $\delta g$ -continuous closed injection and  $Y$  is a normal, then  $X$  is  $\delta g$ -normal.

Proof is routine and hence omitted.

We, define the following.

**Definition 3.38:** A space  $X$  is said to be  $\delta g^*$ -normal if for any pair of disjoint closed sets  $A$  and  $B$  of  $X$ , there exist disjoint  $\delta g$ -open sets  $U$  and  $V$  such that  $A \subset U$  and  $B \subset V$ .

Clearly, every  $\delta g^*$ -normal space is Gr-normal space.

We, prove the following.

**Theorem 3.39:** For a space  $X$ , the following are equivalent :

- (i)  $X$  is  $\delta g$ -normal.
- (ii) For every  $\delta g$ -closed set  $A$  and every  $\delta g$ -open set  $U$  containing  $A$ , there is a clopen set  $V$  such that  $A \subset V \subset U$ .

**Proof :** (i)  $\rightarrow$  (ii). Let  $A$  be  $\delta g$ -closed set and  $U$  be  $\delta g$ -open set containing  $A$ . Now, we have  $A \cap (X - U) = \emptyset$ , hence there exist disjoint open sets  $G$  and  $H$  such that  $A \subset G$  and  $(X - U) \subset H$ .

If we put  $V = Cl(G)$ , then  $V$  is clopen set satisfying  $A \subset V \subset U$ .

(ii)  $\rightarrow$  (i). Obvious.

In view of definition of gr-closed sets, it is observed that the closed sets and gr-closed sets are in general independent and so we have the following is introduced :

We, recall the following.

**Definition 3.40 [2]:** A space  $X$  is a PS-space iff each preopen subset of  $X$  is semiopen.

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It means that , a space  $X$  is PS-space if  $PO(X) \subset SO(X)$ .

We, define the following.

**Definition 3.41:** A space  $X$  is a GRO-space iff each gr-open subset of  $X$  is open.

It means that , a space  $X$  is GRO-space if  $GRO(X) \subset O(X)$ .

**Definition 3.42:** A space  $X$  is a GGR-space iff each g-open subset of  $X$  is gr-open.

It means that , a space  $X$  is GGR-space if  $GO(X) \subset GRO(X)$ .

**Definition 3.43:** A space  $X$  is a GRDG-space iff each gr-open subset of  $X$  is  $\delta g$ -open.

It means that , a space  $X$  is GRDG-space if  $GRO(X) \subset DGO(X)$ .

In view of Definition-3.41,3. 42 & 3.43 , we have the following :

**Lemma 3.44 :** Let  $X$  be GRO-space , then the following hold :

- (i) If  $f : X \rightarrow Y$  is gr-continuous , then it is continuous.
- (ii) If  $f : X \rightarrow Y$  is strongly gr-continuous , then it is strongly continuous.
- (iii) If  $X$  is gr-normal space , then it is normal.
- (iv) If  $X$  is Gr-normal space ,then it is normal.

**Lemma 3.45:** Let  $X$  be GGR-space , then the following hold :

- (i) If  $f : X \rightarrow Y$  is g-continuous , then it is gr-continuous.
- (ii) If  $X$  is gr-normal space , then it is g-normal.

**Lemma 3.46:** Let  $X$  be GRDG-space , then the following hold :

- (i) If  $f : X \rightarrow Y$  is gr-continuous , then it is  $\delta g$ -continuous.
- (ii) If  $X$  is gr-normal space , then it is  $\delta g$ - normal.
- (iii) If  $X$  is Gr-normal space ,then it is  $\delta g^*$ -normal.

## IV.PROPERTIES OF ULTRA GR-NORMAL SPACES

We, define the following.

**Definition 4.1 :** A function  $f : X \rightarrow Y$  is called gr-open if the image of each open set of  $X$  is gr-open in  $Y$ .

**Definition 4.2 :** A function  $f : X \rightarrow Y$  is called pre-Gr-open if the image of each gr-open set of  $X$  is preopen in  $Y$ .

**Definition 4.3:** A function  $f : X \rightarrow Y$  is said to be always gr-open if the image of each gr-open if the image of each gr-open set of  $X$  is gr-open in  $Y$ .

**Definition 4.4 :** A function  $f : X \rightarrow Y$  is said to be strongly gr-continuous if the inverse image of each gr-open set of  $Y$  is open in  $X$ .

We prove the following.

**Theorem 4.5:** A surjective function  $f : X \rightarrow Y$  is pre-Gr-open if and only if for each subset  $B$  of  $Y$  and each gr-closed set  $F$  of  $X$  containing  $f^{-1}(B)$ , there exists preclosed set  $H$  of  $Y$  such that  $B \subset H$  and  $f^{-1}(H) \subset F$ .

**Proof:** Necessity: Suppose  $f$  is pre-Gr-open. Let  $B$  be any subset of  $Y$  and  $F$  is gr-closed set of  $X$  containing  $f^{-1}(B)$ . Put  $H = Y - f(X - F)$ . Then,  $H$  is preclosed in  $Y$ ,  $B \subset H$  and  $f^{-1}(H) \subset F$ .

Sufficiency : Let  $U$  be any gr-open set in  $X$ . Put  $B = Y - f(U)$ , then we have  $f^{-1}(B) \subset X - U$  and  $X - U$  is gr-closed such that  $B = Y - f(U) \subset H$  and  $f^{-1}(H) \subset X - U$ . Therefore we obtain  $f(U) = Y - H$  and hence  $f(U)$  is preopen in  $Y$ . This shows that  $f$  is pre-Gr-open.

**Theorem 4.6 :** A surjective function  $f : X \rightarrow Y$  is gr-open if and only if for each subset  $B$  of  $Y$  and each closed set  $F$  of  $X$  containing  $f^{-1}(B)$ , there exists gr-closed set  $H$  of  $Y$  such that  $B \subset H$  and  $f^{-1}(H) \subset F$ .

Proof is routine and hence omitted.

**Theorem 4.7 :** A surjective function  $f : X \rightarrow Y$  is always gr-open if and only if for each subset  $B$  of  $Y$  and each gr-closed set  $F$  of  $X$  containing  $f^{-1}(B)$ , there exists gr-closed set  $H$  of  $Y$  such that  $B \subset H$  and  $f^{-1}(H) \subset F$ .

Proof is routine and hence omitted.

We,define the following.

**Definition 4.8:** A space  $X$  is called ultra-p-normal if for each pair of disjoint closed sets  $A$  and  $B$  of  $X$ , there exist disjoint preclosed sets  $U$  and  $V$  of  $X$  such that  $A \subset U$  and  $B \subset V$ .

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**Definition 4.9 :** A space  $X$  is called ultra -  $g$ -normal if for each pair of disjoint  $g$ -closed sets  $A$  and  $B$  of  $X$ , there exist disjoint closed sets  $U$  and  $V$  of  $X$  such that  $A \subset U$  and  $B \subset V$ .

**Definition 4.10:** A space  $X$  is called ultra - $gr$ - normal if for each pair of disjoint  $gr$ -closed sets  $A$  and  $B$ , there exist disjoint closed sets  $U$  and  $V$  of  $X$  such that  $A \subset U$  and  $B \subset V$ .

Clearly, every ultra - $g$ -normal space is ultra- $gr$ -normal

**Definition 4.11:** A space  $X$  is called ultra - $(gr,p)$ - normal if for each pair of disjoint  $gr$ -closed sets  $A$  and  $B$ , there exist disjoint preclosed sets  $U$  and  $V$  of  $X$  such that  $A \subset U$  and  $B \subset V$ .

Clearly, every ultra - $gr$ -normal space is ultra- $(gr,p)$ -normal

**Definition 4.12 :** A space  $X$  is called ultra -  $(gr, \delta)$ -normal if for each pair of disjoint  $gr$ -closed sets  $A$  and  $B$  of  $X$ , there exist disjoint  $\delta$ -closed sets  $U$  and  $V$  of  $X$  such that  $A \subset U$  and  $B \subset V$ .

Clearly, every ultra - $(gr, \delta)$ -normal space is ultra- $gr$ -normal

**Definition 4.13 :** A space  $X$  is called ultra -  $(\delta, gr)$ -normal if for each pair of disjoint  $\delta$ -closed sets  $A$  and  $B$  of  $X$ , there exist disjoint  $gr$ -closed sets  $U$  and  $V$  of  $X$  such that  $A \subset U$  and  $B \subset V$ .

**Definition 4.14 :** A space  $X$  is called ultra - $Gr$ -normal if for each pair of disjoint closed sets  $A$  and  $B$  of  $X$ , there exist disjoint  $gr$ -closed sets  $U$  and  $V$  of  $X$  such that  $A \subset U$  and  $B \subset V$ .

**Definition 4.15:** A space  $X$  is called ultra -  $GR$ -normal if for each pair of disjoint  $gr$ -closed sets  $A$  and  $B$  of  $X$ , there exist disjoint  $gr$ -closed sets  $U$  and  $V$  of  $X$  such that  $A \subset U$  and  $B \subset V$ .

Clearly, every ultra - $(gr, \delta)$  - normal space is ultra- $GR$ -normal space

**Theorem 4.16:** The following properties are true for a space  $X$  :

- (i)  $X$  is ultra- $p$ -normal ,
- (ii) For any pair of disjoint closed sets  $A, B$  of  $X$ , there exist  $gp$ -closed sets  $F, H$  in  $X$  such that  $A \subset F$  and  $B \subset H$ .
- (iii) For any pair of disjoint closed sets  $A, B$  of  $X$ , there exist  $gsp$ -closed sets  $F, H$  in  $X$  such that  $A \subset F$  and  $B \subset H$ .

Proof is easy since every preclosed set is  $gp$ -closed and every  $gp$ -closed set is  $gsp$ -closed.

**Theorem 4.17:** The following properties are true for a space  $X$  :

- (i)  $X$  is ultra- $g$ -normal ,
- (ii) For any pair of disjoint  $g$ -closed sets  $A, B$  of  $X$ , there exist semiclosed sets  $F, H$  in  $X$  such that  $A \subset F$  and  $B \subset H$ .
- (iii) For any pair of disjoint  $g$ -closed sets  $A, B$  of  $X$ , there exist  $gs$ -closed sets  $F, H$  in  $X$  such that  $A \subset F$  and  $B \subset H$ .

Proof is easy since every closed set is semiclosed and every semiclosed set is  $gs$ -closed.

**Theorem 4.18:** The following properties are true for a space  $X$  :

- (i)  $X$  is ultra- $gr$ -normal ,
- (ii) For any pair of disjoint  $gr$ -closed sets  $A, B$  of  $X$ , there exist preclosed sets  $F, H$  in  $X$  such that  $A \subset F$  and  $B \subset H$ .
- (iii) For any pair of disjoint  $gr$ -closed sets  $A, B$  of  $X$ , there exist  $gp$ -closed sets  $F, H$  in  $X$  such that  $A \subset F$  and  $B \subset H$ .

Proof is easy since every closed set is preclosed and every preclosed set is  $gp$ -closed.



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**Theorem 4.19:** The following properties are true for a space  $X$  :

- (i)  $X$  is ultra-gr-normal ,
- (ii) For any pair of disjoint gr-closed sets  $A, B$  of  $X$  , there exist  $g$ -closed sets  $F, H$  in  $X$  such that  $A \subset F$  and  $B \subset H$ .
- (iii) For any pair of disjoint gr-closed sets  $A, B$  of  $X$  , there exist  $rg$ -closed sets  $F, H$  in  $X$  such that  $A \subset F$  and  $B \subset H$ .

Proof is easy since every closed set is preclosed and every preclosed set is  $gp$ -closed.

Next , we prove the following.

**Lemma 4.20:** If function  $f: X \rightarrow Y$  is continuous  $gr$ -open surjection and  $X$  is ultra -normal ,then  $Y$  is ultra - $gr$ -normal.

**Proof :** Let  $A$  and  $B$  be any disjoint closed sets of  $Y$ . Then  $f^{-1}(A)$  and  $f^{-1}(B)$  are disjoint closed sets in  $X$  as  $f$  is continuous surjection. Since  $X$  is ultra-normal , there exist disjoint closed sets  $F$  and  $H$  such that  $f^{-1}(A) \subset F$  and  $f^{-1}(B) \subset H$ . Then by Th.3.9 above ,there exist  $gr$ -closed sets  $G$  and  $M$  of  $Y$  such that  $A \subset G, B \subset M, f^{-1}(G) \subset F, f^{-1}(M) \subset H$ . Then , we have  $f^{-1}(G) \cap f^{-1}(M) = \emptyset$  and hence  $G \cap M = \emptyset$ . Then from by definition ,  $Y$  is ultra  $gr$ -normal.

Routine proofs of the following are omitted :

**Lemma 4.21:** If function  $f: X \rightarrow Y$  is  $gr$ -irresolute , pre- $Gr$ -open surjection and  $X$  is ultra-( $gr,p$ ) -normal ,then  $Y$  is ultra -( $gr,p$ )-normal.

**Lemma 4.22:** If function  $f: X \rightarrow Y$  is strongly  $gr$ -continuous , pre- $Gr$ -open surjection and  $X$  is ultra - $Gr$ -normal ,then  $Y$  is ultra -( $gr,p$ )-normal.

**Lemma 4.23:** If function  $f: X \rightarrow Y$  is strongly  $gr$ -continuous,  $gr$ -open surjection and  $X$  is ultra -normal ,then  $Y$  is ultra - $GR$ -normal.

**Lemma 4.24:** If function  $f: X \rightarrow Y$  is  $gr$ -irresolute ,pre- $Gr$ -open surjection and  $X$  is ultra - $GR$ -normal ,then  $Y$  is ultra -( $gr,p$ )-normal.

**Lemma 4.25:** If function  $f: X \rightarrow Y$  is continuous  $gr$ -open surjection and  $X$  is ultra -normal ,then  $Y$  is ultra - $Gr$ -normal.

**Lemma 4.26:** If function  $f: X \rightarrow Y$  is  $gr$ -continuous  $gr$ -open surjection and  $X$  is ultra - $gr$ -normal , then  $Y$  is ultra - $Gr$ -normal.

**Lemma 4.27:** If function  $f: X \rightarrow Y$  is  $\delta$ -irresolute, always- $gr$ -open surjection and  $X$  is ultra -( $\delta, gr$ )-normal ,then  $Y$  is ultra -( $\delta, gr$ )-normal.

## V. CONCLUSION

In this paper , we defined and studied the concepts like-  $gr$ -normal spaces ,  $Gr$ -normal spaces ,  $GR$ -normal spaces and allied  $gr$ -normal spaces via  $gr$ -open sets due S.Bhattacharya (Haldar). Also, we studied the ultra  $gr$ -normal spaces , ultra  $Gr$ -normal normal spaces , ultra  $GR$ -normal spaces and allied ultra  $gr$ -normal spaces . For the study of preservation theorems on these  $gr$ -normality axioms , we introduced a new class of functions called pre- $Gr$ -open functions and pre- $Gr$ -closed functions. It is also observed from the definition of  $gr$ -closed sets that closed sets and  $gr$ -closed sets are in general independent each other , so we introduced  $GRO$ -spaces , in which  $GRO(X) \subset O(X)$  , where  $GRO(X)$  is the family of all  $gr$ -open sets of  $X$  and  $O(X)$  is the family of all open sets of  $X$ .

## REFERENCES

- [1] A. Al-Omari and M.S. Noorani, On Generalized  $b$ -closed sets, Bull. Malays. Math. Sci. Soc., 32(1) ,(2009), 19-30.
- [2] T. Aho and T. Nieminen, Spaces in which preopen subsets are semiopen, Recerela. Mat., 43(1994), 45-59.
- [3] D. Andrijevic , Semipreopen sets, Mat. Vesnik ,38 (1886), 24-32.
- [4] D. Andrijevic, On  $b$ -open sets, Mat. Vesnik , 48 (1996), 59-64.
- [5] S.P. Arya and T. Nour, Characterizations of  $s$ -Normal spaces, Indian J. Pure & Appl. Math. 21(1990), N0.8, 717-719.

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- [6] K. Balachandran, P. Sundaram and H. Maki, On generalized continuous maps in topological spaces, Mem.Fac. Sci. Kochi. Univ., 12 (1991), 5-13.
- [7] S. Bhattacharya (Halder), 'On Generalized Regular closed sets', Int. J. Contempt. Math. Sciences, Vol.6, 2011, no.3, 145-152.
- [8] N.Biswas, "On characterization of semi-continuous functions", Rendiconti, Accademia Nazionale Dei Lincei, April 1970.
- [9] S.G.Crossely and S.K.Hildebrand , Semiclosure,Texas J.Sci., 22(1971),99-112.
- [10] J.Dontchev and M.Ganster, On  $\delta$ -generalized closed sets and  $T_{3/2}$ -spaces, Mem.Fac. Sci.,Kochi Univ.Ser.A,Math., 17(1996),15-31.
- [11] S.N.El-Deeb,I.A.Hasanein, A.S.Mashhour and T.Noiri , On p-regular spaces, Bull.Math.Soc. Sci .Math. R.S.Roumania (N.S), 27(75), (1983), 311-315.
- [12] N.Levine, Generalized Closed Sets in Topology. Rend. Circ. Mat. Palermo. 19(2) (1970), 89-96.
- [13] N.Levine , Semiopen sets and semicontinuity in topological spaces,Amer.Math.Monthly ,70(1963),36-41.
- [14] N.Levine , Strong continuity in topological spaces, Amer.Math.Monthly, 67(1960),269.
- [15] S.I.Mahmood, 'On Generalized Regular Continuous Functions In Topological Spaces', Ibn Al-Haitham Journal for Pure and Applied Science, No.3, Vol. 25, (2012), 377-385.
- [16] A. S. Mashhour, M.E.Abd El-Monsef and S.N.El-Deeb, On Precontinuous and weak precontinuous mappings, Proc. Math. Phys. Soc. Egypt, 53(1982) 47-53.
- [17] B.M.Munshi , Separation Axioms, Acta Ciencia Indica , Vol.XII ,Nr. 2, (1986), 140-145.
- [18] Govindappa Navalagi , P-normal , almost p-normal and mildly p-normal spaces,Pacific-Asian Journal of Mathematics (2006).
- [19] G.B.Navalagi , Generalization of preirresolute functions , T.A. Preprints (2000), #420.
- [20] Govindappa Navalagi and Sujata Mookanagoudar , Properties of GR-regular spaces in topology , IJARIT , Vol.4, Issue 4 , (July-Agust , 2018) , 452-456.
- [21] A.Narmada ,N.Nagaveni and T.Noiri , On reglar b-open sets in topological spaces,Int.J.of Math.Analysis , Vol.7 Nr.19,(2013),937-948.
- [22] T. Noiri, H.Maki And J. Umehara, Generalized preclosed functions. Mem. Fac. Sci. Kochi Univ. Ser. A(Math.), Vol.19(1998), 13-20.
- [23] N.Palniappan and K. Chandrashekar Rao, 'Regular Generalized Closed Sets', Kyungpook Mathematical Journal, Vol.33, No.2,(1993), 211-219.
- [24] I.L.Reilly and M.K.Vamanmurthy , On  $\alpha$ -continuity in topological spaces, Acta. Math. Hungar., 45(1-2), (1985) , 27-32.
- [25] R.Staum , The algebra of bounded continuous functions into a nonarchimedean field, Pacific J.Math., 50(1974),169-185.
- [26] M.H.Stone, Applications of Boolean rings to general topology, Trans.Amer.Math. Soc., 41(1937),375-381.
- [27] N.V.Velicko , H-closed topological spaces,Amer.Math. Soc.Transl.,(2) 78 (1968),103-118.