

# Lyapunov-type Inequality for Higher Order Fractional Boundary Value Problem with Caputo Derivative

Nimisha Pathak

Assistant Professor, Department of Applied Mathematics, The M. S. University of Baroda, Vadodara, Gujarat, India

**ABSTRACT:** In this paper, we present the Lyapunov-type inequality for fractional boundary value problem of order  $\alpha, 3 < \alpha \leq 4$  defined in terms of Caputo fractional derivative. We obtain fractional Green's functions for the corresponding boundary value problem and use a property of the Green's function to obtain the Lyapunov-type inequality. We use this inequality in two applications; first, to find lower bounds for the lowest eigenvalues, and second, to find the domains in which certain combination of Mittag-Leffler functions have no zeros. We further use the Cauchy-Schwarz inequality to improve the lower bounds for the smallest eigenvalues and stretch the domains in which certain combinations of Mittag-Leffler functions have no real zeros.

**KEYWORDS:** Lyapunov inequality, Caputo fractional derivative, Mittag-Leffler function, Cauchy-Schwarz inequality.

## I. INTRODUCTION

The Lyapunov inequality [1] has proved to be very useful in the study of spectral properties of ordinary differential equations [2],[3]. This inequality can be stated as follows:

**Theorem 1.1.** (See [1]) A necessary condition for the boundary value problem (BVP)

$$u''(t) + q(t)u(t) = 0, a < t < b$$

$$u(a) = 0, u(b) = 0$$

(1.1)

to have nontrivial solutions is that

$$\int_a^b |q(s)| ds > 4/(b-a),$$

(1.2)

where  $q$  is a real and continuous function. The constant 4 in equation (1.2) is sharp so that it cannot be replaced by a larger number.

The research on Lyapunov-Type Inequalities (LTIs) for Fractional Boundary Value Problems (FBVPs) has begun since 2013. Ferreira [4], [5], Jleli and Samet[6], Pathak [7], [8] have established Lyapunov-type inequalities for FBVPs of order  $\alpha, \alpha \in (1,2]$  and different boundary conditions. In [9], we obtained LTI for FBVP of order  $\alpha, \alpha \in (2,3]$ . In [10], we established the Cauchy-Schwarz inequality for FBVP of order  $\alpha, \alpha \in (3,4]$  and used it to obtain the lower bound estimate of the eigen value problem using the Cauchy-Schwarz inequality. The results show that it provides better lower bound estimate for the eigenvalues than the bound using Lyapunov-type inequality.

In this paper, we consider the FBVP having the Caputo derivative given by

$$({}_0^C D_t^\alpha u)(t) - q(t)u(t) = 0, 0 < t < 1, \alpha \in (3,4]$$

(1.3)

with the boundary conditions

$$u(0) = 0, u'(0) = 0, u(1) = u'(1) = 0,$$

(1.4)

where  $q: [a, b] \rightarrow \mathbb{R}$  is a continuous function.

Note that for integer order case  $\alpha = 4$ , FBVP (1.3)-(1.4) represents the BVP with clamped beam conditions. We use following notations in this paper:

# International Journal of Innovative Research in Science, Engineering and Technology

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$L(0,1)$ : The space of all Lebesgue integrable functions defined on  $(0,1)$ ,  
 $C(0,1)$ : The space of continuously differentiable functions on  $(0,1)$ .

**Definition 1.1.** The fractional integral of order  $\alpha > 0$  of function  $f: (0, \infty) \rightarrow \mathbb{R}$ , is given by

$$I_{0^+}^\alpha = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds \tag{1.5}$$

provided the right side is pointwise defined on  $(0, \infty)$ .

**Definition 1.2.** The Caputo fractional derivative of order  $\alpha > 0$  of a continuous function  $f: (0, \infty) \rightarrow \mathbb{R}$ , is given by

$${}^C_0 D_t^\alpha u f(t) = I_{0^+}^{n-\alpha} f(t) D^n f(t), t > 0.$$

where  $n = [\alpha] + 1$ , provided that the right side is pointwise defined on  $(0, \infty)$ .

The results given below in Lemma 1.1 is standard within the fractional calculus theory involving Caputo differential operator (see [11], [12]).

**Lemma 1.1.** Let  $\alpha > 0$ . If  $u \in C(0,1) \cap L^1[0,1]$  then

$$I_{0^+}^\alpha {}^C_0 D_t^\alpha u(t) = u(t) - c_0 - c_1 t - c_2 t^2 - \dots - c_{n-1} t^{n-1}, c_i \in \mathbb{R}, i = 1, 2, \dots, n, \tag{1.6}$$

where  $n$  is the smallest positive integer greater than or equal to  $\alpha$ .

## II. MAIN RESULTS

We begin by writing problem (1.3) -(1.4) in an equivalent Fredholm integral equation of second kind.

**Lemma 2.1.**  $u \in C[0, 1]$  is a solution to the boundary value problem (1.3) -(1.4) if and only if  $u$  satisfies the integral equation

$$u(t) = \int_0^1 G(t,s) q(s) u(s) ds, \tag{2.1}$$

where the Green's function is given by

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} [(\alpha-1)(1-s)^{\alpha-2} - 3(1-s)^{\alpha-1}]t^2 + [-(\alpha-1)(1-s)^{\alpha-2} + 2(1-s)^{\alpha-1}]t^3 + (t-s)^{\alpha-1}, & 0 \leq s \leq t \leq 1, \\ [(\alpha-1)(1-s)^{\alpha-2} - 3(1-s)^{\alpha-1}]t^2 + [-(\alpha-1)(1-s)^{\alpha-2} + 2(1-s)^{\alpha-1}]t^3, & 0 \leq t \leq s \leq 1. \end{cases} \tag{2.2}$$

**Proof.** From Lemma 1.1, we write equation (1.3) in an equivalent integral form:

$$u(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} q(s) u(s) ds, \tag{2.3}$$

where  $c_0, c_1, c_2$  and  $c_3$  are some real constants. Since  $u(0) = 0$ , we get  $c_0 = 0$ . Taking time derivative of equation (2.3), and setting  $u'(0) = u'(1) = 0$ , we obtain  $c_1 = 0$ ,

$$c_2 = \frac{1}{\Gamma(\alpha)} \int_0^1 [(\alpha-1)(1-s)^{\alpha-2} - 3(1-s)^{\alpha-1}] q(s) u(s) ds, \text{ and}$$

$$c_3 = \frac{1}{\Gamma(\alpha)} \int_0^1 [-(\alpha-1)(1-s)^{\alpha-2} + 2(1-s)^{\alpha-1}] q(s) u(s) ds. \text{ Hence, equality (2.3) becomes}$$

$$u(t) = \frac{1}{\Gamma(\alpha)} \left[ \int_0^1 \{ [(\alpha-1)(1-s)^{\alpha-2} - 3(1-s)^{\alpha-1}]t^2 + [-(\alpha-1)(1-s)^{\alpha-2} + 2(1-s)^{\alpha-1}]t^3 \} q(s) u(s) ds + \int_0^t (t-s)^{\alpha-1} q(s) u(s) ds \right],$$

which can be written as equation (2.1) with  $G(t,s)$  given by equation (2.2). This concludes the proof.

**Lemma 2.2** The function  $G$  defined in equation (2.2) satisfies the following property

$$|G(t,s)| \leq |G(1,s)| = \frac{27(\alpha-2)(\alpha-1)^{\alpha-1}}{\Gamma(\alpha)(\alpha+2)^{\alpha+2}} \text{ for } (t,s) \in [0,1] \times [0,1]. \tag{2.4}$$

**Proof.** It follows from (2.2) that for  $0 \leq t < s \leq 1$ ,  $G$  is an increasing function of  $t$  and

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$$|G(t, s)| \leq |G(s, s)| \leq \frac{s^2(1-s)^{\alpha-1}}{\Gamma(\alpha)} [(\alpha-2)s - (4-\alpha)(1-s)] \leq \frac{(\alpha-2)s^3(1-s)^{\alpha-1}}{\Gamma(\alpha)} \quad (2.5)$$

For  $0 < s < t \leq 1$ , we have  $(1-s)^{\alpha-2} > (1-s/t)^{\alpha-2}$ ,  $2-3t < 1$  and  $1-s > 1-t$ . Now, for a fixed  $s \in (0, 1)$  let us define

$$\Gamma(\alpha) \omega_s(t) = (t-s)^{\alpha-1} + (1-s)^{\alpha-2} \{t^2[(\alpha-1) - 3(1-s)] + t^3[2(1-s) - (\alpha-1)]\}.$$

Consider the similar approach as in [9]; taking the time derivative of this equation and after some calculations gives

$$\lim_{t \rightarrow s^+} \omega_s(t) = |G(s, s)|.$$

Hence, for  $0 < s < t \leq 1$ , we have  $|G(t, s)| \leq \max\{|\omega_s(s)|, |\omega_s(1)|\}$ .

However,  $\omega_s(1) = 0$ . Hence, we get

$$\max_{t \in [0,1]} |G(t, s)| \leq |G(s, s)|. \quad (2.6)$$

From equation (2.5),  $\phi(s) = s^3(1-s)^{\alpha-1}$ ,  $s \in [0,1]$ . Which indicates that  $\phi(s)$  is maximum at  $s = s^*$  and

$$\max_{t \in [0,1]} \phi(s) = \phi(s^*) = \frac{27(\alpha-1)^{\alpha-1}}{(\alpha+2)^{\alpha+2}}. \text{ Which together with (2.5) and (2.6) proves (2.4).}$$

**Theorem 2.1.** If a nontrivial continuous solution of the FBVP (1.3)-(1.4) exists, then the Lyapunov-type inequality is given by

$$\int_0^1 |q(s)| ds \geq \frac{\Gamma(\alpha)(\alpha+2)^{\alpha+2}}{27(\alpha-2)(\alpha-1)^{\alpha-1}} \quad (2.7)$$

and for integer order case  $\alpha = 4$

$$\int_0^1 |q(s)| ds \geq 192. \quad (2.8)$$

**Proof.** Let  $B = C[0, 1]$  be a Banach space endowed a norm

$$\|u\|_\infty \leq \max_{0 \leq t \leq 1} |u(t)|, u \in B.$$

Hence, from equation (2.1) we get  $\|u\|_\infty \leq \int_0^1 |G(t, s)q(s)| ds \|u\|_\infty$ , or equivalently,

$$1 \leq \sup_{0 \leq t \leq 1} \int_0^1 |G(t, s)q(s)| ds. \quad (2.9)$$

Using (2.4) in (2.9) proves (2.7) and setting  $\alpha = 4$  in equation (2.7), we get equation (2.8). This completes the proof of Theorem 2.1.

We note that the inequality in (2.8) is the Lyapunov-type inequality for integer order boundary value problem (IOBVP), which is obtained by Yang in [14]. Now, consider the following linear Fractional Eigenvalue Problem (FEP) and the boundary conditions,

$$({}^C_0 D_t^\alpha u)(t) - \lambda u(t) = 0, \quad 0 < t < 1, \quad \alpha \in (3,4) \quad (2.10)$$

$$u(0) = 0, u'(0) = 0, u(1) = u'(1) = 0, \quad (2.11)$$

We refer to the definitions of Lyapunov Inequality Lower Bound (LILB) and Cauchy-Schwarz Inequality Lower Bound (CSILB) form articles [7-10] and obtain the following results.

**Corollary 2.1.** The LILB for the smallest eigenvalue of FEP (1.10)-(1.11) for  $\alpha \in (3, 4]$  is given by

$$\lambda \geq \frac{\Gamma(\alpha)(\alpha+2)^{\alpha+2}}{27(\alpha-2)(\alpha-1)^{\alpha-1}}. \quad (2.12)$$

In particular for IOEP this bound is  $\lambda \geq 192$ .

**Proof.** Using equation (1.3) and (2.7) with  $q(t) = \lambda$  we obtain the inequality (2.12). In equation (2.12) taking  $\alpha = 4$  gives the lower bound for IOEP.

**Lemma 2.3.** If  $\lambda$  is the lowest eigenvalue and  $G(t, s)$  is the Green's function of the FEP (1.10)-(1.11) then

$$\lambda \geq \left[ \int_0^1 \int_0^1 G(t, s) ds dt \right]^{-\frac{1}{2}}. \quad (2.13)$$

**Proof.** Using equation (2.1) with  $q(t) = \lambda$  and the Cauchy-Schwarz inequality we obtain

$$|u(t)|^2 \leq \lambda^2 \left[ \int_0^1 |G(t, s)|^2 ds \right] \left[ \int_0^1 u^2(s) ds \right].$$

Integrating this inequality with respect to  $t$  over  $[0, 1]$  and then dividing the result by  $\|u\|_2$ , we get equation (2.13).

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**Theorem 2.2.** If a nontrivial continuous solution of the eigenvalue problem (2.10) -(2.11) exists, then CSILB is given by

$$|\lambda| \geq \Gamma(\alpha) \left[ \frac{2(\alpha-1)^2 - 11(2\alpha-3)}{210(2\alpha-3)} + \frac{32\alpha+35}{70\alpha(2\alpha-1)} + 2B(1, \alpha)[C_1(\alpha) - (\alpha-1)C_2(\alpha)] \right]^{-\frac{1}{2}}, \quad (2.14)$$

where  $C_1(\alpha) = \int_0^1 (2t^3 - 3t^2)t^\alpha {}_2F_1(1-\alpha, 1; \alpha+1; t)dt$  and  $C_2(\alpha) = \int_0^1 (t^3 - t^2)t^\alpha {}_2F_1(2-\alpha, 1; \alpha+1; t)dt$

**Proof.** Using the Green's function from Lemma 2.1 in equation (2.13) for the Cauchy-Schwarz inequality proves the inequality (2.14).

**Theorem 2.3.** The FEP (1.10) -(1.11) has infinite eigenvalues, and they are the roots of the combination of Mittag-Leffler functions  $E_{\alpha,2}(z)E_{\alpha,4}(z) - [E_{\alpha,3}(z)]^2$ , i.e. the eigenvalues satisfy

$$E_{\alpha,2}(\lambda)E_{\alpha,4}(\lambda) - [E_{\alpha,3}(\lambda)]^2 = 0. \quad (2.15)$$

**Proof.** To prove this, we take Laplace transform of equation (2.10), which after some manipulations lead to

$$U(s) = \frac{a_0 s^{\alpha-3}}{s^{\alpha}-\lambda} + \frac{a_1 s^{\alpha-4}}{s^{\alpha}-\lambda}, \quad (2.16)$$

where  $U(s)$  is the Laplace transform [11], [12] of  $u(t)$  and  $a_i = (D_{0^+}^{\alpha-i-1}u)(0), i = 0, 1, 2, 3$ . Taking inverse Laplace transform of equation (2.16), we obtain

$$u(t) = a_0 t^2 E_{\alpha,3}(\lambda t^\alpha) + a_1 t^3 E_{\alpha,4}(\lambda t^\alpha).$$

Using the boundary conditions given by equation (2.11), we obtain that the eigenvalues of the FEP (1.10) -(1.11) are the roots of equation (2.15). It follows from Hadamard's factorization theory that an entire function of fractional order has infinitely many zeros (Ahlfors [13]). Consequently, all Mittag-Leffler functions  $E_{\alpha,\beta}(z)$  will have an infinite number of zeros. This completes the proof of Theorem 2.3.

We use MATHEMATICA to find the smallest eigenvalues of combinations of the Mittag-Leffler functions. For solving equation (2.15) for  $\lambda$ , we examine using MATHEMATICA that equation (2.15) has no solution for  $\alpha = 3.1$  to  $\alpha = 3.469391976$ . Whereas for  $\alpha = 3.469391977$  to  $\alpha = 4$ , equation (2.15) has solutions. For  $\alpha = 4$  the smallest eigenvalue of (2.15) is 500.56390. Hence, for the combination of Mittag-Leffler functions (2.15), we calculate the eigenvalues for  $\alpha = 3.469391977, 3.5, 3.6, 3.7, 3.8, 3.9$  and 4. Which is shown in figure 2.1.

For comparison purpose, we compute the smallest eigenvalues for FEP (2.10) -(2.11) from equation (2.15) and its LILB, and CSILB for different  $\alpha, \alpha \in (3, 4]$  from equations (2.12) and (2.14). The results are shown in Figure 2.1.

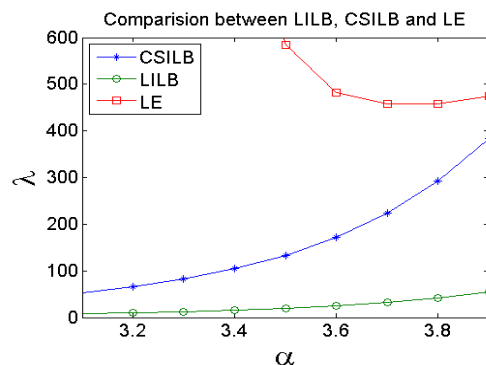


Figure 2.1: Comparison of the lower bounds for  $\lambda$  obtained from Lyapunov-type and Cauchy-Schwarz inequalities with the lowest eigenvalue. (-o-: LILB; -\*: CSILB; -□-: LE - the Lowest Eigenvalue  $\lambda$ )

This figure clearly demonstrates that among the three estimates considered here, the LILB provides the worse estimate and the CSILB provides the best estimate for the smallest eigenvalues of FEP (2.10) -(2.11). We now consider an application of the lower bounds for the smallest eigenvalues of FEP (2.10) -(2.11) found in Corollary 2.1 and Theorem 2,2 and 2.3.

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**Theorem 2.4.** Let  $3 < \alpha \leq 4$  then based on the LILB and CSILB inequalities, the combination of Mittag-Leffler functions  $E_{\alpha,2}(z)E_{\alpha,4}(z) - [E_{\alpha,3}(z)]^2$  has no real zeros in the following domains:

LILB inequality:

$$z \in \left(-\frac{\Gamma(\alpha)(\alpha+2)^{\alpha+2}}{27(\alpha-2)(\alpha-1)^{\alpha-1}}, 0\right], \quad (2.17)$$

CSILB inequality:

$$z \in \left(-\Gamma(\alpha) \left[ \frac{2(\alpha-1)^2 - 11(2\alpha-3)}{210(2\alpha-3)} + \frac{32\alpha+35}{70\alpha(2\alpha-1)} + 2B(1, \alpha)[C_1(\alpha) - (\alpha-1)C_2(\alpha)] \right]^{-\frac{1}{2}}, 0\right]. \quad (2.18)$$

**Proof.** Let  $\lambda$  be the smallest eigenvalue of the FEP (2.10)-(2.11), then  $z = \lambda$  is the smallest value of  $z$  for which  $E_{\alpha,2}(z)E_{\alpha,4}(z) - [E_{\alpha,3}(z)]^2$ . If there is another  $z$  smaller than  $\lambda$  for which  $E_{\alpha,2}(z)E_{\alpha,4}(z) - [E_{\alpha,3}(z)]^2$ , then it will contradict that  $\lambda$  is the smallest eigenvalue. Therefore,  $E_{\alpha,2}(z)E_{\alpha,4}(z) - [E_{\alpha,3}(z)]^2$  has no real zero for  $z \in (-\lambda, 0]$ . Now, the results of LILB and CSILB, from (2.12) and (2.14) prove equations (2.17) and (2.18).

From Figure 2.1, it is clear that among the two inequalities discussed in the paper, LILB provides the smallest interval and CSILB provide the largest interval in which the combination of Mittag-Leffler functions  $E_{\alpha,2}(z)E_{\alpha,4}(z) - [E_{\alpha,3}(z)]^2$  have no real zero.

### III. CONCLUSION

In this paper, we established a Lyapunov-type inequality for a fractional boundary value problem of order  $\alpha$  for  $\alpha \in (3, 4]$ . For the integer and fractional eigenvalue problems, we determined lower bounds for the first eigenvalue from Lyapunov-type and Cauchy-Schwarz type inequalities. We showed that the Cauchy-Schwarz type inequality provides the best and the Lyapunov type inequality provides the worse estimate for the smallest eigenvalue of the FEP. We used these bounds for the smallest eigenvalue to find the domain in which the combination of Mittag-Leffler functions  $E_{\alpha,2}(z)E_{\alpha,4}(z) - [E_{\alpha,3}(z)]^2$  have no zero. Results showed that Cauchy-Schwarz type inequality provides the largest domain.

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